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Almost α-Hardy-Rogers-F-contractions and their applications

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Abstract. The aim of this article is to introduce the notion of almost α-Hardy-Rogers-F-contractions in the partial metric space and utilize it to establish the existence of a unique fixed point. Some examples are given to demonstrate the validity of our main result. Our results generalize classical and newer results in the literature. As an application, we solve the initial value problem of damped harmonic oscillator and a nonlinear fractional differential equation satisfying periodic boundary conditions, which demonstrates the importance of our contraction and provides motivation for such investigations.

Key Words: α-admissible, 0-complete, almost α-Hardy-Rogers-F-contraction, partial metric space.
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Introduction

Matthews [5] initiated the study of partial metric spaces, as a simplification of metric spaces defined by Maurice Rèn Frèchet in 1906, which has extensive applications in numerous branches of Mathematics in addition to the field of the Computer science, for instance, Domain theory and denotational semantics of programming languages. Partial metrics are more adaptable than usual metrics and lead to partial orders. Further, topological characteristics of partial metrics are more widespread than those of metrics, as the self distance is not essentially zero in it. Diversity of nonlinear problems in real world necessitates the exploration of newer and more superior tools for the survival of the fixed point, consequently, it is imperative, from a reasonable perspective, that the fixed point exists and is possibly unique. Acknowledging the works of Wardowski [9] and Samet et al. [8], the aim of this paper is to introduce the notion of "almost α-Hardy-Rogers-F-contraction" in partial metric spaces and then establish an adequate environment for the existence of a single fixed point. From this particular almost α-Hardy-Rogers-F-contraction, we acquire several contractions of known type, for instance:

Kannan contraction[3], Chatterjea contraction[1], Reich contraction[6], Wardowski contraction[9], and so on. We solve the equation of damped harmonic oscillator since in the real world frictional forces (such as air resistance) will slow or dampen the motion of an object and damped harmonic oscillator is a decent model for numerous physical structures (for instance, shock absorbers in automobiles, carpet pads, circuits with an inductor and capacitor). Further, we solve nonlinear fractional differential equations since they have applications in numerous fields of science.

1 Prelimanaries

Definition 1 *[4, 5] Let $X \neq \phi$ and $\tilde{p} : X \times X \to [0, \infty)$ satisfy the following conditions:*
(i) $\rho = \eta \Leftrightarrow \tilde{p}(\rho, \rho) = \tilde{p}(\eta, \eta) = \tilde{p}(\rho, \eta)$ (0-separation),
(ii) $\tilde{p}(\rho, \rho) \leq \tilde{p}(\rho, \eta)$ (small self-distances),
(iii) $\tilde{p}(\rho, \eta) = \tilde{p}(\eta, \rho)$ (symmetry),
(iv) $\tilde{p}(\rho, \eta) \leq \tilde{p}(\rho, \mu) + \tilde{p}(\mu, \eta) - \tilde{p}(\mu, \mu)$ (modified triangle inequality),
for all $\rho, \eta, \mu \in X$. Then we say that $(X, \tilde{p})$ is a partial metric space.

Clearly, if $\tilde{p}(\rho, \eta) = 0$, then $\rho = \eta$, however, if $\rho = \eta$, then $\tilde{p}(\rho, \eta)$ is not essentially 0. In fact, $\tilde{p}(\rho, \rho)$ is the size or weight of ρ, which is utilized to portray the measure of data contained in ρ. Further, ρ is completely defined if $\tilde{p}(\rho, \rho) = 0$. One may notice that the partial metric is equivalent to metric plus size.

Moreover, the functions $\tilde{p}^s, \tilde{p}^w : X \times X \to \mathbb{R}^+$ defined by

$$\tilde{p}^s(\rho, \eta) = 2\tilde{p}(\rho, \eta) - \tilde{p}(\rho, \rho) - \tilde{p}(\eta, \eta) \tag{1}$$

and

$$\tilde{p}^w(\rho, \eta) = \tilde{p}(\rho, \eta) - \min\{\tilde{p}(\rho, \rho), \tilde{p}(\eta, \eta)\}$$

are the (usual) metrics on X.

Definition 2 *([7]) The sequence $\{\rho_n\}$ in $(X, \tilde{p})$*
(i) 0-converges to $\rho \in X$ if $\lim_{n\to\infty} \tilde{p}(\rho_n, \rho) = \tilde{p}(\rho, \rho) = \lim_{n\to\infty} \tilde{p}(\rho_n, \rho_n) = 0$,
(ii) 0-Cauchy if $\lim_{n,m\to\infty} \tilde{p}(\rho_n, \rho_m) = 0$.

Definition 3 *[7] If every 0-Cauchy sequence converges to $\rho \in X$ and $\tilde{p}(\rho, \rho) = 0$, then $(X, \tilde{p})$ is 0-complete.*

In a partial metric space every 0-Cauchy sequence is a Cauchy sequence, however, the opposite is not always valid.

Example 1 *Let $X = \{a, b\}$, define*

$$\tilde{p}(\rho,\eta) = \begin{cases} 1, & \rho = \eta, \\ 3, & otherwise. \end{cases}$$

The sequence $\{\rho_n = a, n \geq 1\}$ is not 0-Cauchy but converges to a. Hence $\{\rho_n\}$ is a Cauchy sequence.

Definition 4 *[9] By $\mathcal{F}$ denote the family of functions $F : (0, +\infty) \to \mathbb{R}$ satisfying*

(F1) F is strictly increasing,

(F2) for all $\{a_n\} \in \mathbb{R}^+$, $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} F(a_n) = -\infty$,

(F3) there exists a number $\lambda \in (0, 1)$ satisfying $\lim_{a\to 0^+} a^\lambda F(a) = 0$.

Definition 5 *[8] A self mapping T on X is α-admissible if there exists a function $\alpha : X \times X \to \mathbb{R}^+$ such that $\alpha(\rho, \eta) \geq 1$ implies $\alpha(T\rho, T\eta) \geq 1$ for all $\rho, \eta \in X$.*

Example 2 *[8] Let $X = [0, \infty)$. Define $T : X \to X$ as*

$$T\rho = \begin{cases} ln|\rho|, & \rho \neq 0, \\ 3, & otherwise, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(\rho,\eta) = \begin{cases} 3, & \rho \geq \eta, \\ 0, & otherwise. \end{cases}$$

Then T is α-admissible.

2 Main Results

Now we introduce the notion of "almost α-Hardy-Rogers-F-contraction" in partial metric spaces, and prove the existence and uniqueness of the fixed point of this mapping. For the sake of convenience, we assume that an expression $-\infty \cdot 0$ has the value $-\infty$.

Definition 6 *A self mapping T on $(X, \tilde{p})$ is called an almost α-Hardy-Rogers-F-contraction if there exist $\tau > 0$, $F \in \mathcal{F}$, and $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that $\tilde{p}(T\rho, T\eta) > 0$ implies*

$$\tau + \alpha(\rho, \eta)F(\tilde{p}(T\rho, T\eta)) \leq F(m(\rho, \eta)) + L\tilde{p}^w(\eta, T\rho)$$

for all $\rho, \eta \in X$, where

$$m(\rho, \eta) = A\tilde{p}(\rho, \eta) + B\tilde{p}(\rho, T\rho) + C\tilde{p}(\eta, T\eta) + D\tilde{p}(\rho, T\eta) + E\tilde{p}(\eta, T\rho),$$

$A + B + C + D + E = 1$, $C \neq 1$, $E \geq 0$, and $L \geq 0$.

For $A = D = E = 0$, $B + C = 1$, $C \neq 1$ in Definition 6 we get

Definition 7 *A self mapping T on $(X, \tilde{p})$ is called an almost α-Kannan-F-contraction if there exist $\tau > 0$, $F \in \mathcal{F}$, and $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that $\tilde{p}(T\rho, T\eta) > 0$ implies*

$$\tau + \alpha(\rho, \eta)F(\tilde{p}(T\rho, T\eta)) \leq F(m(\rho, \eta)) + L\tilde{p}^w(\eta, T\rho),$$

for all $\rho, \eta \in X$, where $m(\rho, \eta) = B\tilde{p}(\rho, T\rho) + C\tilde{p}(\eta, T\eta)$, $B + C = 1$, $C \neq 1$, and $L \geq 0$.

For $A = B = C = 0$, $D + E = 1$, $E > 0$ in Definition 6 we get

Definition 8 *A self mapping T on $(X, \tilde{p})$ is called an almost α-Chatterjee-F-contraction if there exist $\tau > 0$, $F \in \mathcal{F}$, and $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that $\tilde{p}(T\rho, T\eta) > 0$ implies*

$$\tau + \alpha(\rho, \eta)F(\tilde{p}(T\rho, T\eta)) \leq F(m(\rho, \eta)) + L\tilde{p}^w(\eta, T\rho),$$

for all $\rho, \eta \in X$, where $m(\rho, \eta) = D\tilde{p}(\rho, T\eta) + E\tilde{p}(\eta, T\rho), D + E = 1$, $E > 0$ and $L \geq 0$.

For $A + B + C = 1$, $C \neq 1$, $D = E = 0$ in Definition 6 we get

Definition 9 *A self mapping T on $(X, \tilde{p})$ is called an almost α-Chatterjee-F-contraction if there exist $\tau > 0$, $F \in \mathcal{F}$, and $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that $\tilde{p}(T\rho, T\eta) > 0$ implies*

$$\tau + \alpha(\rho, \eta)F(\tilde{p}(T\rho, T\eta)) \leq F(m(\rho, \eta)) + L\tilde{p}^w(\eta, T\rho),$$

for all $\rho, \eta \in X$, where $m(\rho, \eta) = A\tilde{p}(\rho, \eta) + B\tilde{p}(\rho, T\rho) + C\tilde{p}(\eta, T\eta)$, $A + B + C = 1$, $C \neq 1$ and $L \geq 0$.

For $B = C = D = E = 0$ and $A = 1$ in Definition 6 we get

Definition 10 *A self mapping T on $(X, \tilde{p})$ is called an almost α-Wardowski-F-contraction if there exist $\tau > 0$, $F \in \mathcal{F}$, and $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ such that $\tilde{p}(T\rho, T\eta) > 0$ implies*

$$\tau + \alpha(\rho, \eta)F(\tilde{p}(T\rho, T\eta)) \leq F(m(\rho, \eta)) + L\tilde{p}^w(\eta, T\rho)$$

for all $\rho, \eta \in X$, where $m(\rho, \eta) = \tilde{p}(\rho, \eta)$ and $L \geq 0$.

Theorem 1 *Let an almost α-Hardy-Rogers-F-contraction T on a 0-complete partial metric space $(X, \tilde{p})$ satisfy the following conditions*

(i) there exists $\rho_0 \in X$ such that $\alpha(\rho_0, T\rho_0) \geq 1$,

(ii) T is α-admissible,

(iii) T is continuous.

Then T has a unique fixed point $\rho^* \in X$ such that for all $\rho_0 \in X$ the sequence $\{T^n\rho_0\}_{n\in\mathbb{N}}$ is convergent to ρ^*.

Proof. Let $\rho_0 \in X$ be such that $\alpha(\rho_0, T\rho_0) \geq 1$. Let a sequence $\{\rho_n\}_{n\in\mathbb{N}} \in X$ be defined as $\rho_{n+1} = T\rho_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\rho_{n_0+1} = \rho_{n_0}$, i.e., $T\rho_{n_0} = \rho_{n_0}$, then the conclusion follows. Hence, assume that $\rho_{n+1} \neq \rho_n$ for all $n \in \mathbb{N}$. Therefore, from condition (i), we get

$$\alpha(\rho_0, \rho_1) = \alpha(\rho_0, T\rho_0) \geq 1.$$

Using condition (ii), we get

$$\alpha(\rho_1, \rho_2) = \alpha(T\rho_0, T\rho_1) \geq 1.$$

Again, using condition (ii), we get

$$\alpha(\rho_2, \rho_3) = \alpha(T\rho_1, T\rho_2) \geq 1$$

and so on. Thus, by mathematical induction we get

$$\alpha(\rho_n, \rho_{n+1}) \geq 1, \text{for all } n \in \mathbb{N}. \tag{2}$$

Since T is an almost α-Hardy-Rogers-F-contraction, it holds

$$F(\tilde{p}(\rho_n, \rho_{n+1})) = F(\tilde{p}(T\rho_{n-1}, T\rho_n)) \leq \alpha(\rho_{n-1}, \rho_n)F(\tilde{p}(T\rho_{n-1}, T\rho_n)),$$

and hence

$$\begin{aligned}
&\tau + F(\tilde{p}(\rho_n, \rho_{n+1})) \leq \tau + \alpha(\rho_{n-1}, \rho_n)F(\tilde{p}(T\rho_{n-1}, T\rho_n)) \\
&\leq F\Bigg(A\tilde{p}(\rho_{n-1}, \rho_n) + B\tilde{p}(\rho_{n-1}, T\rho_{n-1}) + C\tilde{p}(\rho_n, T\rho_n) + D\tilde{p}(\rho_{n-1}, T\rho_n) \\
&+E\tilde{p}(\rho_n, T\rho_{n-1})\Bigg) + L\tilde{p}^w(\rho_n, T\rho_{n-1}) \\
&\leq F\Bigg((A+B)\tilde{p}(\rho_{n-1}, \rho_n) + C\tilde{p}(\rho_n, \rho_{n+1}) + D\tilde{p}(\rho_{n-1}, \rho_{n+1}) + E\tilde{p}(\rho_n, \rho_n)\Bigg) \\
&+L\tilde{p}^w(\rho_n, \rho_n).
\end{aligned}$$

By using the modified triangular inequality, we get

$$\tilde{p}(\rho_{n-1}, \rho_{n+1}) \leq \tilde{p}(\rho_{n-1}, \rho_n) + \tilde{p}(\rho_n, \rho_{n+1}) - \tilde{p}(\rho_n, \rho_n).$$

Also $\tilde{p}^w$ is a metric on X, therefore

$$\tilde{p}^w(\rho_n, \rho_n) = \tilde{p}(\rho_n, \rho_n) - \min\Bigg\{\tilde{p}(\rho_n, \rho_n), \tilde{p}(\rho_n, \rho_n)\Bigg\} = 0.$$

Hence,

$$\tau + F(\tilde{p}(\rho_n, \rho_{n+1}) \leq F\Bigg((A+B)\tilde{p}(\rho_{n-1}, \rho_n) + C\tilde{p}(\rho_n, \rho_{n+1})$$
$$+D\Bigg(\tilde{p}(\rho_{n-1}, \rho_n) + \tilde{p}(\rho_n, \rho_{n+1}) - \tilde{p}(\rho_n, \rho_n)\Bigg) + E\tilde{p}(\rho_n, \rho_n)\Bigg)$$
$$= F\Bigg((A+B+D)\tilde{p}(\rho_{n-1}, \rho_n) + (C+D)\tilde{p}(\rho_n, \rho_{n+1}) + (E-D)\tilde{p}(\rho_n, \rho_n)\Bigg).$$

Since in a partial metric $\tilde{p}(\rho_n, \rho_n) \leq \tilde{p}(\rho_n, \rho_{n+1})$, we have

$$F(\tilde{p}(\rho_n, \rho_{n+1}) \leq F\Bigg((A+B+D)\tilde{p}(\rho_{n-1}, \rho_n) + (C+E)\tilde{p}(\rho_n, \rho_{n+1})\Bigg) - \tau$$
$$< F\Bigg((A+B+D)\tilde{p}(\rho_{n-1}, \rho_n) + (C+E)\tilde{p}(\rho_n, \rho_{n+1})\Bigg).$$

Note that F is strictly increasing, and hence

$$\tilde{p}(\rho_n, \rho_{n+1}) < (A+B+D)\tilde{p}(\rho_{n-1}, \rho_n) + (C+E)\tilde{p}(\rho_n, \rho_{n+1}).$$

Since $A+B+C+D+E = 1$ and $C \neq 1$, we have $1-C-E > 0$ and

$$\tilde{p}(\rho_n, \rho_{n+1}) < \frac{(A+B+D)}{(1-C-E)}\tilde{p}(\rho_n, \rho_{n-1}) = \tilde{p}(\rho_n, \rho_{n-1}),$$

for all $n \in \mathbb{N}$. Consequently,

$$F(\tilde{p}(\rho_n, \rho_{n+1})) \leq F(\tilde{p}(\rho_{n-1}, \rho_n)) - \tau \text{ for each n} \in \mathbb{N}.$$

Hence,

$$F(\tilde{p}(\rho_n, \rho_{n+1})) \leq F(\tilde{p}(\rho_{n-2}, \rho_{n-1})) - 2\tau.$$

By generalizing, we get

$$F(\tilde{p}(\rho_n, \rho_{n+1})) \leq F(\tilde{p}(\rho_0, \rho_1)) - n\tau \text{ for all } n \in \mathbb{N}. \tag{3}$$

Taking the limit as $n \to \infty$ and using $(F2)$, we get

$$\lim_{n\to\infty} \tilde{p}(\rho_n, \rho_{n+1}) = 0. \tag{4}$$

Using $(F3)$, we prove that there exists $\lambda \in (0,1)$ such that

$$\lim_{n\to\infty} (\tilde{p}(\rho_n, \rho_{n+1}))^\lambda F(\tilde{p}(\rho_n, \rho_{n+1})) = 0. \tag{5}$$

Therefore, from inequality (3) it follows that

$$(\tilde{p}(\rho_n, \rho_{n+1}))^\lambda \Big(F(\tilde{p}(\rho_n, \rho_{n+1})) - F(\tilde{p}(\rho_0, \rho_1)) \Big) \leq -(\tilde{p}(\rho_n, \rho_{n+1}))^\lambda n\tau \leq 0. \tag{6}$$

Using equations (4) and (5), we obtain

$$\lim_{n\to\infty} (n(\tilde{p}(\rho_n, \rho_{n+1}))^\lambda) = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that $n(\tilde{p}(\rho_n, \rho_{n+1}))^\lambda \leq 1$, i.e.,

$$\tilde{p}(\rho_n, \rho_{n+1}) \leq n^{-\frac{1}{\lambda}}$$

for each $n \geq n_1$. Therefore, for all $m > n > n_1$, we have

$$\begin{aligned} \tilde{p}(\rho_n, \rho_m) &\leq \sum_{i=n}^{m-1} \tilde{p}(\rho_i, \rho_{i+1}) - \sum_{i=n}^{m-2} \tilde{p}(\rho_{i+1}, \rho_{i+1}) \\ &= \sum_{i=n}^{m-1} \tilde{p}(\rho_i, \rho_{i+1}) < \sum_{i=n}^{\infty} \tilde{p}(\rho_i, \rho_{i+1}) \leq \sum_{i=1}^{\infty} i^{-\frac{1}{\lambda}}. \end{aligned}$$

By the p-series test, $\sum_{i=1}^{\infty} i^{-\frac{1}{\lambda}}$ is convergent for $\lambda \in (0,1)$, i.e., $1/\lambda > 1$. Hence, $\lim_{n,m\to\infty} \tilde{p}(\rho_n, \rho_m) = 0$, and $\{\rho_n\}$ is a 0-Cauchy sequence. From 0-completeness, it follows that there exists $\rho^* \in X$ such that $\lim_{n\to\infty} \rho_n = \rho^*$. But T is a continuous function, therefore

$$\tilde{p}(\rho^*, T\rho^*) = \lim_{n\to+\infty} \tilde{p}(\rho_n, T\rho_n) = \lim_{n\to+\infty} \tilde{p}(\rho_n, \rho_{n+1}) = 0,$$

i.e., ρ^* is a fixed point.

Let ρ^* and v be fixed points of T, but yet $\rho^* \neq v$. Using almost α-Hardy-Rogers-F-contraction for $\rho = \rho^*$ and $\eta = v$, we obtain that $\tilde{p}(T\rho^*, Tv) > 0$ implies

$$\begin{aligned} \tau + \alpha(\rho^*, v) F(\tilde{p}(T\rho^*, Tv)) \leq F(&A\tilde{p}(\rho^*, v) + B\tilde{p}(\rho^*, T\rho^*) + C\tilde{p}(v, Tv) \\ &+ D\tilde{p}(\rho^*, Tv) + E\tilde{p}(v, T\rho^*)) + L\tilde{p}^w(v, T\rho^*), \end{aligned}$$

$$\tau + \alpha(\rho^*, v)F(\tilde{p}(\rho^*, v)) \leq F(A\tilde{p}(\rho^*, v) + B\tilde{p}(\rho^*, \rho^*) + C\tilde{p}(v, v) + D\tilde{p}(\rho^*, v)$$

$$+ E\tilde{p}(v, \rho^*)) + L\bigg(\tilde{p}(v, \rho^*) - \min\{\tilde{p}(v, v), \tilde{p}(\rho^*, \rho^*)\}\bigg),$$

$$\tau+\alpha(\rho^*, v)F(\tilde{p}(\rho^*, v)) \leq F((A+B+C+D+E)\tilde{p}(\rho^*, v))+L(\tilde{p}(v, \rho^*)-\tilde{p}(\rho^*, \rho^*)),$$

and

$$F(\tilde{p}(\rho^*, v)) \leq F(\tilde{p}(\rho^*, v)),$$

since $A + B + C + D + E = 1$. Taking into account that F is strictly increasing, we get

$$\tilde{p}(\rho^*, v) \leq \tilde{p}(\rho^*, v),$$

which is possible iff $\tilde{p}(\rho^*, v) = 0$, i.e., $\rho^* = v$. Thus, the fixed point of T is unique. Further,

$$\lim_{n\to\infty} T^n\rho_0 = \lim_{n\to\infty} T^{n-1}\rho_1 = ... = \lim_{n\to\infty} \rho_n = \rho^*.$$

□

Now, we display an example to provide motivation for investigating non-linear almost α-Hardy-Rogers-F-contractions. It is interesting to note that the exhibited example does not have any significant bearing to the F-contraction.

Example 3 *Let* $X = [0, 4]$ *and*

$$\tilde{p}(\rho, \eta) = \begin{cases} \max\{\rho, \eta\}, & \rho \neq \eta, \\ 0, & \rho = \eta. \end{cases}$$

The mapping $T : X \to X$ *defined by* $T\rho = (\rho + 1)/2$ *for each* $\rho \in X$ *is continuous. Let us define the function* α *by*

$$\alpha(\rho, \eta) = \begin{cases} 1, & \rho \geq \eta, \\ 0, & \textit{otherwise}. \end{cases}$$

When $0 < \eta < \rho < 4$, *we have*

$$\tilde{p}(T\rho, T\eta) = \max\left\{\frac{\rho + 1}{2}, \frac{\eta + 1}{2}\right\} = \frac{\rho + 1}{2}.$$

Clearly, T *is almost* α*-Hardy-Rogers-*F*-contraction mapping with* $A = 0.1$, $B = 0.3$, $C = D = E = 0.2$, $\tau = 0.01$, $L = 1$, *and* $F(\alpha) = -1/\alpha$ *for all* $\rho, \eta \in X$. *Also, for* $\rho_0 = 1$ *we have*

$$\alpha(1, T1) = \alpha(1, 1) = 1.$$

Further, $1, 2 \in X$, *and* $\alpha(2, 1) = 1$ *implies* $\alpha(T2, T1) = \alpha(1.5, 1) = 1$, *i.e.,* T *is* α*-admissible.*

Now a sequence $\{\rho_n\} = \left\{\frac{1}{3n}\right\}_{n\in\mathbb{N}}$ *is* 0*-Cauchy as*

$$\lim_{n\to\infty} \tilde{p}\left(\frac{1}{3n}, 0\right) = \tilde{p}(0, 0) = \lim_{n,m\to\infty} \tilde{p}\left(\frac{1}{3n}, \frac{1}{3m}\right).$$

Since $0 \in X$, X *is* 0*-complete. Therefore, all the hypotheses of Theorem 1 are verified, and* $\rho = 1$ *is the unique fixed point of* T*. Again, for all* $\rho_0 \in X$*, there exists a sequence* $\{\rho_n\} = \left\{1 + \frac{\rho_0}{n}\right\}_{n\in\mathbb{N}}$ *such that* $\{T^n\rho_0\}_{n\in\mathbb{N}}$ *converges to* 1.

If $\alpha(\rho, \eta) = 1$ for all $\rho, \eta \in X$, and $L = 0$, then Theorem 1 is an improvement of the result by Cosentino and Vetro [2].

Theorem 2 *Let an almost* α*-Kannan-*F*-contraction* T *on a* 0*-complete partial metric space* $(X, \tilde{p})$ *satisfy the following conditions*

(i) there exists $\rho_0 \in X$ *such that* $\alpha(\rho_0, T\rho_0) \geq 1$,

(ii) T *is* α*-admissible,*

(iii) T *is continuous.*

Then T *has a unique fixed point* $\rho^* \in X$ *such that for all* $\rho_0 \in X$ *the sequence* $\{T^n\rho_0\}_{n\in\mathbb{N}}$ *is convergent to* ρ^*.

Proof. The proof follows immediately by taking $A = D = L = 0$ and $B = C \in [0, 0.5]$ in Theorem 1.□

Theorem 3 *Let an almost* α*-Chatterjee-*F*-contraction* T *on a* 0*-complete partial metric space* $(X, \tilde{p})$ *satisfy the following conditions*

(i) there exists $\rho_0 \in X$ *such that* $\alpha(\rho_0, T\rho_0) \geq 1$,

(ii) T *is* α*-admissible,*

(iii) T *is continuous.*

Then T *has a unique fixed point* $\rho^* \in X$ *such that for all* $\rho_0 \in X$ *the sequence* $\{T^n\rho_0\}_{n\in\mathbb{N}}$ *is convergent to* ρ^*.

Proof. The proof follows immediately by taking $A = B = C = 0$, $D + E = 1$, and $E \geq 0$ in Theorem 1.□

Theorem 4 *Let an almost* α*-Reich-*F*-contraction* T *on a* 0*-complete partial metric space* $(X, \tilde{p})$ *satisfy the following conditions* *(i) there exists $\rho_0 \in X$ such that $\alpha(\rho_0, T\rho_0) \geq 1$,*

(ii) T is α-admissible,

(iii) T is continuous.

Then T has a unique fixed point $\rho^* \in X$ such that for all $\rho_0 \in X$ the sequence $\{T^n \rho_0\}_{n\in\mathbb{N}}$ is convergent to ρ^*.

Proof. The proof follows immediately by taking $D = E = 0$, $A+B+C = 1$, and $C \neq 1$ in Theorem 1.□

Theorem 5 *Let an almost α-Wardowski-F-contraction T on a 0-complete partial metric space $(X, \tilde{p})$ satisfy the following conditions*

(i) there exists $\rho_0 \in X$ such that $\alpha(\rho_0, T\rho_0) \geq 1$,

(ii) T is α-admissible,

(iii) T is continuous.

Then T has a unique fixed point $\rho^* \in X$ such that for all $\rho_0 \in X$ the sequence $\{T^n \rho_0\}_{n\in\mathbb{N}}$ is convergent to ρ^*.

Proof. The proof follows immediately by taking $B = C = D = E = 0$ and $A = 1$ in Theorem 1.□

Example 4 *Let $X = [0, 4]$ and*

$$\tilde{p}(\rho, \eta) = \begin{cases} \max\{\rho, \eta\}, & \rho \neq \eta, \\ 0, & \rho = \eta. \end{cases}$$

The mapping $T : X \to X$ defined by $T\rho = (\rho + 1)/2$ for all $\rho \in X$ is continuous. Let us define the function α by

$$\alpha(\rho, \eta) = \begin{cases} 1, & \rho \geq \eta, \\ 0, & \textit{otherwise}. \end{cases}$$

When $0 < \eta < \rho < 4$,

$$\tilde{p}(T\rho, T\eta) = \max\left\{\frac{\rho+1}{2}, \frac{\eta+1}{2}\right\} = \tilde{p}(\rho, \eta).$$

Clearly, T is almost α-Wardowski-F-contraction mapping with $\tau = 0.01$, $L = 1$, and $F(\alpha) = -1/\alpha$ for all $\rho, \eta \in X$. Also, for $\rho_0 = 1$, we have

$$\alpha(1, T1) = \alpha(1, 1) = 1.$$

Further, $1, 2 \in X$*, and* $\alpha(2, 1) = 1$ *implies* $\alpha(T2, T1) = \alpha(1.5, 1) = 1$*, i.e.,* T *is* α*-admissible.*

Now a sequence $\{\rho_n\} = \{1/(n+1)\}_{n\in\mathbb{N}}$ *is* 0*-Cauchy as*

$$\lim_{n\to\infty} \tilde{p}\left(\frac{1}{n+1}, 0\right) = \tilde{p}(0, 0) = \lim_{n,m\to\infty} \tilde{p}\left(\frac{1}{n+1}, \frac{1}{m+1}\right).$$

Since $0 \in X$*,* X *is* 0*-complete. Hence, all the hypotheses of Theorem 5 are verified, and* $\rho = 1$ *is the unique fixed point of* T*. Again, for all* $\rho_0 \in X$*, the sequence* $\{\rho_n\} = \left\{1 + \frac{\rho_0}{n}\right\}_{n\in\mathbb{N}}$ *converges to* 1*.*

Different biological, industrial and economic phenomena involving threshold operations are discontinuous. In particular, neurons in a neural net either fires (function value equals to 1) or does not fire (function value equals to 0) conditional to the fact that whether the input crosses a certain threshold or not. Numerous industrial censors, bandpasses filters and the diode also work in this manner. As a result, discontinuous mappings are of strong interest for the scientists working in the fields of Natural sciences, Engineering, Economics, Neural networking, etc.

We establish our next result for discontinuous mappings satisfying almost α-Hardy- Rogers-F-contractions.

Theorem 6 *Let an almost* α*-Hardy-Rogers-*F*-contraction* T *on a* 0*-complete partial metric space* $(X, \tilde{p})$ *satisfy the following conditions*

(i) there exists $\rho_0 \in X$ *such that* $\alpha(\rho_0, T\rho_0) \geq 1$*,*

(ii) T *is* α*-admissible,*

(iii) if $\{\rho_n\}$ *is a sequence in* X *satisfying* $\alpha(\rho_n, \rho_{n+1}) \geq 1$ *and* $\rho_n \to \rho$ *as* $n \to +\infty$*, then* $\alpha(\rho_n, \rho) \geq 1$*, for all* $n \in \mathbb{N}$*.*

Then T *has a unique fixed point* $\rho^* \in X$*, and for all* $\rho_0 \in X$ *the sequence* $\{T^n\rho_0\}_{n\in\mathbb{N}}$ *is convergent to* ρ^**.*

Proof. Let $\rho_0 \in X$ be such that $\alpha(\rho_0, T\rho_0) \geq 1$, and let $\rho_{n+1} = T\rho_n$ for each $n \in \mathbb{N}$. Following the same pattern as in Theorem (1), $\{\rho_n\}$ is a 0-Cauchy sequence in $(X, \tilde{p})$. Hence, there exists $\rho^* \in X$ such that $\rho_n \to \rho^*$ as $n \to +\infty$. From the hypothesis (iii), we get

$$\alpha(\rho_n, \rho^*) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Case I: Let there exist $i_n \in \mathbb{N}$ such that $\rho_{i_n+1} = T\rho^*$ and $i_n > i_{n-1}$ for each $n \in \mathbb{N}$. Then

$$\rho^* = \lim_{n\to\infty} \rho_{i_n+1} = \lim_{n\to\infty} T\rho^* = T\rho^*,$$

and hence ρ^* is a fixed point of T.
Case II: Let $\rho_{n+1} \neq T\rho^*$, i.e., $\tilde{p}(T\rho_n, T\rho^*) > 0$ for each $n \geq n_0$. As T is an almost α-Hardy-Rogers-F-contraction on X, using $(F1)$ we obtain that

$$\tau + F(\tilde{p}(\rho_{n+1}, T\rho^*)) = \tau + F(\tilde{p}(T\rho_n, T\rho^*)) \leq \tau + \alpha(\rho_n, \rho^*)F(\tilde{p}(T\rho_n, T\rho^*))$$

$$\leq F\Bigg(A\tilde{p}(\rho_n, \rho^*) + B\tilde{p}(\rho_n, T\rho_n) + C\tilde{p}(\rho^*, T\rho^*) + D\tilde{p}(\rho_n, T\rho^*)$$

$$+E\tilde{p}(\rho^*, T\rho_n)\Bigg) + L\tilde{p}^w(\rho^*, T\rho_n)$$

$$\leq F\Bigg(A\tilde{p}(\rho_n, \rho^*) + B\tilde{p}(\rho_n, \rho_{n+1}) + C\tilde{p}(\rho^*, T\rho^*) + D\tilde{p}(\rho_n, T\rho^*)$$

$$+E\tilde{p}(\rho^*, \rho_{n+1})\Bigg) + L\tilde{p}^w(\rho^*, \rho_{n+1}).$$

Now,

$$\tilde{p}(\rho_n, T\rho^*) \leq \tilde{p}(\rho_n, \rho^*) + \tilde{p}(\rho^*, T\rho^*) - \tilde{p}(\rho^*, \rho^*),$$

and therefore

$$\tau + F(\tilde{p}(\rho_{n+1}, T\rho^*)) \leq F\Bigg(A\tilde{p}(\rho_n, \rho^*) + B\tilde{p}(\rho_n, \rho_{n+1}) + C\tilde{p}(\rho^*, T\rho^*)$$

$$+D\Bigg\{\tilde{p}(\rho_n, \rho*) + \tilde{p}(\rho^*, T\rho^*) - \tilde{p}(\rho^*, \rho^*)\Bigg\} + E\tilde{p}(\rho^*, \rho_{n+1})\Bigg)$$

$$+L\Bigg\{\tilde{p}(\rho^*, \rho_{n+1}) - \min\{\tilde{p}(\rho^*, \rho^*), \tilde{p}(\rho_{n+1}, \rho_{n+1})\}\Bigg\}.$$

If $\tilde{p}(\rho^*, T\rho^*) > 0$ then

$$\lim_{n\to\infty} \tilde{p}(\rho_n, \rho^*) = \lim_{n\to\infty} \tilde{p}(\rho_{n+1}, \rho^*) = \tilde{p}(\rho^*, \rho^*).$$

Taking the limit as $n \to \infty$ in the inequality above, we obtain

$$\tau + F(\tilde{p}(\rho^*, T\rho^*)) \leq F\Bigg((C + D)\tilde{p}(\rho^*, T\rho^*)\Bigg).$$

Since $A + B + C + D + E = 1$ and $C \neq 1$, we have $C + D < 1$, and hence

$$\tau + F(\tilde{p}(\rho^*, T\rho^*)) \leq F(\tilde{p}(\rho^*, T\rho^*)),$$

which contradicts $\tau > 0$. Therefore, our assumption is wrong, and hence $\tilde{p}(\rho, T\rho^*) = 0$, i.e., ρ^* is a fixed point of T.

Uniqueness of the fixed point follows easily, utilising an almost α-Hardy-Rogers-F-contraction. Further,

$$\lim_{n\to\infty} T^n\rho_0 = \lim_{n\to\infty} T^{n-1}\rho_1 = ... = \lim_{n\to\infty} \rho_n = \rho^*.$$

□

Example 5 *Let* $X = [0,\infty)$ *and*

$$\tilde{p}(\rho,\eta) = \begin{cases} \max\{\rho,\eta\}, & \rho \neq \eta, \\ 0, & \rho = \eta. \end{cases}$$

The mapping $T : X \to X$ *defined as*

$$T(\rho) = \begin{cases} \dfrac{\rho+2}{3}, & 1 \leq \rho \leq 4, \\ \\ \dfrac{\rho}{3}, & \rho > 4, \end{cases}$$

for all $\rho \in X$ *is discontinuous. Let us define the function* α *by*

$$\alpha(\rho,\eta) = \begin{cases} 1, & \rho,\eta \in [1,4], \\ 0, & \textit{otherwise}. \end{cases}$$

When $0 < \eta < \rho < 4$,

$$\tilde{p}(T\rho, T\eta) = \max\left\{\frac{\rho+2}{3}, \frac{\eta+2}{3}\right\} = \frac{\rho+2}{3}.$$

Clearly, T *is an almost* α*-Hardy-Rogers-*F*-contraction mapping with* $A = 0.1$, $B = 0.3$, $C = D = E = 0.2$, $\tau = 0.01$, $L = 1$, *and* $F(\alpha) = -1/\alpha$ *for all* $\rho,\eta \in X$. *Also, for* $\rho_0 = 1$ *we have*

$$\alpha(1, T1) = \alpha(1,1) = 1.$$

Further, $1, 2 \in X$, *and* $\alpha(2,1) = 1$ *implies that* $\alpha(T2, T1) = \alpha\left(\frac{4}{3}, \frac{4}{3}\right) = 1$, *i.e.,* T *is* α*-admissible.*

Now, a sequence $\{\rho_n\} = \{1/3n\}_{n\in\mathbb{N}}$ *is* 0*-Cauchy as*

$$\lim_{n\to\infty} \tilde{p}\left(\frac{1}{3n}, 0\right) = \tilde{p}(0,0) = \lim_{n,m\to\infty} \tilde{p}\left(\frac{1}{3n}, \frac{1}{3m}\right).$$

Also, $0 \in X$, *and hence* X *is* 0*-complete.*

Thus, all hypotheses of Theorem (6) are verified, and $\rho = 1$ *is unique fixed point. Again, for all* $\rho_0 \in X$ *there exists a sequence* $\{\rho_n\} = \left\{1 + \frac{\rho_0}{n}\right\}_{n\in\mathbb{N}}$ *such that* $\{T^n\rho_0\}_{n\in\mathbb{N}}$ *converges to* 1.

Remark 3.1 Following the same pattern, we may establish the existence of a fixed point for almost α-Kannan-F-contraction, almost α-Chatterjee-F-contraction, almost α-Reich-F-contraction, and almost α-Wardowski-F-contraction in a 0-complete partial metric space for a discontinuous mapping.

3 Application to Damped Harmonic Oscillators

Most of the structures in the real world follow Hooke's law when perturbed near an equilibrium point, besides losing energy as they decay back to equilibrium. The equation of damped harmonic oscillator can model entities oscillating whilst immersed in a fluid, as well as they can model more abstract structures where quantities oscillate while losing energy. In real oscillators, friction decelerates the motion of the system, consequently, the decrease in velocity is proportional to the acting frictional force. Hence, the balance of forces for damped harmonic oscillators is:

$$F = F_{ext} - k\rho - c\frac{d\rho}{dt} = m\frac{d^2\rho}{dt^2}.$$

When there is no external force (i.e., $F_{ext} = 0$),

$$\frac{d^2\rho}{dt^2} + 2\zeta\omega\frac{d\rho}{dt} + \omega^2\rho = 0, \tag{7}$$

where $\omega = \sqrt{k/m}$ is the undamped angular frequency of the oscillator and $\zeta = \dfrac{c}{2\sqrt{mk}}$ is the damping ratio. A damped harmonic oscillator is critically damped for $\zeta = 1$, i.e., the system comes back to the steady state as quickly as time permits without oscillation (however, overshoot can happen as in the case of doors).

Green's function associated to equation (7) in case of critically damped motion under conditions $\rho(0) = 0$, $\rho'(0) = a$ is given by

$$G(t,v) = \begin{cases} -ve^{\tau(v-t)}, & 0 \le v \le t \le 1, \\ \\ -te^{\tau(v-t)}, & 0 \le t \le v \le 1, \end{cases} \tag{8}$$

where $\tau > 0$ is a constant, calculated in terms of ζ and ω. Let $X = C([0,1], \mathbb{R}^+)$. For an arbitrary $u \in X$, we define

$$\|u\|_\tau = \sup_{u\in[0,1]} \{|\rho(u)|e^{-2\tau u}\}. \tag{9}$$

Further, define $\tilde{p} : X \times X \to \mathbb{R}^+$ by

$$\tilde{p}(\rho,\eta) = \|\rho - \eta\|_\tau = \sup_{u\in[0,1]} \{|\rho(u) - \eta(u)|e^{-2\tau u}\}. \tag{10}$$

In the next theorem, we consider the equation of critically damped harmonic oscillators in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position:

Theorem 7 *Let $T : X \to X$ be a self mapping on 0-complete partial metric space $(X, \tilde{p})$ such that*

1. $\alpha(\rho_0(u), T\rho_0(u)) \geq 1$ *and*

$$T\rho(u) = \int_0^u G(t, v)k(v, \rho(v))dv \qquad \text{for } \rho_0 \in X,$$

where $k : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is an increasing function which satisfies

$$|k(v, \rho(v)) - k(v, \eta(v))| \leq \tau^2 e^{-\tau} \tilde{p}(\rho, \eta)$$

for $F \in \mathcal{F}$, $\tau \in \mathbb{R}^+$, $v \in [0, 1]$, and ρ, $\eta \in \mathbb{R}^+$,

2. $\alpha(\rho(u), \eta(u)) \geq 1$ *implies* $\alpha(T\rho(u), T\eta(u)) \geq 1$ *for $\rho, \eta \in X$,*
3. *if $\{\rho_n\}$ is a sequence in X such that $\rho_n \to \rho \in X$ and $\alpha(\rho_n(u), \rho_{n+1}(u)) \geq 1$, then $\alpha(\rho_n(u), \rho(u)) \geq 1$ for all $n \in \mathbb{N}$.*

Then equation (7) *has a solution.*

Proof. Finding solution of equation (7) is equivalent to solving the integral equation

$$\rho(u) = \int_0^u G(t, v)k(v, \rho(v))dv.$$

Hence, ρ^* is a solution of equation (7) iff ρ^* is a fixed point of T. Using condition (1), for all $\rho, \eta \in X$ we have

$$|T\rho(u) - T\eta(u)| \leq \int_0^u G(t, v)|k(v, \rho(v)) - k(v, \eta(v))|dv$$

$$\leq \int_0^u G(t, v)\tau^2 e^{-\tau} \tilde{p}(\rho, \eta)dv = \int_0^u \tau^2 e^{-\tau} e^{2\tau v} e^{-2\tau v} \tilde{p}(\rho, \eta)G(t, v)dv$$

$$\leq \tau^2 e^{-\tau} \|\tilde{p}(\rho, \eta)\|_\tau \times \int_0^u e^{2\tau s} G(t, v)dv$$

$$= \tau^2 e^{-\tau} \|\tilde{p}(\rho, \eta)\|_\tau \times [-\tfrac{e^{2\tau t}}{\tau^2}(2\tau t - \tau t e^{-\tau t} + e^{-\tau t} - 1)],$$

i.e.,

$$|T\rho(u) - T\eta(u)|e^{-2\tau u} \leq e^{-\tau} \|\tilde{p}(\rho, \eta)\|_\tau \times (1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}).$$

Since $(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}) \leq 1$,

$$\|T\rho(u) - T\eta(u)\|_\tau \leq e^{-\tau} \|\tilde{p}(\rho, \eta)\|_\tau.$$

Hence,

$$\tilde{p}(T\rho, T\eta) \leq e^{-\tau}\|\tilde{p}(\rho,\eta)\|_{\tau}.$$

Taking the logarithm, we obtain

$$\tau + ln(\tilde{p}(T\rho, T\eta)) \leq ln\|\tilde{p}(\rho,\eta)\|_{\tau}.$$

Note that if $F(\rho) = \ln(\rho)$ then $F \in \mathcal{F}$. Further, let $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ be defined as

$$\alpha(\rho,\eta) = \begin{cases} 1, & if\, \alpha(\rho(u),\eta(u)) > 0, u \in [0,1], \\ \\ -\infty, & otherwise. \end{cases}$$

Then

$$\tau + \alpha(\rho,\eta)F(\tilde{p}(T\rho,T\eta)) \leq F(\tilde{p}(\rho,\eta)) + L\tilde{p}^{w}(\eta, T\rho)$$

for each $\rho,\eta \in X$, $\tilde{p}(T\rho,T\eta) \geq 1$, and $L \geq 0$. Hence, T is an almost α-Wardowski-F-contraction. According to condition (1), there exists $\rho \in X$ such that $\alpha(\rho, T\rho) \geq 1$. Further, using condition (2), we obtain

$$\alpha(T\rho(u), T\eta(u)) \geq 1$$

for each $\rho,\eta \in X$ and $u \in [0,1]$.

Thus, T is α-admissible, and using condition (3), we see that all the hypotheses of Theorem 6 (taking $B = C = D = E = 0$, $A = 1$) are verified, and $\rho^* = T\rho^*$. Hence, ρ^* is a solution of the problem (7) in case of critically damped harmonic oscillator. $\square$

4 Application to fractional calculus

Fractional differential equations occur in numerous scientific problems, since the mathematical modeling of systems and processes involves derivatives of fractional order. However, for these nonlinear fractional differential equations, it is difficult to get exact solutions. An efficient technique for solving such equations is required.

Now, we apply Theorem 5 to solve the following nonlinear fractional differential problem

$$^{c}D^{\beta}(\rho(u)) + f(u,\rho(u)) = 0, \tag{11}$$

where $\rho(0) = 0 = \rho(1)$, $(0 \leq u \leq 1, \beta < 1)$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The Caputo derivative is defined by

$$^{c}D^{\beta}(g(u)) = \frac{1}{\Gamma(n-\beta)} \int_{0}^{u} (u-v)^{n-\beta-1} g^{n}(v)dv, (n = [\beta]+1),\ n \in \mathbb{N},$$

where Γ is the gamma function and β is the fractional order of the Caputo derivative. It is worth mentioning here that the fractional order derivatives depend not only on local conditions of the evaluated time, but also on all the history of the function. Further, the Caputo derivative of a constant is equal to zero, as in the integer-order case. The associated Green's function is

$$G(u,v) = \begin{cases} (u(1-v))^{\rho-1} - (u-v)^{\rho-1}, & 0 \leq v \leq u \leq 1, \\ \dfrac{(u(1-v))^{\rho-1}}{\Gamma\rho}, & 0 \leq u \leq v \leq 1. \end{cases}$$

Let $X = C([0,1],\mathbb{R})$ be the set of all continuous real functions on $[0,1]$. We define $\|\rho\|_\infty = \sup_{u\in[0,1]} |\rho(u)|$ for all $\rho \in X$ and define $\tilde{p} : X \times X \to \mathbb{R}^+$ by $\tilde{p}(\rho,\eta) = \|\rho - \eta\|_\infty$.

Theorem 8 *Let $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $T : X \to X$ be a self mapping of a 0-complete partial metric space $(X, \tilde{p})$. Suppose that for all $u \in [0,1]$*

1. *$\alpha(\rho_0(u), T\rho_0(u)) \geq 1$ and*

$$T\rho(u) = \int_0^1 G(u,v) f(v,\rho(v)) dv, \qquad \rho_0 \in X,$$

   where $|f(v,\rho) - f(v,\eta)| \leq e^{-\tau}|\rho - \eta|$, $\rho, \eta \in X$,

2. *$\alpha(\rho(u), \eta(u)) \geq 1$ implies $\alpha(T\rho(u), T\eta(u)) \geq 1$, for $\rho, \eta \in X$,*
3. *if $\{\rho_n\}$ is a sequence in X such that $\rho_n \to \rho$ in X and $\alpha(\rho_n(u), \rho_{n+1}(u)) \geq 1$, then $\alpha(\rho_n(u), \rho(u)) \geq 1$ for all $n \in \mathbb{N}$.*

Then equation (11) *has a solution.*

Proof. Clearly, finding a solution of equation (11) is equivalent to finding $\rho^* \in X$, i.e., a fixed point of T.

We have

$$|T\rho(u) - T\eta(u)| = |\int_0^1 G(u,v)[f(v,\rho(v)) - f(v,\eta(v))]dv|$$

$$\leq \int_0^1 G(u,v) e^{-\tau} |\rho(v) - \eta(v)| dv$$

$$\leq e^{-\tau}\|\rho - \eta\|_\infty \sup_{u\in I} \int_0^1 G(u,v)dv \leq e^{-\tau}\|\rho - \eta\|_\infty$$

Consequently,

$$\|T\rho(u) - T\eta(u)\|_\infty \leq e^{-\tau}\|\rho - \eta\|_\infty$$

or

$$\tilde{p}(T\rho, T\eta) \leq e^{-\tau}\tilde{p}(\rho, \eta).$$

Taking the logarithm, we get

$$\tau + log(\tilde{p}(T\rho, T\eta)) \leq log(\tilde{p}(\rho, \eta)).$$

Note that $F(\rho) = \ln(\rho)$ is such that $F \in \mathcal{F}$. Let $\alpha : X \times X \to \{-\infty\} \cup (0, \infty)$ be defined as

$$\alpha(\rho, \eta) = \begin{cases} 1, & if\ \alpha(\rho(u), \eta(u)) > 0, u \in [0, 1], \\ -\infty, & otherwise. \end{cases}$$

Then

$$\tau + \alpha(\rho, \eta)F(\tilde{p}(T\rho, T\eta)) \leq F(\tilde{p}(\rho, \eta)) + L\tilde{p}^{w}(\eta, T\rho)$$

for each $\rho, \eta \in X$ with $\tilde{p}(T\rho, T\eta) \geq 1$ and $L \geq 0$. Hence, T is an almost α-Wardowski-F-contraction. According to condition (1), there exists $\rho_0 \in X$ such that $\alpha(\rho_0, T\rho_0) \geq 1$. Further, by condition (2), for all $\rho, \eta \in X$ and any $u \in [0, 1]$ we have $\alpha(T\rho(u), T\eta(u)) \geq 1$. Hence, T is α-admissible, and tacking into account condition (3), we see that all the hypotheses of Theorem (6) (with $B = C = D = E = 0$, $A = 1$) are verified, and $\rho^* = T\rho^*$. Hence, ρ^* is a solution of the problem (11). □

It is worth mentioning here that similar applications can be given for other known results as well.

Conclusion. In this article, we introduced the notion of almost α-Hardy-Rogers-F-contraction, independent of the existing definitions of contractions, and established novel results for continuous as well as discontinuous mappings in nonlinear analysis. From these results we deduced results for almost α-Kannan-F-contractions, almost α-Chatterjee-F-contractions, almost α-Reich-F-contractions, and almost α-Wardowski-F-contractions. These novel ideas prompt further examinations and applications. Our study is encouraged by possible applications to partial metric spaces, diversity of nonlinear problems in the real world, e.g. for real oscillators (like the case of the door), and numerous engineering and scientific disciplines.

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