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The Jordan-Postnikov Normal Form of a Real Linear Operator

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Abstract. We give a direct proof of the following result: for any real linear operator there exists a basis in which it has a Jordan-Postnikov matrix. This matrix for a real linear operator is determined uniquely up to the order of direct summands on the diagonal.

Key words: Linear operator, Jordan-Postnikov matrix, generalized Jordan forms of the first and second kinds.

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This paper is an extended variant of the report on the First International Algebra and Geometry Confer., 16-20 May, 2007, Yerevan, Armenia ([**5**]).

The theory of diverse normal (canonical) forms for linear operators on finite-dimensional vector spaces goes back to Weierstrass [**10**] and Jordan [**8**] (see [**2**], Historical notes to Ch. VI and VII). The Jordan normal form and the rational canonical form are the most known. M. Artin in [**1**] writes: "Jordan form is much nicer than rational canonical

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form, ... but it [rational canonical form] is the best form available for an arbitrary field" (pp. 479-480). Note that Jordan normal forme is available for an algebraically closed field or, more generally, when the characteristic polynomial (equivalently, the minimal polynomial) of the operator has only linear irreducible divisors.

Using the Jordan theory and a procedure of complexification to a real linear operator, Postnikov [**9**] obtains a nice canonical form for diagonalizable real operators. The generalization of this method allows to get the correspondent result in the case of an arbitrary field and for linear operators with an arbitrary characteristic polynomial ([**4**]). Emphasize that this proof is indirect, it uses the Jordan normal form. Here we give a direct geometric proof for an arbitrary real operator. This result can be obtained also from the rational canonical form, but the proof of the theorem on the rational canonical form of a linear operator is non-geometric and use many special notions.

A Postnikov's block is a matrix

$$\mathbf{P} = \begin{pmatrix} [c] & e & \dots & 0 & 0 \\ 0 & [c] & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & [c] & e \\ 0 & 0 & \dots & 0 & [c] \end{pmatrix}, \tag{1.1}$$

where

$$[c] = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is the matrix representation of the complex number $c = a + bi$ and e is the unit matrix of order two.

Definition 1. A Jordan-Postnikov *matrix is a direct sum of Postnikiv's blocks and Jordan's blocks.*

Theorem 1. *For an arbitrary linear operator $\mathcal{A}$ of a finite-dimensional real linear space L there exists a basis e such that the matrix $\mathcal{A}_e$ of $\mathcal{A}$ in e is a Jordan-Postnikov matrix. The Jordan-Postnikov* *matrix for a real linear operator $\mathcal{A}$ is defined uniquely up to ordering of Jordan's and Postnikov's blocks on the diagonal.*

One can derive the theorem 1 from the following theorem.

THEOREM 2. (see [**3**]) *Let* $\mathbf{k}$ *be an arbitrary field, and* $\mathbf{A}$ *be a matrix over* $\mathbf{k}$. *Suppose that all irreducible over* $\mathbf{k}$ *divisors of the characteristic polynomial* $F(t)$ *of* $\mathbf{A}$ *are separable and let* K *be the field of the decomposition of* $F(t)$. *Then there exists a matrix* $\mathbf{Q}$ *with elements in* K *such that*

$$\mathbf{J} = \mathbf{Q^{-1}AQ} = \mathbf{J_1} \oplus \ ... \oplus \mathbf{J}_\tau \tag{1.2}$$

is a generalized Jordan matrix of the first kind. *The generalized Jordan blocks of the first kind* $\mathbf{J}_1, \ldots \mathbf{J}_\tau$ *are determined uniquely (i.e. don't depend on the choice of* $\mathbf{Q}$*, for which (1.2) takes a place).*

A generalized Jordan block of the first kind has a form

$$\begin{pmatrix} [P(t)] & \mathbf{E} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [P(t)] & \mathbf{E} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [P(t)] & \ldots & \mathbf{0} & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & [P(t)] & \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & [P(t)] \end{pmatrix}, \tag{1.3}$$

where

$$[P(t)] = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -p_0 & -p_{\kappa 1} & -p_2 & \ldots & -p_{m-2} & -p_{m-1} \end{pmatrix}.$$

is the *companion matrix* of an irreducible polynomial

$$P(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_{m-2} t^{m-2} + p_{m-1} t^{m-1} + t^m,$$

and $\mathbf{E}, \mathbf{0}$ are respectively unit and zero blocks of the same order m that $[P(t)]$. Emphasize that in theorem 2 $P(t)$ run all irreducible divisors of $F(t)$.

The theorem 2 may be used for deriving of the theorem 1 because non-separable polynomials exist only over fields of finite characteristic, but the field $\mathbb{R}$ of the real numbers has zero characteristic.

The irreducible polynomials over $\mathbb{R}$ are or linear, or quadratic with negative discriminant. Consequently, the generalized Jordan blocks $\mathbf{J}_s$ in this case are respectively or Jordan blocks, or matrices of the form (1.1), where the diagonal block is the companion matrix

$$[P_s(t)] = \begin{pmatrix} 0 & 1 \\ -q_s & -p_s \end{pmatrix}$$

of an irreducible polynomial

$$P_s(t) = t^2 + p_s t + q_s.$$

Suppose that in this case $c_s = a_s + b_s i$ is a complex root of $P_s(t)$. Then the characteristic polynomials of $[c_s]$ and of the companion matrix $[P_s(t)]$ equal to $P_s(t)$. Consequently, there exists a matrix w_s such that

$$[c_s] = w_s^{-1}[P_s(t)]w_s.$$

Let $\deg \mathbf{J}_s = 2n_s$, $\mathbf{W}_s$ be the matrix $w_s \oplus ... \oplus w_s$ where the number of direct summands is n_s. Let $\mathbf{W} = \mathbf{W}_1 \oplus ... \oplus \mathbf{W}_\tau$ where the direct summand $\mathbf{W}_s$ is as above if the correspondent irreducible divisor of the characteristic matrix is quadratic, and $\mathbf{W}_s$ is a unit matrix elsewhere. Then $\mathbf{W}^{-1}\mathbf{J}\mathbf{W}$ is a Jordan-Postnikov matrix and the theorem 1 is proved.

Remark. The condition on the characteristic of the field in the theorem 2 is essential (a counterexample is in [**4**], the end of item 1). In the case of a field of an arbitrary characteristic it is necessary to consider the *generalized Jordan forms of the second kind* (see [**4**]). *A generalized Jordan matrix of the second kind* is a direct sum of generalized Jordan blocks of the second kind. A *generalized Jordan block of* *the second kind* has the form (1.3), but the matrices $\mathbf{F}$, which consist of 0 and have a unique 1 at the left down angle, are used instead of the unit matrices $\mathbf{E}$.

Now we give a sketch of a direct proof of the theorem 2. The full proof for the general case when $\mathcal{A}$ is a linear operator over an arbitrary field $\mathbf{k}$ see in [**6**]. (For theorem 2 in the text below $\mathbf{k} = \mathbb{R}$.)

In the first step we obtain a primary decomposition for a linear operator $\mathcal{A}$.

Definition 2. *Let $p(t)$ be a prime polynomial over $\mathbf{k}$. A linear operator $\mathcal{A}$ of a vector space L over $\mathbf{k}$ is called $p(t)$-*primary*, if the minimal polynomial $m_{\mathcal{A}}(t) = p(t)^{\mu}$ for some positive integer μ. The positive integer μ is called* the height *of $p(t)$-primary operator $\mathcal{A}$.*

Theorem 3 (Primary Decomposition Theorem ([**7**], Ch. 6.8.)). *Let $\mathcal{A}$ be an arbitrary linear operator of a vector space L over $\mathbf{k}$, $m_{\mathcal{A}}(t) = p_1^{\mu_1}(t) \ldots, p_l^{\mu_l}(t)$ be the (unitary) irreducible decomposition of $m_{\mathcal{A}}(t)$ over $\mathbf{k}$, $L_k = \ker p_k(\mathcal{A})^{\mu_k}$ be the kernel of $p_k(\mathcal{A})^{\mu_k}$, $\mathcal{A}_k = \mathcal{A}|_{L_k}$ be the restriction of $\mathcal{A}$ on L_k $(k = 1, \ldots, l)$. Then each $\mathcal{A}_k$ is a $p_k(t)$-primary operator and $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_l$. If in a direct decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_l$ the summands $\mathcal{A}_k$ are $p_k(t)$-primary operators of height μ_k $(k = 1, \ldots, l)$, then $m_{\mathcal{A}}(t) = p_1^{\mu_1}(t) \ldots p_l^{\mu_l}(t)$. Consequently, the primary decomposition of $\mathcal{A}$ is unique up to order of direct summands.*

The $p_k(t)$-primary summands are not *indecomposable* in the general case: it is possible decompose its in direct sum of non-zero operators. One can obtain a decomposition of any $p_k(t)$-primary operator $\mathcal{A}_k$ of the vector space L_k in a direct sum of indecomposable summands as follows. Further we don't use here the index k, thus, $\mathcal{A}$ is below a $p(t)$-primary operator of the vector space L.

Set $L_i = \ker p(\mathcal{A})^i$ and consider a chain of non-increasing subspaces

$$L = L_{\mu} \supset L_{\mu-1} \supset \ldots \supset L_1 \supset L_0 = \{0\}.$$

This leads to a chain of $\mathbf{k}$-linear mappings

$$L = L_\mu \to L_{\mu-1} \to \ldots \to L_1 \to L_0 = \{0\},$$

where $x \in L_i$ maps to $p(\mathcal{A})(x) \in L_{i-1}$, and to an induced chain

$$L_\mu/L_{\mu-1} \to L_{\mu-1}/L_{\mu-2} \to \ldots \to L_1/L_0 = L_1, \tag{1.4}$$

where $x + L_i \in L_{i+1}/L_i$ maps to $p(\mathcal{A})(x) + L_{i-1} \in L_i/L_{i-1}$. One can consider each $\mathbf{k}$-linear space L_i/L_{i-1} as a linear space over the quotient-field $K = \mathbf{k}[t]/(p(t))$. All arrows of chain 1.4 are injections.

For any $e \in L$ and for any subspace L_0 of L, set $\overline{e} = e + L_0 \in L/L_0$. Choose a basis in the chain of quotient-spaces (1.4) over K as follows.

1) Fix an arbitrary basis $\overline{e}_{\mu 1}, \ldots, \overline{e}_{\mu n_\mu}$ in $L_\mu/L_{\mu-1}$.

2) In each L_i/L_{i-1} $(i = \mu - 1, \ldots, 1)$ add to the K-independent vectors, that are the images of the basis vectors of L_{i+1}/L_i with respect to the K-linear map $p(\mathcal{A})$, the vectors $\overline{e}_{i1}, \ldots, \overline{e}_{i(n_i - n_{i+1})}$ such that the obtained system is a basis of the K-linear space L_i/L_{i-1}.

Let $d = \deg p(t)$, $n_i = \dim L_i/L_{i-1}$ $(\mu \geq i \geq 1)$, $n_{\mu+1} = 0$. Denote by $\mathbf{e}_{ij}$ the system of vectors (elements)

$$e_{ij}, \mathcal{A}(e_{ij}), \ldots, \mathcal{A}^{d-1}(e_{ij}); p(\mathcal{A})(e_{ij}), \mathcal{A}p(\mathcal{A})(e_{ij}), \ldots, \mathcal{A}^{d-1}p(\mathcal{A})(e_{ij}); \ldots;$$
$$p(\mathcal{A})^{i-1}(e_{ij}), \mathcal{A}p(\mathcal{A})^{i-1}(e_{ij}), \ldots, \mathcal{A}^{d-1}p(\mathcal{A})^{i-1}(e_{ij})$$

and denote by $\mathbf{e}$ the system consisting of all vectors of the systems $\mathbf{e}_{ij}$, where $i = \mu, \ldots, 1$ and $j = 1, \ldots, n_i - n_{i+1}$.

Then the system $\mathbf{e}$ is a basis of L over $\mathbf{k}$. The matrix $\mathcal{A}_{\mathbf{e}}$ is a direct sum of (irreducible) generalized Jordan blocks of second kind with $[p(t)]$ on the diagonal. These summands are determined uniquely because the number of Jordan blocks of fixed order in such a decomposition depends only on n_i, i.e. on the chain 1.4.

Returning to the theorem 1 and applying this construction to each $p_k(t)$-primary summand $\mathcal{A}_k$ of a general real linear operator $\mathcal{A}$ we get its proof.

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