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On Biorthogonalization of a Dirichlet System Over a Finite Interval

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Abstract. Ultimately aiming to estimate Dirichlet polynomials, a representation problem for special biorthogonal systems of exponentials is explored in $L^2(0,a)$. If $a=+\infty$, a method of construction of such systems through suitable Blaschke products is known, but the method ceases to operate when a is finite.

It turns out that the Blaschke product cannot be even adjusted to maintain the old method for the new situation. The biorthogonal system is then represented by a single determinant of a modified Gram matrix of the original system. Bernstein-type inequalities for Dirichlet polynomials and their higher order derivatives are established. The best constants and extremal polynomials are obtained in terms of the Gram matrix.

Key Words: Dirichlet Polynomials, Biorthogonal Systems, Blaschke Product, Gram Matrix, Bernstein-Type Inequality
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Introduction

Let H be a Hilbert space equipped with an inner product $\langle\cdot\,,\cdot\rangle$. The systems $\{e_k\}_{k=1}^n \subset H$ and $\{\varphi_k\}_{k=1}^n \subset H$ are called biorthogonal if

$$\langle e_i, \varphi_j \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker's delta. Furthermore, if the linear spans of the systems coincide then they are called generated biorthogonal systems, or GBS, in short.

Generated biorthogonal systems play the same important role as orthogonal systems usually do when the question concerns to extremal problems. When orthogonalization of $\{e_k\}_{k=1}^n$ is inconvenient, constructing of $\{\varphi_k\}_{k=1}^n$ can be a good alternative.

When $H = L^2(0,\infty)$ and $\{e_k\}$ is the Dirichlet system $\{e^{-\lambda_k x}\}_{k=1}^n$ or, more generally, the system of Müntz-Szasz,

$$\{e^{-\lambda_k x}, xe^{-\lambda_k x}, \dots, x^{m_k-1}e^{-\lambda_k x}\}_{k=1}^p,$$

where λ_i, $i = 1, 2, \dots, p$ are distinct complex numbers from the right half-plane and m_i, $i = 1, 2, \dots, p$ are natural numbers, the corresponding system of functions φ_k was implicitly created and used by L. Schwartz [1] and M. M. Jerbashian [2], respectively. Later, with use of Blaschke products, an explicit representation of those functions was obtained by V. Kh. Musoyan [3], who elaborated the method of GBS for solving approximation problems. Some of the results benefited from the method can be found in [4]-[6]. These materials mostly refer to fast approximation problems, the area that our recent research has been focused on.

In this paper, we consider the system of Dirichlet in $L^2(0,a)$ instead of $L^2(0,\infty)$. This brings up the question of whether or not the Musoyan's method developed in [3] can be still applied for constructing generated biorthogonal systems. The simplest answer to this question is no, but further analysis is made to show that the Blaschke product cannot be replaced by another function to make the previous method applicable to the case of $L^2(0,a)$. Despite this inconvenience, it turns out that estimation of Dirichlet polynomials is still possible. In the second part of this paper Markov-Bernstein-type inequalities for Dirichlet polynomials and their higher order derivatives are established, where the best constants and extremal polynomials are obtained in terms of the Gram matrix of the original system.

1 Generated Biorthogonal Systems in $L^2(0,a)$

Consider the finite system

$$\{e^{-\lambda_k x}\}_{k=1}^n, \tag{1}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct complex numbers chosen from the right half-plane $\Pi_+ = \{\lambda : \operatorname{Re}\lambda > 0\}$.

For $0 < a \le +\infty$, denote by $\{\varphi_k^{(a)}(x)\}_{k=1}^n$ the system of functions chosen from the linear span of (1) so that

$$\int_0^a \overline{\varphi_k^{(a)}(t)} e^{-\lambda_r t} dt = \delta_{kr},$$

for $1 \le k, r \le n$. In short, this means that the systems (1) and $\{\varphi_k^{(a)}(x)\}_{k=1}^n$ are GBS in $L^2(0,a)$. Let

$$J_k^{(a)}(\lambda) = \int_0^a \overline{\varphi_k^{(a)}(t)} e^{-\lambda t} dt, \quad \lambda \in \Pi_+.$$

For a given value of $a \in (0, +\infty]$, we examine the existence of some function $L(\lambda)$, analytic in Π_+, such that

$$\frac{L(\lambda)}{L'(\lambda_k)(\lambda - \lambda_k)} = J_k^{(a)}(\lambda), \quad \lambda \in \Pi_+. \tag{2}$$

If $a = +\infty$, then the Blaschke product

$$L(\lambda) = \prod_{k=1}^{n} \frac{\lambda - \lambda_k}{\lambda + \overline{\lambda_k}}$$

comfortably fits (2), which enables us (see [3]) to acquire $\varphi_k^{(+\infty)}$ explicitly:

$$\varphi_k^{(+\infty)}(x) = \frac{1}{\overline{L'(\lambda_k)}} \sum_{m=1}^{n} \frac{e^{-\lambda_m x}}{L'(\lambda_m)(\lambda_m + \overline{\lambda_k})}.$$

The following result shows that the above method cannot be extended for finite values of a to construct the GBS explicitly in $L^2(0, a)$.

Theorem 1 *If $0 < a < +\infty$ then there is no function $L(\lambda)$ satisfying the equation* (2).

Proof. Suppose the opposite. Then some function $L(\lambda)$ will satisfy

$$L(\lambda) = L'(\lambda_1)(\lambda - \lambda_1) J_1^{(a)}(\lambda) = L'(\lambda_2)(\lambda - \lambda_2) J_2^{(a)}(\lambda) = \dots$$
$$\dots = L'(\lambda_n)(\lambda - \lambda_n) J_n^{(a)}(\lambda), \ \lambda \in \Pi_+.$$

To simplify notations let us admit $A_i = L'(\lambda_i)$, $i = 1, 2, \dots, n$. Since

$$(\lambda - \lambda_1) J_1^{(a)}(\lambda) = \frac{A_k}{A_1}(\lambda - \lambda_k) J_k^{(a)}(\lambda), \ \ k = 1, 2, \dots, n,$$

we arrive to

$$J_1^{(a)}(\lambda)\frac{\lambda - \lambda_1}{\lambda - \lambda_k} = \frac{A_k}{A_1} \int_0^a \overline{\varphi_k^{(a)}(t)} e^{-\lambda t} dt, \ 1 \le k \le n. \tag{3}$$

On the other hand,

$$J_1^{(a)}(\lambda)\frac{\lambda - \lambda_1}{\lambda - \lambda_k} = J_1^{(a)}(\lambda) - (\lambda_1 - \lambda_k)\frac{J_1^{(a)}(\lambda)}{\lambda - \lambda_k}, \ 1 \le k \le n. \tag{4}$$

Let's fix a and k and denote $\psi(t) = \displaystyle\int_0^t \overline{\varphi_1^{(a)}(u)} e^{-\lambda_k u} du$, where $0 \le t \le a$. Then

$$\frac{J_1^{(a)}(\lambda)}{\lambda - \lambda_k} = \frac{1}{\lambda - \lambda_k} \int_0^a \overline{\varphi_1^{(a)}(t)} e^{-\lambda_k t} e^{-(\lambda - \lambda_k)t} dt = \frac{1}{\lambda - \lambda_k} \int_0^a e^{-(\lambda - \lambda_k)t} d\psi(t).$$

Since for $k \geq 2$ we have $\psi(0) = \psi(a) = 0$, integration by parts yields

$$\frac{J_1^{(a)}(\lambda)}{\lambda - \lambda_k} = \int_0^a \psi(t) e^{\lambda_k t} e^{-\lambda t} dt. \tag{5}$$

From (3), (4), and (5) we conclude

$$\int_0^a \left(\frac{A_k}{A_1} \overline{\varphi_k^{(a)}(t)} - \overline{\varphi_1^{(a)}(t)} + (\lambda_1 - \lambda_k)\psi(t) e^{\lambda_k t} \right) e^{-\lambda t} dt = 0, \ \ \lambda \in \Pi+,$$

and by Laplace transform uniqueness,

$$\frac{A_k}{A_1} \overline{\varphi_k^{(a)}(t)} - \overline{\varphi_1^{(a)}(t)} + (\lambda_1 - \lambda_k)\psi(t) e^{\lambda_k t} = 0,$$

where $2 \leq k \leq n$, $0 \leq t \leq a$. As a result, $\overline{\psi(t)} e^{\overline{\lambda_k} t}$ belongs to the linear span of (1), which we'll show below is not possible.

Let $\varphi_1^{(a)}(t) = \sum_{m=1}^{n} \alpha_m e^{-\lambda_m t}$. Then

$$\overline{\psi(t)} = \int_0^t \varphi_1^{(a)}(u) e^{-\overline{\lambda_k} u} du = \sum_{m=1}^{n} \alpha_m \int_0^t e^{-(\lambda_m + \overline{\lambda_k})u} du = \sum_{m=1}^{n} \alpha_m \frac{1 - e^{-(\lambda_m + \overline{\lambda_k})t}}{\lambda_m + \overline{\lambda_k}},$$

which implies

$$\overline{\psi(t)} e^{\overline{\lambda_k} t} = e^{\overline{\lambda_k} t} \sum_{m=1}^{n} \frac{\alpha_m}{\lambda_m + \overline{\lambda_k}} - \sum_{m=1}^{n} \alpha_m \frac{e^{-\lambda_m t}}{\lambda_m + \overline{\lambda_k}}.$$

Since $\mathrm{Re}(-\overline{\lambda_k}) < 0$, the function $\overline{\psi(t)} e^{\overline{\lambda_k} t}$ belongs to the linear span of (1) if and only if

$$\sum_{m=1}^{n} \frac{\alpha_m}{\lambda_m + \overline{\lambda_k}} = 0,$$

which is equivalent to

$$\int_0^\infty \overline{\varphi_1^{(a)}(t)} e^{-\lambda_k t} dt = 0 \ \ (k \geq 2).$$

Similarly, we'll obtain

$$\int_0^\infty \overline{\varphi_r^{(a)}(t)} e^{-\lambda_k t} dt = 0 \quad (k \neq r).$$

Due to the uniqueness of GBS, there should exist coefficients $c_r^{(a)} \in \mathbb{C} \backslash \{0\}$ such that $\varphi_r^{(a)}(t) = c_r^{(a)} \varphi_r^{(+\infty)}(t)$. Hence, the equations

$$\int_0^a \overline{\varphi_r^{(+\infty)}(t)} e^{-\lambda_k t} dt = 0 \quad (k \neq r)$$

are valid for any positive variable a, and therefore, $\varphi_r^{(+\infty)}(t) \equiv 0$, which is false. □

2 Estimation of Dirichlet Polynomials in $L^2(0,a)$

In general, for any pair of generated biorthogonal systems in a Hilbert space the following lemma takes place.

Lemma 1 *If $\{e_k\}_{k=1}^n$ and $\{\varphi_k\}_{k=1}^n$ are GBS in a Hilbert space then*

$$\varphi_k = \frac{1}{det(G)} \cdot det \begin{bmatrix} \langle e_1, e_1\rangle & \langle e_2, e_1\rangle & \dots & \langle e_n, e_1\rangle \\ \vdots & \vdots & \vdots & \vdots \\ e_1 & e_2 & \dots & e_n \\ \vdots & \vdots & \vdots & \vdots \\ \langle e_1, e_n\rangle & \langle e_2, e_n\rangle & \dots & \langle e_n, e_n\rangle \end{bmatrix}, \qquad (6)$$

where $[\,e_1 \quad e_2 \quad \dots \quad e_n\,]$ stands in the kth row and G denotes the Gram matrix $[\langle e_i, e_j\rangle]_{i,j=1}^n$.

Proof. Let $\varphi_k = \alpha_{k1}e_1 + \alpha_{k2}e_2 + \ldots + \alpha_{kn}e_n$. Then $\alpha_{k1}\langle e_1, e_r\rangle + \alpha_{k2}\langle e_2, e_r\rangle + \ldots + \alpha_{kn}\langle e_n, e_r\rangle = \delta_{kr}$, for $r = 1, 2, \ldots, n$. By Cramer's rule,

$$\alpha_{kj} = \frac{1}{det(G^T)} \cdot det \begin{bmatrix} \langle e_1, e_1\rangle & \dots & 0 & \dots & \langle e_n, e_1\rangle \\ \vdots & \dots & \vdots & \dots & \vdots \\ \langle e_1, e_k\rangle & \dots & 1 & \dots & \langle e_n, e_k\rangle \\ \vdots & \dots & \vdots & \dots & \vdots \\ \langle e_1, e_n\rangle & \dots & 0 & \dots & \langle e_n, e_n\rangle \end{bmatrix},$$

where $[\,0 \quad \dots \quad 1 \quad \dots 0\,]^T$ stands in the jth column. Then by Laplace expansion theorem, $\alpha_{kj} = \dfrac{C_{kj}}{det(G)}$, where C_{kj} is the k,j cofactor of G^T. Thus,

$$\varphi_k = \frac{1}{det(G)} \sum_{j=1}^{n} C_{kj} e_j,$$

which is equivalent to (6). $\square$

From now on, by vectors e_k, $k = 1, 2, \ldots, n$ we will mean the functions of the system (1) in $L^2(0,a)$, where a is finite. The value of a will not variate, and thus the former notation $\varphi_k^{(a)}$ will be simply replaced by φ_k. We have $\langle e_i, \varphi_j\rangle = \delta_{ij}$, $1 \le i, j \le n$, where the inner product of $f, g \in L^2(0,a)$ is defined as

$$\langle f, g\rangle = \int_0^a f(x)\overline{g(x)}dx.$$

The set of Dirichlet polynomials generated by (1) will be denoted by $\mathcal{E} = \text{span}\{e_k\}_{k=1}^n = \text{span}\{\varphi_k\}_{k=1}^n$.

Lemma 2 *If $P \in \mathcal{E}$ then*

$$P(z) = \int_0^a P(t)K(z,t)dt \quad (z \in \mathbb{C}),$$

where

$$K(z,t) = -\frac{1}{det(G)} \cdot det \begin{bmatrix} 0 & e^{-\overline{\lambda_1}t} \dots e^{-\overline{\lambda_n}t} \\ e^{-\lambda_1 z} & \\ \vdots & G \\ e^{-\lambda_n z} & \end{bmatrix}. \tag{7}$$

Proof. Let $P(x)$ be a Dirichlet polynomial generated by (1). Then it is analytically continued to the entire function

$$P(z) = \sum_{k=1}^{n} \langle P, \varphi_k \rangle e^{-\lambda_k z} = \int_0^a P(t) \sum_{k=1}^{n} \overline{\varphi_k(t)} e^{-\lambda_k z} dt.$$

Due to Lemma 1 and the identity $\overline{\langle e_i, e_j \rangle} = \langle e_j, e_i \rangle$, the function $K(z,t) = \sum_{k=1}^{n} \overline{\varphi_k(t)} e^{-\lambda_k z}$ is now representable as

$$K(z,t) = \frac{1}{det(G)} \cdot \sum_{k=1}^{n} det \begin{bmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \dots & \langle e_1, e_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ e^{-\lambda_k z} e^{-\overline{\lambda_1}t} & e^{-\lambda_k z} e^{-\overline{\lambda_2}t} & \dots & e^{-\lambda_k z} e^{-\overline{\lambda_n}t} \\ \vdots & \vdots & \vdots & \vdots \\ \langle e_n, e_1 \rangle & \langle e_n, e_2 \rangle & \dots & \langle e_n, e_n \rangle \end{bmatrix}, \tag{8}$$

where the vector $[\, e^{-\lambda_k z} e^{-\overline{\lambda_1}t} \quad e^{-\lambda_k z} e^{-\overline{\lambda_2}t} \quad \dots \quad e^{-\lambda_k z} e^{-\overline{\lambda_n}t}]$ stands for the kth row of the Gram matrix G. It remains to check that the cofactor expansion of the right-hand side of (7) along the first column is equal to the right-hand side of (8). □

Theorem 2 *Let r be a non-negative integer. Then for any $z \in \mathbb{C}$ the equality*

$$\max_{P \in \mathcal{E}} \frac{|P^{(r)}(z)|}{||P||_{L^2(0,a)}} = \sqrt{\Phi_r(z)}$$

holds, where

$$\Phi_r(z) = -\frac{1}{det(G)} \cdot det \begin{bmatrix} 0 & \overline{\lambda_1^r} e^{-\overline{\lambda_1 z}} \dots \overline{\lambda_n^r} e^{-\overline{\lambda_n z}} \\ \lambda_1^r e^{-\lambda_1 z} & \\ \vdots & G \\ \lambda_n^r e^{-\lambda_n z} & \end{bmatrix}. \tag{9}$$

Moreover, for any fixed value of $z \in \mathbb{C}$ the maximum is reached for polynomial $P_z(t) = \overline{\frac{d^r}{dz^r} K(z,t)}$.

Note: when $r = 0$ we assume $P^{(r)}(z) \equiv P(z)$ and $\overline{\frac{d^r}{dz^r}K(z,t)} = \overline{K(z,t)}$.

Proof. Due to the integral representation provided in Lemma 2, the proof of this theorem in case of $r = 0$ may formally repeat the proof of its analogue obtained earlier in [3] for $L^2(0,\infty)$. Indeed, by Cauchy-Schwartz inequality,

$$|P(z)| \leq ||P||_{L^2(0,a)}||K(z,t)||_{L^2(0,a)}, \quad P \in \mathcal{E}.$$

Then, by taking $P_z(t) = \overline{K(z,t)} \in \mathcal{E}$ we arrive to $\overline{K(z,z)} = ||K(z,t)||^2_{L^2(0,a)}$, which results in $|P(z)| \leq \sqrt{K(z,z)}||P||_{L^2(0,a)} = \sqrt{\Phi_0(z)}||P||_{L^2(0,a)}$.

Now let $r \geq 1$. When $P \in \mathcal{E}$, Lemma 2 yields

$$P^{(r)}(z) = \int_0^a \frac{d^r}{dz^r}K(z,t)P(t)dt \quad (z \in \mathbb{C}), \tag{10}$$

and therefore, again by Cauchy-Schwartz inequality,

$$|P^{(r)}(z)| \leq ||P||_{L^2(0,a)}||\frac{d^r}{dz^r}K(z,t)||_{L^2(0,a)}, \quad P \in \mathcal{E}. \tag{11}$$

Let's notice that $\frac{d^r}{dz^r}K(z,t) = \sum_{k=1}^{n}(-1)^r\lambda_k^r e^{-\lambda_k z}\overline{\varphi_k(t)}$, which means that for any fixed value of $z \in \mathbb{C}$ the function $P_z(t) = \overline{\frac{d^r}{dz^r}K(z,t)}$ belongs to $\mathcal{E}$. Then formula (10) works for $P_z(t)$, i.e.

$$P_z^{(r)}(z) = \int_0^a \frac{d^r}{dz^r}K(z,t)\overline{\frac{d^r}{dz^r}K(z,t)}dt = ||\frac{d^r}{dz^r}K(z,t)||^2_{L^2(0,a)} \quad (z \in \mathbb{C}).$$

Now according to (11), for any $P \in \mathcal{E}$ we will have the inequality

$$|P^{(r)}(z)| \leq \sqrt{P_z^{(r)}(z)}||P||_{L^2(0,a)} \quad (z \in \mathbb{C}),$$

where the equality holds for $P \equiv P_z$.

To complete the proof it remains to show that $P_z^{(r)}(z) = \Phi_r(z)$. The first step to be taken on this direction is the r times differentiation of the equation (8) with respect to z.

$$\frac{d^r}{dz^r}K(z,t) = \frac{(-1)^r}{det(G)} \cdot \sum_{k=1}^{n} \lambda_k^r e^{-\lambda_k z} det \begin{bmatrix} \langle e_1, e_1\rangle & \langle e_1, e_2\rangle & \dots & \langle e_1, e_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ e^{-\overline{\lambda_1}t} & e^{-\overline{\lambda_2}t} & \dots & e^{-\overline{\lambda_n}t} \\ \vdots & \vdots & \vdots & \vdots \\ \langle e_n, e_1\rangle & \langle e_n, e_2\rangle & \dots & \langle e_n, e_n\rangle \end{bmatrix},$$

where the vector $[\, e^{-\overline{\lambda_1} t} \quad e^{-\overline{\lambda_2} t} \quad \dots \quad e^{-\overline{\lambda_n} t}\,]$ stands in the kth row of the determinant. Applying the complex conjugate to both sides of this equation followed by r times differentiation with respect to t we come to an expression for $P_z^{(r)}(t)$. The replacement of t by z in that expression produces

$$P_z^{(r)}(z) =$$

$$= \frac{1}{det(G)} \sum_{k=1}^{n} \overline{\lambda_k^r} e^{-\overline{\lambda_k z}} det \begin{bmatrix} \langle e_1, e_1 \rangle & \langle e_2, e_1 \rangle & \dots & \langle e_n, e_1 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^r e^{-\lambda_1 z} & \lambda_2^r e^{-\lambda_2 z} & \dots & \lambda_n^r e^{-\lambda_n z} \\ \vdots & \vdots & \vdots & \vdots \\ \langle e_1, e_n \rangle & \langle e_2, e_n \rangle & \dots & \langle e_n, e_n \rangle \end{bmatrix}. \qquad (12)$$

On the other hand, expanding the determinant in formula (9) along the first column allows us to represent Φ_r as follows.

$$\Phi_r(z) = \frac{1}{det(G)} \sum_{k=1}^{n} \lambda_k^r e^{-\lambda_k z} det \begin{bmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \dots & \langle e_1, e_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \overline{\lambda_1^r} e^{-\overline{\lambda_1 z}} & \overline{\lambda_2^r} e^{-\overline{\lambda_2 z}} & \dots & \overline{\lambda_n^r} e^{-\overline{\lambda_n z}} \\ \vdots & \vdots & \vdots & \vdots \\ \langle e_n, e_1 \rangle & \langle e_n, e_2 \rangle & \dots & \langle e_n, e_n \rangle \end{bmatrix}. \qquad (13)$$

The right-hand sides of (12) and (13) are complex conjugates of each other. Since $P_z^{(r)}(z)$ is real, we conclude $P_z^{(r)}(z) = \overline{P_z^{(r)}(z)} = \Phi_r(z)$. $\square$

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