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Cayley-type theorems for g-dimonoids

Yu. M. Movsisyan and M. A. Yolchyan

Abstract. In this paper we prove Cayley-type theorems for g-dimonoids using the left (right) acts of sets and concept of dialgebra.

Key Words: g-dimonoid, dimonoid, act of set, dialgebra, morphism of acts, (l, r)-morphism of semigroup.
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Introduction

The concepts of a dimonoid and a dialgebra were introduced by Loday [1]. Dimonoids are a tool to study Leibniz algebras [1]. A dimonoid is a set with two binary associative operations satisfying certain additional identities. A dialgebra is a linear analogy of a dimonoid. The concept of a g-dimonoid (a generalized dimonoid) is introduced in [2]. The Cayley-type theorems for dimonoids are proved in [3].

In this paper two Cayley-type theorems for g-dimonoid are suggested.

Definition 1 *An algebra $(X; \prec, \succ)$ with two binary operations is called g-dimonoid [2] if it satisfies the following four identities of associativity:*

$$(x \prec y) \prec z = x \prec (y \prec z), \qquad (A_1)$$

$$(x \prec y) \prec z = x \prec (y \succ z), \qquad (A_2)$$

$$(x \prec y) \succ z = x \succ (y \succ z), \qquad (A_3)$$

$$(x \succ y) \succ z = x \succ (y \succ z). \qquad (A_4)$$

Definition 2 *A g-dimonoid $(X; \prec, \succ)$ is called dimonoid [1] if it satisfies the following additional identity of associativity, too:*

$$(x \succ y) \prec z = x \succ (y \prec z). \qquad (A_5)$$

Let us give two examples of a dimonoid.

Example 1 *Let* $(X;\prec)$ *be a zero semigroup, that is* $x \prec y = 0$ *for all* $x, y \in X$*, where* 0 *is a fixed element of the set* X*. For fixed elements* a, b *of the set* X*, where* $a \neq 0, b \neq 0, a \neq b$*, we define on* X *the operation* $\succ$*, assuming*

$$x \succ y = \begin{cases} a, & x = y = b, \\ 0, & otherwise, \end{cases}$$

for all $x, y \in X$*. It is easy to see that* $(X;\prec,\succ)$ *is a dimonoid.*

Example 2 *Let* $X \neq \emptyset$*. Define the following operations on the set* X*:*

$$x \succ y = y,$$

$$x \prec y = x,$$

for all $x, y \in X$*. Then the algebra* $(X;\prec,\succ)$ *is a dimonoid.*

Let us give an example of a g-dimonoid which is not a dimonoid.

Example 3 *Let* X *be an arbitrary nonempty set,* $|X| \geq 2$*, and let* X^* *be the set of all finite nonempty words in the alphabet* X*. Denote the first (respectively, the last) letter of a word* $\omega \in X^*$ *by* $\omega^{(0)}$ *(respectively, by* $\omega^{(1)}$*). Define the following operations* $\prec, \succ$ *on* X^**:*

$$\omega \prec u = \omega^{(0)},$$

$$\omega \succ u = u^{(1)},$$

for all $\omega, u \in X^*$*. It is easy to check that the binary algebra* $(X^*;\prec,\succ)$ *is a* g*-dimonoid, but is not a dimonoid.*

Definition 3 *Let* S *be a semigroup,* $X \neq \emptyset$*. The map* $(\cdot) : S \times X \to X; (s, x) \mapsto s \cdot x$ *is called a left S-act of* X *if the identity* $(st) \cdot x = s \cdot (t \cdot x)$ *holds.*

Definition 4 *Let* S *be a semigroup,* $X \neq \emptyset$*. The map* $(\cdot) : X \times S \to X; (x, s) \mapsto x \cdot s$ *is called a right S-act of* X *if the identity* $x \cdot (st) = (x \cdot s) \cdot t$ *holds.*

Definition 5 *Let the map* $(\cdot)$ *be a left S-act of* X *and the map* $(\circ)$ *be a left S-act of* Y*: The map* $\varphi : X \to Y$ *is called morphism of that left S-acts if* $\varphi(s \cdot x) = s \circ \varphi(x)$ *for all* $x \in X$ *and* $s \in S$*.*

A morphism of right S-acts can be defined in a similar way. Let $(S;\bullet)$ be a semigroup, $X \neq \emptyset$. We can interpret the operation $(\bullet)$ as a left and a right S-act of S.

Definition 6 *Let $(S;\bullet)$ be a semigroup, $X \neq \emptyset$, and maps $(\cdot)$ and $(\circ)$ are respectively left and right S-acts of X; then the map $\varphi : X \to S$ is called (l,r)-morphism of semigroup S if the following conditions are valid:*

$$\varphi(s \cdot x) = s \bullet \varphi(x),$$

$$\varphi(x \circ s) = \varphi(x) \bullet s,$$

for all $x \in X$ and $s \in S$.

Definition 7 *A dialgebra is a vector space over a field equipped with two binary bilinear operations satisfying the axioms of a dimonoid* [1].

For the second order formulae (and the second order languages) see [4, 5, 6]. Let us recall, that a hyperidentity [7, 8, 9, 10, 11, 12] (or $\forall(\forall)$-identity) is a second-order formula of the following form:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (\omega_1 = \omega_2),$$

where ω_1, ω_2 are words (terms) in the alphabet of functional variables $X_1, \ldots, X_m$ and objective variables $x_1, \ldots, x_n$. However hyperidentities are usually presented without universal quantifiers: $\omega_1 = \omega_2$. A hyperidentity $\omega_1 = \omega_2$ is said to be satisfied in the algebra $(Q;\Sigma)$ if this equality holds whenever every object variable x_j is replaced by an arbitrary element from Q and every functional variable X_i is replaced by an arbitrary operation of the corresponding arity from Σ. A possibility of such replacement is assumed, that is:

$$\{|X_1|, \ldots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{(Q,\Sigma)} = T_{(\Sigma)},$$

where $|S|$ is the arity of S, and $T_{(Q,\Sigma)}$ is called the arithmetic type of $(Q;\Sigma)$. A T-algebra is an algebra with arithmetic type $T \subseteq N$. A class of algebras is called a class of T-algebras if every algebra in it is a T-algebra.

Definition 8 *An algebra $(Q;\Sigma)$ is called idempotent, if the following hyperidentity of idempotency is valid:*

$$X(\underbrace{x, \ldots, x}_{n}) = x, \tag{id}$$

for all $n \in T_{(Q;\Sigma)}$.

The hyperidentity is said to be non-trivial if $m > 1$, and it is trivial if $m = 1$. The number m is called the functional rank of the given hyperidentity.

A binary algebra $(Q;\Sigma)$ is said to be a q-algebra (e-algebra) if there is an operation $A \in \Sigma$ such that $Q(A)$ is a quasigroup (a groupoid with a unit).

A binary algebra $(Q;\Sigma)$ is called non-trivial if $|\Sigma|>1$. It is known [7, 8] (see also [9, 13]) that if an associative non-trivial hyperidentity is satisfied in a non-trivial q-algebra (e-algebra), then this hyperidentity can only be of the functional rank 2 and of one of the following forms:

$$X(x,Y(y,z))=Y(X(x,y),z), \qquad (asm)_1$$

$$X(x,Y(y,z))=X(Y(x,y),z), \qquad (asm)_2$$

$$Y(x,Y(y,z))=X(X(x,y),z). \qquad (asm)_3$$

Moreover, in the class of q-algebras (e-algebras) the hyperidentity $(asm)_3$ implies the hyperidentity $(asm)_2$ which, in its turn, implies the hyperidentity $(asm)_1$.

The algebra $(Q;\Sigma)$ is called hyperassociative, if it satisfies the hyperidentity of associativity $(asm)_1$.

Example 4 *Let A,B be nonempty sets, Σ be the set of all mappings from B to A, and Q be the set of all mappings from A to B. Then every element $\alpha\in\Sigma$ can be considered as a binary operation on Q,*

$$\alpha(a,b)=a\cdot\alpha\cdot b,$$

where $a,b\in Q$ and $a\cdot\alpha\cdot b$ is the usual superposition of mappings. So we obtain hyperassociative algebra $(Q;\Sigma)$. Moreover, if $A=B$ we obtain second degree (order) algebra $(Q;\Sigma;\cdot)$ in the sense of [14].

Thus, hyperassociative algebras are algebras with semigroup operations. Hyperassociative algebras under the name of Γ-semigroups (gamma-semigroups) or doppelsemigroups were considered by various authors [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

Note that a g-dimonoid $(Q;\prec,\succ)$ is hyperassociative iff (A_5) and the following identitiy of associativity hold:

$$(x\prec y)\succ z=x\prec(y\succ z). \qquad (A_6)$$

1 Main results

A Cayley-type theorem for dimonoids was proved in [3]. In this chapter we prove a more general result.

Theorem 1 *Let S be a semigroup, $X\neq\emptyset$, and maps $(\cdot)$ and $(\circ)$ be left and right S-acts of X respectively, and let the map $\varphi: X\to S$ be a (l,r)-morphism of semigroup S. Define the operations $\succ$ and $\prec$ as follows:*

$$\succ: X\times X\to X, \quad x\succ y:=\varphi(x)\cdot y,\ \ \forall x,y\in X,$$

$$\prec : X \times X \to X, \quad x \prec y := x \circ \varphi(y), \ \forall x, y \in X.$$

Then the algebra $(X; \prec, \succ)$ is a g-dimonoid, which we denote by $C_S(X)$ and call Cayley's g-dimonoid.

Conversely, for any arbitrary g-dimonoid G there exists a semigroup S, and $X \neq \emptyset$, and left and right S-acts of X, and an (l,r)-morphism of semigroup S such that the g-dimonoid G coincides with $C_S(X)$, i.e. $G = C_S(X)$.

Proof. Let us prove the first part of the result. For this, it suffices to verify the following four identities:

(A_1) $(x \prec y) \prec z = (x \circ \varphi(y)) \prec z = (x \circ \varphi(y)) \circ \varphi(z) = x \circ (\varphi(y)\varphi(z)) = x \circ \varphi(y \circ \varphi(z)) = x \prec (y \circ \varphi(z)) = x \prec (y \prec z)$,

(A_2) $(x \prec y) \prec z = (x \prec y) \circ \varphi(z) = (x \circ \varphi(y)) \circ \varphi(z) = x \circ (\varphi(y)\varphi(z)) = x \circ \varphi(\varphi(y) \cdot z) = x \prec (\varphi(y) \cdot z) = x \prec (y \succ z)$,

(A_3) $(x \prec y) \succ z = (x \circ \varphi(y)) \succ z = \varphi(x \circ \varphi(y)) \cdot z = (\varphi(x)\varphi(y)) \cdot z = \varphi(x) \cdot (\varphi(y) \cdot z) = x \succ (\varphi(y) \cdot z) = x \succ (y \succ z)$,

(A_4) $(x \succ y) \succ z = (\varphi(x) \cdot y) \succ z = \varphi(\varphi(x) \cdot y) \cdot z = (\varphi(x)\varphi(y)) \cdot z = \varphi(x) \cdot (\varphi(y) \cdot z) = x \succ (\varphi(y) \cdot z) = x \succ (y \succ z)$.

Hence, the algebra $(X; \prec, \succ)$ is a g-dimonoid and $(X; \prec, \succ) = C_S(X)$.

Now we prove the second part of the result. Let the algebra $D = (D; \prec, \succ)$ be a g-dimonoid. We denote by $\sim$ the least congruence on D for which the quotient g-dimonoid is a semigroup. It is clear that $\sim$ is the congruence generated by the relation $\sigma = \{(a,a), (a \succ b, a \prec b), (a \prec b, a \succ b) : a, b \in D\}$. Let $x \sim y$. According to the definition of the congruence, we can say that there exist $b_1, c_1, ..., b_n, c_n \in D$ and $o_1, o_1', ..., o_n, o_n' \in \{\succ, \prec\}$ such that the following condition holds:

$$x = (b_1 o_1 c_1)\sigma(b_1 o_1' c_1) = (b_2 o_2 c_1)\sigma(b_2 o_2' c_2) = ... = (b_n o_n c_n)\sigma(b_n o_n' c_n) = y.$$

According to the identities A_3, A_4, we have:

$$(b_i o_i c_i) \succ a = (b_i o_i' c_i) \succ a, \ \forall i = 1, ..., n, \ \forall a \in D.$$

Thus,

$$x \succ a = (b_1 o_1 c_1) \succ a = (b_1 o_1' c_1) \succ a = ... = (b_n o_n c_n) \succ a =$$

$$= (b_n o_n' c_n) \succ a = y \succ a \to x \succ a = y \succ a, \ \forall a \in D.$$

According to the identities A_1, A_2, we have $a \prec x = a \prec y$, $\forall a \in D$, i.e.

$$x \sim y \to x \succ a = y \succ a \ \& \ a \prec x = a \prec y, \ \forall a \in D.$$

It means, that the congruence $\sim$ is contained in the congruence:

$$\tau = \{(x, y) \in D \times D \mid x \succ a = y \succ a \,\&\, a \prec x = a \prec y, \forall a \in D\}.$$

Now let us pick any congruence θ which satisfies the condition $\sim \subseteq \theta \subseteq \tau$. Hence D/θ is an algebra with two binary operations, which are equal, because θ contains the congruence $\sim$. So D/θ is a semigroup, which we denote by S, and for arbitrary $x \in D$ we denote by $[x]$, the class of the congruence θ that contains x. Define the maps:

$$\cdot : S \times D \to D, \;\; [x] \cdot y := x \succ y, \tag{1}$$

$$\circ : D \times S \to D, \;\; x \circ [y] := x \prec y. \tag{2}$$

The correctness of definitions of the maps $\cdot$ and $\circ$ follows from the fact that $\theta \subseteq \tau$. We now check that the maps $\cdot$ and $\circ$ are left and right S-acts of D, respectively:

$$([a][b]) \cdot x = [a \prec b] \cdot x \overset{(1)}{=} (a \prec b) \succ x \overset{(A_3)}{=}$$
$$a \succ (b \succ x) \overset{(1)}{=} a \succ ([b] \cdot x) \overset{(1)}{=} [a] \cdot ([b] \cdot x),$$

$$x \circ ([a][b]) = x \circ [a \succ b] \overset{(2)}{=} x \prec (a \succ b) \overset{(A_2)}{=}$$
$$(x \prec a) \prec b \overset{(2)}{=} (x \circ [a]) \prec b \overset{(2)}{=} (x \circ [a]) \circ [b].$$

We now consider the map $\Phi : D \to S, \Phi(x) = [x]$ and show that it is a (l, r)-morphism of S:

$$\Phi([a] \cdot x) = \Phi(a \succ x) = [a \succ x] = [a][x] = [a]\Phi(x),$$

$$\Phi(x \circ [a]) = \Phi(x \prec a) = [x \prec a] = [x][a] = \Phi(x)[a].$$

Note, that the conditions of the first part of the theorem are satisfied, so $(D; \prec', \succ') = C_S(D)$ where:

$$\succ' : D \times D \to D, x \succ' y := \Phi(x) \cdot y, \;\; \forall x, y \in D,$$

$$\prec' : D \times D \to D, x \prec' y := x \circ \Phi(y), \;\; \forall x, y \in D.$$

Also note that:

$$\succ' = \succ,$$

$$\prec' = \prec,$$

hence $(D; \prec', \succ') = C_S(D)$. $\square$

Let K be a field, X be a nonempty set: $X = \{x_i \mid i \in I\}$. Denote $G := \{\{\alpha_i x_i \mid i \in I\} \mid \exists J \subset I\ ,\ |I \setminus J| < \infty,\ \alpha_j = 0,\ \forall j \in J, \alpha_i \in K\}$, where $\alpha_i x_i$ is just symbol. For any $x_t \in X$, considering $x_t = \{\alpha_i x_i \mid i \in I\}$, where $\alpha_t = 1$, and $\alpha_i = 0$ if $i \neq t$, we obtain that $X \subseteq G$. Now we define the operation $+$ on G:

$$\{\alpha_i x_i \mid i \in I\} + \{\beta_i x_i \mid i \in I\} = \{(\alpha_i + \beta_i)x_i \mid i \in I\}.$$

Because $K(+)$ is an abelian group, $G(+)$ will be an abelian group, too. We now define the operation $\cdot : K \times G \to G$ as follows:

$$\alpha \cdot \{\alpha_i x_i \mid i \in I\} = \{(\alpha\alpha_i)x_i \mid i \in I\}.$$

Then $G(+)$ with operation $\cdot$ will be a free K-module, which we denote by $K[X]$. It is easy to see, that if D is a dimonoid, then $K[D]$ is a dialgebra. This fact is used in the proof of the following result.

Theorem 2 *Let S be a semigroup satisfying the following additional identity:*

$$xyz = tlp. \qquad (*)$$

Define the operations $\succ$ and $\prec$ on $S \times S$ in the following way:

$$(g, h) \prec (k, l) = (gk, hk), \qquad (op_1)$$

$$(g, h) \succ (k, l) = (gk, gl), \qquad (op_2)$$

where $(g, h), (k, l) \in S \times S$. Then $\overline{S} = (S \times S; \prec, \succ)$ is a g-dimonoid, which satisfies the following hyperidentity:

$$X(x, Y(y, z)) = Z(T(t, l), p). \qquad (**)$$

In particular the g-dimonoid $\overline{S}$ will be hyperassociative.

Conversely, for any g-dimonoid D with the hyperidentity $(**)$ there exists a semigroup H with the identity $(*)$ such that D is isomorphically embedded into $\overline{H}$.

Proof. For proving the first part of the assertion, we use the previous theorem. Namely, we define the left and right S-acts of $S \times S$:

$$a \cdot (b, c) = (ab, ac),$$

$$(b, c) \circ a = (ba, ca).$$

We also define the operation $\varphi : S \times S \to S, (a, b) \mapsto a$ and check that the map φ is a (l, r)-morphism of S:

$$\varphi(a \cdot (b, c)) = \varphi((ab, ac)) = ab = a\varphi((b, c)),$$

$$\varphi((b,c)\circ a)=\varphi((ba,ca))=ba=\varphi((b,c))a.$$

Then

$$(g,h)\prec(k,l)=(gk,hk)=(g,h)\circ k=(g,h)\circ\varphi(k,l),$$

$$(g,h)\succ(k,l)=(gk,gl)=g\cdot(k,l)=\varphi(g,h)\cdot(k,l),$$

for all $(g,h),(k,l)\in S\times S$. Now according to Theorem 1, we obtain, that $\overline{S}$ is a g-dimonoid. Using the identity $(*)$, it is easy to check that $\overline{S}$ satisfies the hyperidentity $(**)$. We should check the following cases:

1. $(g,h)\prec((k,l)\succ(p,t))=((m,n)\prec(r,s))\succ(u,v)$,
2. $(g,h)\prec((k,l)\succ(p,t))=((m,n)\prec(r,s))\prec(u,v)$,
3. $(g,h)\prec((k,l)\succ(p,t))=((m,n)\succ(r,s))\succ(u,v)$,
4. $(g,h)\prec((k,l)\succ(p,t))=((m,n)\succ(r,s))\prec(u,v)$,
5. $(g,h)\succ((k,l)\succ(p,t))=((m,n)\prec(r,s))\succ(u,v)$,
6. $(g,h)\succ((k,l)\succ(p,t))=((m,n)\prec(r,s))\prec(u,v)$,
7. $(g,h)\succ((k,l)\succ(p,t))=((m,n)\succ(r,s))\succ(u,v)$,
8. $(g,h)\succ((k,l)\succ(p,t))=((m,n)\succ(r,s))\prec(u,v)$,
9. $(g,h)\prec((k,l)\prec(p,t))=((m,n)\prec(r,s))\succ(u,v)$,
10. $(g,h)\prec((k,l)\prec(p,t))=((m,n)\prec(r,s))\prec(u,v)$,
11. $(g,h)\prec((k,l)\prec(p,t))=((m,n)\succ(r,s))\succ(u,v)$,
12. $(g,h)\prec((k,l)\prec(p,t))=((m,n)\succ(r,s))\prec(u,v)$,
13. $(g,h)\succ((k,l)\prec(p,t))=((m,n)\prec(r,s))\succ(u,v)$,
14. $(g,h)\succ((k,l)\prec(p,t))=((m,n)\prec(r,s))\prec(u,v)$,
15. $(g,h)\succ((k,l)\prec(p,t))=((m,n)\succ(r,s))\succ(u,v)$,
16. $(g,h)\succ((k,l)\prec(p,t))=((m,n)\succ(r,s))\prec(u,v)$.

Indeed:

1.
$$(g,h)\prec((k,l)\succ(p,t))=(g,h)\prec(kp,kt)=(gkp,hkp)\overset{(*)}{=}$$
$$(mru,mrv)=(mr,nr)\succ(u,v)=((m,n)\prec(r,s))\succ(u,v);$$

2.
$$(g,h) \prec ((k,l) \succ (p,t)) = (g,h) \prec (kp,kt) = (gkp,hkp) \overset{(*)}{=}$$
$$(mru,nru) = (mr,nr) \prec (u,v) = ((m,n) \prec (r,s)) \prec (u,v);$$

3.
$$(g,h) \prec ((k,l) \succ (p,t)) = (g,h) \prec (kp,kt) = (gkp,hkp) \overset{(*)}{=}$$
$$(mru,mrv) = (mr,ms) \succ (u,v) = ((m,n) \succ (r,s)) \succ (u,v);$$

4.
$$(g,h) \prec ((k,l) \succ (p,t)) = (g,h) \prec (kp,kt) = (gkp,hkp) \overset{(*)}{=}$$
$$(mru,msu) = (mr,ms) \prec (u,v) = ((m,n) \succ (r,s)) \prec (u,v);$$

5.
$$(g,h) \succ ((k,l) \succ (p,t)) = (g,h) \succ (kp,kt) = (gkp,gkt) \overset{(*)}{=}$$
$$(mru,mrv) = (mr,nr) \succ (u,v) = ((m,n) \prec (r,s)) \succ (u,v);$$

6.
$$(g,h) \succ ((k,l) \succ (p,t)) = (g,h) \succ (kp,kt) = (gkp,gkt) \overset{(*)}{=}$$
$$(mru,nru) = (mr,nr) \prec (u,v) = ((m,n) \prec (r,s)) \prec (u,v);$$

7.
$$(g,h) \succ ((k,l) \succ (p,t)) = (g,h) \succ (kp,kt) = (gkp,gkt) \overset{(*)}{=}$$
$$(mru,mrv) = (mr,ms) \succ (u,v) = ((m,n) \succ (r,s)) \succ (u,v);$$

8.
$$(g,h) \succ ((k,l) \succ (p,t)) = (g,h) \succ (kp,kt) = (gkp,gkt) \overset{(*)}{=}$$
$$(mru,msu) = (mr,ms) \prec (u,v) = ((m,n) \succ (r,s)) \prec (u,v);$$

9.
$$(g,h) \prec ((k,l) \prec (p,t)) = (g,h) \prec (kp,lp) = (gkp,hkp) \overset{(*)}{=}$$
$$(mru,mrv) = (mr,nr) \succ (u,v) = ((m,n) \prec (r,s)) \succ (u,v);$$

10.
$$(g,h) \prec ((k,l) \prec (p,t)) = (g,h) \prec (kp,lp) = (gkp,hkp) \overset{(*)}{=}$$
$$(mru,nru) = (mr,nr) \prec (u,v) = ((m,n) \prec (r,s)) \prec (u,v);$$

11.
$$(g,h) \prec ((k,l) \prec (p,t)) = (g,h) \prec (kp,lp) = (gkp,hkp) \overset{(*)}{=}$$
$$(mru,mrv) = (mr,ms) \succ (u,v) = ((m,n) \succ (r,s)) \succ (u,v);$$

12.
$$(g,h) \prec ((k,l) \prec (p,t)) = (g,h) \prec (kp,lp) = (gkp,hkp) \overset{(*)}{=}$$
$$(mru,msu) = (mr,ms) \prec (u,v) = ((m,n) \succ (r,s)) \prec (u,v);$$

13.
$$(g,h) \succ ((k,l) \prec (p,t)) = (g,h) \succ (kp,lp) = (gkp,glp) \overset{(*)}{=}$$
$$(mru,mrv) = (mr,nr) \succ (u,v) = ((m,n) \prec (r,s)) \succ (u,v);$$

14.
$$(g,h) \succ ((k,l) \prec (p,t)) = (g,h) \succ (kp,lp) = (gkp,glp) \overset{(*)}{=}$$
$$(mru,nru) = (mr,nr) \prec (u,v) = ((m,n) \prec (r,s)) \prec (u,v);$$

15.
$$(g,h) \succ ((k,l) \prec (p,t)) = (g,h) \succ (kp,lp) = (gkp,glp) \overset{(*)}{=}$$
$$(mru,mrv) = (mr,ms) \succ (u,v) = ((m,n) \succ (r,s)) \succ (u,v);$$

16.
$$(g,h) \succ ((k,l) \prec (p,t)) = (g,h) \succ (kp,lp) = (gkp,glp) \overset{(*)}{=}$$
$$(mru,msu) = (mr,ms) \prec (u,v) = ((m,n) \succ (r,s)) \prec (u,v).$$

Now, we turn to the proof of the second part of the theorem. Let $D = (D; \prec, \succ)$ be a g-dimonoid satisfying the hyperidentity $(**)$. By D^0 we denote D with an adjoined element 0 (called zero) such that $x*0 = 0 = 0*x$, for all $x \in D^0$, and $* \in \{\succ, \prec\}$. Let K be a field and $H = (D/\theta)^0 \times K[D^0]$ (we invoke the notation from the proof of Theorem 1). On H we define a binary operation $\bullet$:

$$([a], x) \bullet ([b], y) = ([a \succ b], (a \succ y) + (x \prec b)),$$

for all $a, b, x, y \in D^0$. This operation is well-defined, because for a congruence θ we have:

$$a\theta a' \to a \succ y = a' \succ y \,\&\, x \prec a = x \prec a',$$

for all $x, y \in D$.

We now check, that $\bullet$ is associative:

$$(([a], x) \bullet ([b], y)) \bullet ([c], z) = ([a \succ b], (a \succ y) + (x \prec b)) \bullet ([c], z) =$$
$$([a \succ b \succ c], (a \succ b \succ z) + (((a \succ y) + (x \prec b)) \prec c)) =$$
$$([a \succ b \succ c], (a \succ b \succ z) + ((a \succ y) \prec c) + (x \prec b \prec c))) \overset{(A_5)}{=}$$
$$([a \succ b \succ c], (a \succ b \succ z) + (a \succ (y \prec c)) + (x \prec b \prec c)) =$$
$$([a \succ b \succ c], (a \succ ((b \succ z) + (y \prec c))) + (x \prec b \prec c)) =$$
$$([a], x) \bullet ([b \succ c], (b \succ z) + (y \prec c)) =$$
$$([a], x) \bullet (([b], y) \bullet ([c], z)).$$

Then $H(\bullet)$ is a semigroup with zero element $([0], 0)$. Define a map $f : D \to \overline{H}$ in the following way:

$$f(x) = (([x], 0), ([0], x)).$$

It is easy to see, that f is an injective map. Furthermore:

$$\begin{gathered} f(a) \succ f(b) = (([a], 0), ([0], a)) \succ (([b], 0), ([0], b)) = \\ (([a], 0) \bullet ([b], 0), ([a], 0) \bullet ([0], b)) = \\ (([a \succ b], (a \succ 0) + (0 \prec b)), ([0], (a \succ b) + (0 \prec 0))) = \\ (([a \succ b], 0), ([0], a \succ b)) = f(a \succ b). \end{gathered}$$

We obtain $f(a) \prec f(b) = f(a \prec b)$ in a similar way. Hence, f is a monomorphism of the g-dimonoids. Then D is isomorphically embedded into $\overline{H}$. Using the hyperidentity $(**)$, it is easy to show that the semigroup $\overline{H}$ satisfies the identity $(*)$. Indeed:

$$\begin{gathered} ([g], x_g) \bullet ([k], x_k) \bullet ([t], x_t) = ([g \succ k], (g \succ x_k) + (x_g \prec k)) \bullet ([t], x_t) = \\ ([g \succ k \succ t], (g \succ k \succ x_t) + ((g \succ x_k) \prec t) + ((x_g \prec k) \prec t)) \overset{(**)}{=} \\ ([h \succ r \succ l], (h \succ r \succ x_l) + ((h \succ x_r) \prec l) + ((x_h \prec r) \succ l)) = \\ ([h \succ r], (h \succ x_r) + (x_h \prec r)) \bullet ([l], x_l) = ([h], x_h) \bullet ([r], x_r) \bullet ([l], x_l). \end{gathered}$$

□

2 Corollaries

Corollary 1 ([3]) Let S be a semigroup and X be both a left S-act and a right S-act with commuting actions, and let $\varphi : X \to S$ be a (l, r)-morphism of S. Then the set X with two binary operations $\prec$ and $\succ$ defined by the following rules:

$$x \prec y := x \circ \varphi(y),$$

$$x \succ y := \varphi(x) \cdot y,$$

for all $x, y \in X$, is a dimonoid. Conversely, any dimonoid can be so constructed.

Corollary 2 (Cayley-type theorem for idempotent g-dimonoids) Let S be a semigroup and $X \neq \emptyset$, let the maps $(\cdot)$ and $(\circ)$ be left and right S-acts respectively, let the map $\varphi : X \to S$ be a (l, r)-morphism of S which satisfies the following condition:

$$\varphi(x) \cdot x = x \circ \varphi(x) = x, \forall x \in X. \qquad (\#)$$

In this case C_S(X) is an idempotent g-dimonoid.

Conversely, for any arbitrary idempotent g-dimonoid G there exists a semigroup S, and $X \neq \emptyset$, a left and a right S-acts of X, and a (l,r)-morphism of S, which satisfies (#) such that G coincides with C_S(X), i.e. $G = C_S(X)$.

Corollary 3 (Cayley-type theorem for hyperassociative g-dimonoids) Let S be a semigroup and $X \neq \emptyset$, let the maps $(\cdot)$ and $(\circ)$ be respectively left and right S-acts, let the map $\Phi : X \to S$ be a (l,r)-morphism of S which satisfies the following conditions:

$$(\varphi(x)\varphi(y)) \cdot z = x \circ (\varphi(y)\varphi(z)), \ \forall x, y, z \in X, \qquad (\#\#)$$

$$(\varphi(x) \cdot y) \circ \varphi(z) = \varphi(x) \cdot (y \circ \varphi(z)), \ \forall x, y, z \in X. \qquad (\#\#\#)$$

In this case $C_S(X)$ is a hyperassociative g-dimonoid.

Conversely, for any arbitrary hyperassociative g-dimonoid G there exists a semigroup S, a non empty set X, and a left and a right S-acts of X, and a (l,r)-morphism of S, which satisfies (##) and (###), that G coincides with C_S(X), i.e. $G = C_S(X)$.

Corollary 4 *Let S be a commutative semigroup. Define the operations $\succ$ and $\prec$ on $S \times S$ as in (op_1) and (op_2). Then $\overline{S} = (S \times S; \prec, \succ)$ is a g-dimonoid, which satisfies the following identity:*

$$x \succ y = y \prec x. \qquad (***)$$

Conversely, for any g-dimonoid D with the condition $(***)$ there exists a commutative semigroup H such that D is isomorphically embedded into $\overline{H}$.

Definition 9 *A g-dialgebra is a vector space over a field equipped with two binary bilinear operations satisfying the axioms of a g-dimonoid.*

Open problem 1. Prove a Cayley-type theorem for g-dialgebras.
Open problem 2. Characterize the free g-dialgebras.

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Yuri Movsisyan
Yerevan State University,
Alex Manoogian 1, 0025 Yerevan, Armenia.
movsisyan@ysu.am

Marlen Yolchyan
Yerevan State University,
Alex Manoogian 1, 0025 Yerevan, Armenia.
marlen.yolchyan94@gmail.com

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