Armenian Journal of Mathematics
Volume 1, Number 2, 2008, 30–36

Definition of Strong Equality of Tautologies and Universal System for Various Propositional Logics

An. Chubaryan, Arm. Chubaryan
Yerevan State University, Yerevan, Armenia
achubaryan@ysu.am

Abstract. Earlier we have introduced a definition of strong equality of classical tautologies, according to which two tautologies are equal iff they have the same hardness. The strong equality implies well known equality, but not vice versa. The strong equality is based on the notion of determinitive conjunct, using of which some new deduction system for classical propositional logic were defined. Here the notions of strong equality of tautologies for various logics are suggested and the idea of construction of universal deduction system for various propositional logics is given.

Key words: Strong equality of tautologies, determinative conjunct, determinative bijunction normal form, universal system for propositional logic

Mathematics Subject Classification 2000: 03B22 03F20

*** Presented at the First International Algebra and Geometry Conference 16–20 May, 2007, Yerevan, Armenia**

1. Introduction

We would like to discuss a conceptual question: in what case two tautologies can be considered as equal. Let φ and ψ be propositional formulae (logical functions) and let each of them depend on the propositional variables $p_1, p_2, \ldots, p_n$. It is well-known, that φ and ψ are equal iff for every $\sigma = (\sigma_1, \ldots, \sigma_n)$ ($\sigma_i \in \{0, 1\}$, $1 \leq i \leq n$) $\varphi(\sigma_1, \ldots, \sigma_n) = \psi(\sigma_1, \ldots, \sigma_n)$. By this conception all classical tautologies are equal to each other. In our opinion this thesis is not entirely correct.

In fact, the tautology $\varphi_k = (p_1 \supset (p_2 \supset (p_3 \supset \ldots \supset (p_k \supset p_1)\ldots)))$ is very "simple". It is easy to notice that (i) if the value of p_1 is 1, then, because of its second occurrence, the value of φ is equal to 1 without taking into consideration the values of the remaining variables, and (ii) dually if the value of p_1 is 0, then the value of φ is 1 because of the first occurrence of p_1. So, only the variable p_1 is "important" in this formula, while the other variables are absolutely unimportant. In some tautologies several variables are "important", and there are also tautologies where nearly all variables are "important". It is natural, that such tautologies are "harder".

In [**1**] the notions of determinative conjunct and determinative disjunctive normal form were introduced. On the basis of these notions some deduction system for classical propositional logic were defined in [**1**] and the notion of strong equality of classical tautologies was suggested in [**2**].

In this paper we generalize this notion for various propositional logics and suggest the idea of construction of universal deduction system for various propositional logics.

2. Preliminary

We must recall some notion and notations, given in [**1**] and [**2**]. If we deal with classical propositional logic, we shall use generally accepted concepts of unit Boolean cube (E^n), logical function, propositional formula, tautology, conjunct and disjunctive normal form (DNF).

Let $\varphi(p_1, p_2, \ldots, p_n)$ be a propositional formula. By N_φ we denote the set of $\sigma = (\sigma_1, \ldots, \sigma_n) \in E^n$, for which $\varphi(\sigma_1, \ldots, \sigma_n) = 1$.

It is well-known, that two propositional formulae φ and ψ are equal iff $N_\varphi = N_\psi$. By this conception all tautologies are equal, but from the point of view of the hardness of validity this conception is wrong.

In fact one cannot consider, that the tautologies

$$\varphi_n = \bigvee_{(\varepsilon_1 \ldots \varepsilon_n) \in E^n} \bigwedge_{j=1}^{2^n-1} \bigvee_{i=1}^{n} p_{ij}^{\varepsilon_i}$$

and

$$\psi_n = (p_{11} \vee \bar{p}_{11}) \vee \bigvee_{j=1}^{2^n-1} \bigvee_{i=1}^{n} p_{ij}$$

are equal, because ψ_n can be valid very "easy", while the tautologies φ_n are "hard".

In [**2**] it was shown, what is the main difference between these formulae.

In the ordinary terminology we call variables and negated variables *literals*; the conjunct $\mathcal{K}$ can be represented simply as the sets of literals and is called *clause* (no clause contains both a variable and its negation). A formula in DNF can be expressed as a set of clauses $\{\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_\ell\}$.

Let $\{p_1, p_2, \ldots, p_n\}$ be the set of variables of the propositional formula φ. For $\sigma = (\sigma_1, \ldots, \sigma_m)$ $(\sigma_j \in \{0,1\},\ 1 \le j \le m,\ 1 \le m \le n)$ the conjunct $K = \left\{p_{i_1}^{\sigma_1}, p_{i_2}^{\sigma_2}, \ldots, p_{i_m}^{\sigma_m}\right\}$ is called *φ-determinative* if the assignment of values σ_j to each p_{i_j} $(1 \le j \le m)$ induces in real time the value for φ, independently of values of the other variables. DNF $\mathcal{D}_\varphi = \{K_1, K_2, \ldots, K_s\}$ is called *φ-determinative DNF* if every K_i $(1 \le i \le s)$ is φ-determinative and $\mathcal{D}_\varphi = \varphi$.

It is not difficult to see that for above mentioned formulae φ_n and ψ_n every φ_n-determinative conjuct contains $2^n - 1$ literals, while every ψ_n-determinative conjunct contains only one literal.

Some arguments for the following definition were given in [**2**].

Definition. *The classical tautologies φ and ψ are strongly equal if every φ-determinative conjunct is also ψ-determinative and vice versa.*

Using the notion of φ-determinative DNF, some deduction system $\mathcal{E}$ for classical propositional logic is introduced in [**1**].

Let $\mathcal{D} = \{\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_\ell,\}$ be a DNF.

The *elimination-rule* (ε-rule) infers $\mathcal{K}' \cup \mathcal{K}''$ from clauses $\mathcal{K}' \cup \{p\}$ and $\mathcal{K}'' \cup \{\bar{p}\}$, where $\mathcal{K}'$ and $\mathcal{K}''$ are clauses and p is a propositional variable.

We would like to say that the conjunct K is deduced from the DNF $\mathcal{D}$ if there is a finite sequence of such clauses, that every clause in the sequence is one of the clauses of $\mathcal{D}$ or is inferred from earlier clauses in the sequence by ε-rule, and the last clause is $\mathcal{K}$.

DNF $\mathcal{D}$ is called *full* (tautology) if the empty conjunct (Λ) can be deduced from $\mathcal{D}$.

The system $\mathcal{E}$ is defined as follow. Axioms are not fixed. For every formula φ every φ-determinative conjunct from some φ-dererminative DNF $\mathcal{D}_\varphi$ can be considered in the capacity of axioms. The inference rule is ε-rule.

It is interesting how the notions of φ-determinative conjuct and φ-determinative DNF can be generalized for nonclassical logic.

As the intuitionistic (minimal) validity is determined only by derivability in some intuitionistic (minimal) propositional calculus and $\overline{\overline{p}}$ is not equivalent to p in nonclassical logic, above notion of φ-determinative conjunct is not directly applicable for these systems. The analogies of the φ-determinative conjunct for intuitionistic and minimal logics are constructed in [**3**].

For fuzzy logic the notion of φ-determinative conjunct is suggested by D. Alanakyan in [**4**].

For some other logics the analogies of the φ-determinative conjunt can be also suggested, therefore the above definition of strong equality can be valid for various logics, and, it seems, some universal system for various propositional logics can be constructed.

3. Main results and ideas

Here we show how can be constructed φ-determinative conjuncts and φ-determinative DNF's (therefore the corresponding deduction system, based on the φ-determinative DNF and elimenation rule) for various non-classical logics.

φ-determinative conjunct and φ-determinative DNF for intuitionistic and minimal (Jnhansson's) propositional logics are constructed in [**3**].

- The literals for **intuitionistic** logic are p, $\bar{p}$ and $\bar{\bar{p}}$ (note that $\bar{\bar{p}} \sim p$ is not derivable in intuitionistic logic). The contrary pairs of literals are both $p\ \bar{p}$ and $\bar{p}$, $\bar{\bar{p}}$. For every intuitionistic validity formula φ φ-determinative conjuncts and φ-determinative DNF are constracted on the basis of intuitionistic resolution refutation of φ. Two ε-rules for corresponding deduction system eliminate the contrary pairs.
- The literals for **minimal** logic are p, $p \supset\perp$ and $(p \supset\perp) \supset\perp$. The contrary pairs of literals are both p, $p \supset\perp$ and $p \supset\perp$, $(p \supset\perp) \supset\perp$. For every minimal validity formula φ φ-determinative conjunct and φ-determinative DNF are constructed on the basis of minimal resolutiuon refutation of φ. Two ε-rules for corresponding deduction system eliminate the contrary pairs.

Recall that there a well-known notion of positive and negative occurences of subformulas (or variables) in the formula or in the sequent (see for example [**3**]). If a variable p has negative occurence in some subformula, which in its turn has negative occurence in the formula, we say that the variable p has double negative occurence in this formula.

- The literals for **positive** and **monotone** propositional logics can be p^+, p^- and p^{--} (positive, negative and double negative). We suppose that for construction of φ-determinative conjuncts and φ-determinative DNF's for positive and monotone propositional logics it is necessary to give the definitions of resolution-type systems for these propositional logics.
- The notions of φ-determinative conjuncts and φ-determinative DNF's for some systems of **fuzzy** logic are defined in [**4**] and in the same place they are constructed on the basis of corresponding resolution systems. The literals are p^α for each number α from interval $[0,1]$ and p^β for each interval β, which is the subinterval of interval $[0,1]$. The pair of literals p^β and

$p^{\beta'}$ is contrary iff $\beta' \cup \beta'' = [0,1]$. The corresponding ε-rules eliminate the contrary pairs.

A study of the other interesting logics is in progress.

References

[1] A. A. Chubaryan, *On complexity of the proofs in Frege system*, CSIT Conference, Yerevan, 2001, 129–132.

[2] An. Chubaryan, Arm. Chubaryan, *A new conception of Equality of Tautologies*, L&PS, Vol. V, No. 1, Triest, Italy, 2007, 3-8.

[3] An. Chubaryan, Arm. Chubaryan, S. Sayadyan, *Relative efficiency propositional proof systems for Classical and Nonclassical Propositional Logic*, Perspectives on Universal Logic, Polimetrica, ISP, Monza, Italy, 2007, 265-275.

[4] D. Alanakyan, *Definition and study of φ-determinative DNF for Fuzzy logic*, The Magister Thesis, Dept. of Informatic and Applied Mathematic, YSU, 2007, Manuscript (on Armenian).