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Normal Subgroups in Free Burnside Groups of Odd Period

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Abstract. In the current paper we announce a positive answer for all prime numbers $n > 997$ to the following problem set by Adian in Kourovka Notebook: Is it true that all proper normal subgroups of the group $B(m, n)$ of prime period $n > 665$ are not free periodic groups? The current result also strengthens a similar result of Olshanskiy for sufficiently large odd numbers n ($n > 10^7 7$).

Key words: Burnside groups, normalizer of subgroup, variety of n-periodic groups, non-abelian simple group

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Problem 7.1 in Kourovka Notebook [1], set by S. I. Adian asks: "It is known that free periodic groups $B(m, n)$ of prime period $n > 665$ have many properties similar to the properties of free groups (see. [2]). Is it true that all proper normal subgroups of the group $B(m, n)$ of prime period $n > 665$ are not free periodic groups?"

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For odd $n \geq 665$ the free 2-generator periodic group $B(2, n)$ of period n contains isomorphic copies of free periodic groups $B(m, n)$ of any finite rank $m \geq 1$, which first was proved by Adian in [3]. Later Shirvanyan in [4] proved that the group $B(2, n)$ also contains a subgroup isomorphic to the free Burnside groups $B(\infty, n)$ of infinite rank. We proved (see. [5],[6],[7]), that for odd $n \geq 1003$ every non-cyclic subgroup of the group $B(2, n)$ contains an isomorphic copy of the free Burnside group $B(\infty, n)$.

Besides the listed results there are a few other known important properties of free periodic groups, similar to the properties of absolutely free groups. As it is established by Adian:

- for odd $n \geq 665$ and $m > 1$ the group $B(m, n)$ has exponential growth ([2], VI.2.5),
- in the group $B(m, n)$ the problem of conjugency is solvable,
- the center of the group $B(m, n)$ is trivial,
- the centralizer of any non-trivial element in $B(m, n)$ is a cyclic group ([2], VI.3.2-VI.3.5),
- the group $B(m, n)$ in a non-amenable group (see. [8]).

Thus, the mentioned questions stresses that we need not expect full analogy with absolutely free groups.

The positive answer to the Problem 7.1 [1] for sufficiently large odd n (where $n > 10^{78}$) was given by Ol'shanskiy in [9]. He proved ([9], Theorem 1.1), that for sufficiently large odd n the normalizer of any free periodic subgroup N of rank $r \geq 1$ in free periodic group $B(m, n)$ of period n and of any rank $m \geq 1$ coincides with N (the rank $m \geq 1$ may also be infinite). When rank r of free periodic subgroup N is equal to 1, and $n \geq 665$ is an odd number, then coincidence of the normalizer of N with N immediately follows from the following theorem of Adian: each finite subgroup of the group $B(m, n)$ is cyclic (see. [2], Theorem VII. 1.8). Let us notice that for sufficiently large composite n ($n > 2 \cdot 10^{77}$) the statement of the mentioned Theorem 1.1 was earlier proved by

Ivanov in [10]. Nevertheless, that prove is no longer valid for prime numbers n.

We proved:

THEOREM 1. *Let $n \geq 1003$ be an odd number, and N be a non-trivial subgroup of the free Burnside group $B(\mathcal{U}, n)$ with a set of free generators $\mathcal{U}$. Assume the subgroup N is isomorphic to a free periodic group $B(\mathcal{V}, n)$. Then N coincides with its normalizer in the group $B(\mathcal{U}, n)$.*

COROLLARY 1. *Let $n \geq 1003$ be an odd number, and N be a non-trivial subgroup of the free Burnside group $B(\mathcal{U}, n)$ with a set of free generators $\mathcal{U}$. Then if N is isomorphic to a free periodic group, Then $N = B(\mathcal{U}, n)$.*

From this a positive answer to Problem 7.1 [1] immediately follows for all $n > 997$.

As it was proved by Adian in [2], [11], [12], groups $B(m, n)$ are rich in normal subgroups. In particular, it is proved in [12] that for odd $n \geq 665$ and $m > 65$ the group $B(m, n)$ does not satisfy min and max conditions for normal subgroups (see. also Theorem VI.3.9[2]), it is also proved that the group $B(m, n)$ contains continuum distinct normal subgroups. In article [13] proved are similar properties of the group $B(m, n)$ for all odd $n \geq 1003$ and $m \geq 1$.

COROLLARY 2. *For any odd $n \geq 1003$ and $m \geq 2$ the group $B(m, n)$ contains continuum distinct subgroups which are not free in the variety of all n-periodic groups.*

Ol'shanskiy proved in [9] , that to guarantee Theorem 1 it is sufficient to prove that if $N = B(\mathcal{V}, n)$ is a free Burnside group of sufficiently large period with base $\mathcal{V} = \{a_1, a_2, ...\}$, and the word $v = v(a_1, a_2, ..., a_m)$ is not conjugate in $N = B(\mathcal{V}, n)$ with the powers of the generators $a_1, a_2, ...,$, then there exists a non-abelian simple factor-group ${}^{N}/_{L}$ such that the canonical image of the generator a_1 in ${}^{N}/_{L}$ has order n, and images v and a_1 are not conjugate with respect to any automorphism of the group ${}^{N}/_{L}$ (see Lemma 2.3 [9]).

We proved:

PROPOSITION 1. *(Stronger version of Lemma 2.3 [9]). Let* $N = B(\mathcal{V}, n)$ *is a free Burnside group of the odd period* $n \geq 1003$ *with base* $\mathcal{V} = \{a_1, a_2, \ ...\}$. *Assume the word* $v = v(a_1, a_2, \ ..., a_m)$ *is not conjugate in* $N = B(\mathcal{V}, n)$ *with a power of the generator* a_1. *Then there exists a non-abelian simple factor-group* ${}^{N}/_{L}$ *such that:*

1. *canonical image of the generator* a_1 *in* ${}^{N}/_{L}$ *has order* n,
2. $\psi(a_1) \neq v(a_1, a_2, \ ..., a_m)$ *for any automorphism* $\psi : {}^{N}/_{L} \to {}^{N}/_{L}$.

To compare the results let us notice that in Theorem 3 [13] and non-cyclic free Burnside group of odd period $n \geq 1003$ is a residually non-abelian simple group (in which the image of the generator a_1 has order n). Proving Theorem 2 we seriously use the work of Adian and Lisyonok [14], where for any odd $n \geq 1003$ an infinite group of period n with cyclic subgroups is built (the "Tarski monster").

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