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Algebraic groups and Lie groups with few factors

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Abstract. Algebraic groups are treated here from a group theoretical point of view and the obtained results are compared with the analogous issues in the theory of Lie groups. The main body of the text is devoted to a classification of algebraic groups and Lie groups having only few subgroups or few factor groups of different type. In particular, the diversity of the nature of algebraic groups over fields of positive characteristic and over fields of characteristic zero is emphasized. This is revealed by the plethora of three-dimensional unipotent algebraic groups over a perfect field of positive characteristic, as well as, by many concrete examples which cover an area systematically. In the final section, algebraic groups and Lie groups having many closed normal subgroups are determined.

Key words: Cohomology theory, Group varieties, Nilpotent and solvable Lie groups, Solvable, nilpotent Lie algebras, Chains and lattices of subgroups

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One essential task of group theory is the description of a given group by the way in which it is decomposed from more elementary subgroups.There are two kinds of most elementary Lie groups and algebraic groups. One class is formed by such groups that have a dense cyclic subgroup; these goups are commutative. The other kind of elementary connected Lie groups or connected algebraic groups are those which have a chain as lattice of their connected subgroups. If such groups do not have dimension one, then in the class of Lie groups they are Shafarevich extensions of simple complex tori. In the class of algebraic groups over fields of characteristic zero,they are either simple abelian varieties or extensions of a one-dimensional affine goup by a simple abelian variety. In the class of algebraic groups over fields of positive characteristic the situation is much more complicated. Besides simple abelian varieties and extensions of a onedimensional torus by a simple abelian variety there are also affine algebraic groups having a chain as lattice of connected closed subgroups. If they are commutative, then they are Witt groups. If they are not commutative, then they forms, as I want to indicate in my talk. a very rich family of unipotent groups, called chains. The knowledge of threedimensional chains allow us to classify, up to isogenies, all threedimensional unipotent groups over perfect fields of characteristic greater than two . The classification of chains having a onedimensional commutator subgroup yields a classification of connected algebraic groups G over perfect fields of characteristic $p > 2$ such that G has a central subgroup of codimension one. These groups have a representation as an almost direct product of a commutative group and a group which is a direct group of chains with amalgamated factor group. Moreover, in an algebraic group G having a central maximal connected subgroup the commutator subgroup G_0 is a central vector group, Conversely, if G_0 is a central vector group and G/zG is isogenous to a Witt group, then the center zG has codimension one in G, a situation which cannot occur for algebraic group over a field of characteristic zero.

The lattice of normal connected algebraic subgroups of unipotent algebraic groups for which the nilpotency class is equal to their dimension n forms a chain of length n. Such unipotent groups occur only over fields of positive characteristic and play an opposite role to those of chains which cannot have maximal nilpotency class. The nilpotency class of $n$dimensional algebraic groups over fields of characteristic zero as well as ndimensional real or complex Lie groups is at most $n-1$. The simple structure of the lattice of connected subgroups of an algebraic or analytic chain motivated us to study how much individual properties of chains restrict the structure of algebraic and analytic groups. Most of these properties remain invariant under isogenies. For example, we investigate connected algebraic groups and connected Lie groups having exactly one maximal connected closed subgroup (unimaximal groups) as well as connected algebraic groups and connected Lie groups having exactly one minimal connected closed subgroup (uniminimal groups). The description of nonaffine algebraic groups, respectively complex Lie groups, which are uniminimal or unimaximal easily reduces to extensions of affine groups of dimension smaller or equal one by abelian varieties, respectively to toroidal groups. Connected affine algebraic groups which are uniminimal or unimaximal and have dimension greater than one are unipotent algebraic groups over fields of positive characteristic.

A noncommutative connected unipotent algebraic group G is uni maximal if and only if the commutator subgroup of every proper connected algebraic subgroup of G is smaller than the commutator subgroup of G. Any group in which the commutator subgroup is a maximal connected subgroup is unimaximal; in particular the unipotent algebraic groups over fields of positive characteristic having maximal nilpotency class are of such type.

Uniminimal noncommutative groups G turn out to be products of chains, where at most only one factor C has dimension greater than two; if C is not commutative then the commutator subgroup of G coincides with the commutator subgroup of C This shows that the structure of uniminimal groups is less complicated than the structure of unimaximal groups.

The conditions to be uniminimal and unimaximal are strong enough to characterise the chains over fields of characteristic greater than two. Moreover, a connected affine algebraic group over a field of arbitrary prime characteristic, which contains a chain M as a maximal connected algebraic subgroup, is either a chain or a product of M with a chain of dimension at most two.

In chains with a onedimensional commutator subgroup any connected algebraic subgroup as well as any proper epimorphic image is commutative. For algebraic groups over fields of positive characteristic none of these two conditions is sufficient for a concrete description. In contrast to this, for real or complex Lie groups and for algebraic groups over fields of characteristic zero.the assumption of commutativity of all proper connected subgroups as well as the dual condition of commutativity of all proper epimorphic images is strong enough for a classification. If G is a noncommutative connected affine algebraic group over a field of characteristic zero such that any connected algebraic subgroup is commutative then G is at most threedimensional. Real and complex Lie groups having only proper connected commutative subgroups exist in any dimension; they are precisely the extraspecial real or complex Lie groups. A connected nonsimple noncommutative affine algebraic group over a field of characteristic zero of dimension at least three such that every epimorphic image of G is commutative is a Heisenberg group. A connected real or complex nonsimple non commutative Lie group of dimension greater than three having only commutative proper epimorphic images is a extraspecial complex Lie group having as center a simple complex torus of dimension at least two.

An affine chain of dimension n has exactly only one connected algebraic subgroup for any dimension $d \leq n$ and any two epimorphic images of the same dimension are isogenous. Investigating these two properties for connected algebraic groups, respectively for real or complex

Lie groups we call any such group aligned if any two proper connected closed subgroups of the same dimension are isomorphic, respectively coaligned if all epimorphic images of the same dimension are isogenous. The property to be coaligned is to weak to obtain a concrete description for such groups algebraich groups or Lie groups. In contrast to the condition to be coaligned the property to be aligned is strong. This documents the fact that a noncommutative connected affine algebraic group over a field of characteristic zero or a linear complex Lie group is aligned if and only if it is unipotent and has dimension three. However, a noncommutative connected real Lie group of dimension $n \geq 4$ is aligned if and only if it is locally isomorphic to one of the following compact Lie groups: $SO_2(R) \times SO_3(R)$, $SO_3(R) \times SO_3(R)$, $SU_3(C, 0), SO_5(R)$ and the 14-dimensional exceptional Lie group G_2.

The chains can be characterized by the fact that they have only few nonisogenous factors. Namely a connected unipotent algebraic group is a chain if and only if it has only finitely many connected algebraic subgroups. This result allows far reaching generalisations. For instance, a noncommutative unipotent algebraic group G is a chain if and only if every epimorphic image of G is isogenous to a subgroup of G and any two connected algebraic subgroups of G of the same dimension are isogenous. The dual conditions also give a characterisation of noncommutative unipotent chains.

A connected algebraic group is called hamiltonian if all its connected algebraic subgroups are normal. A connected algebraic $k$group over a field k of characteristic zero such that any connected $k$subgroup is normal is commutative. But the situation changes drastically if we consider hamiltonian groups over fields of positive characteristic. Any noncommutative chain, more generally any uniminimal connected algebraical group, is hamiltonian. Other examples of connected hamiltonian algebraic groups are the groups in which the centre has co dimension one.

A connected algebraic $k$subgroup P of a connected affine algebraic group G defined over an infinite perfect field k such that $PH = HP$ for any $k$closed subgroup H of G is normal in G. Because of this result we call a connected algebraic subgroup Q of an algebraic group G quasinormal if it is permutable with every connected algebraic subgroup of G. Quasinormal, but not normal algebraic subgroups exist only in algebraic groups G over fields of positive characteristic. Essentially they are contained in the unipotent radical of G.

A connected closed subgroup Q of a topological group G is topologically permutable with a closed subgroup P of G if the sets QP and PQ have the same closure in G. Using a suitable closure operator on the set of subgroups of G it is possible to prove that in a connected real or complex Lie group G any connected closed subgroup which is topologically permutable with every closed connected subgroup of G must be normal in G. Moreover, every connected real or complex Lie group in which every connected closed subgroup is topologically permutable with any other connected closed subgroup must be commutative. Our point of view has also the advantage that p-adic Lie groups are included in the considerations provided we modify topological permutability to locally topological permutability.

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