Armenian Journal of Mathematics
Volume 10, Number 9, 2018, 1–22

On an Over-Convergence Phenomenon for Fourier series. Basic Approach.

Anry Nersessian

Dedicated to the memory of my teacher,
academician Mkhitar Djrbashyan

Abstract. This paper is devoted to the acceleration of the convergence of the partial sums of the classical Fourier series for the sufficiently smooth functions. Some universal and adaptive algorithms are constructed and studied. It is shown that the use of a finite number of Fourier coefficients makes it possible exact approximation of a given function from an infinite-dimensional set of quasi-polynomials. In this sense, we call the corresponding essentially nonlinear algorithms as over-convergent.

The proposed algorithms are implemented using *Wolfram Mathematica* system. Numerical results demonstrate their effectiveness.

Key Words: Fourier series, biorthogonalization, acceleration of convergence, spectral methods, adaptive algorithms, over-convergence, detection of periodicities.
Mathematics Subject Classification 2010: 42B05, 65B99

1 Introduction

One of the classical tools of mathematics is the apparatus of Fourier series, based on the orthogonal system $\{e^{i\pi nx}\}$, $n = 0, \pm 1, \pm 2, \ldots$, complete in $L_2[-1, 1]$. However, in practice its usage is extremely limited because of the poor approximation of a piecewise smooth function. Thus, in the case when the function has discontinuity points (taking into account also discontinuities at the ends of the interval $[-1, 1]$[1], an intense oscillation arises in their neighborhood (Gibbs phenomenon), and the uniform convergence is absent,

[1] It is accepted that $1 + 0 = -1 + 0$.

although within the intervals of high smoothness of the decomposing function, a good convergence is observed. This leads to a slow L_2-convergence on the entire segment $[-1, 1]$.

Respectively, a direction appeared in the scientific literature which is characterized, in particular, as the "overcoming of Gibbs phenomenon" and "the acceleration of convergence of Fourier series". We shortly give the main milestones of such studies.

1.1

The pioneer of the "overcoming of the Gibbs phenomenon", for the Fourier series, is undoubtedly the outstanding scientist and engineer A. N. Krylov, who as far back as at the dawn of the twentieth century ([1], 1907) proposed the methods that he later developed in many times republished classical work [2]. He developed an approach based on calculation the jumps of approximated piecewise smooth functions and their derivatives. Unfortunately, in the international arena the ideas of A.N. Krylov remained in the shadow for a long time.

However, difficulties in calculation of the above mentioned jumps directly by the approximated function f, limits the scope of the practical application of the Krylov method. In [6] (1993) Knut Eckhoff developed a "spectral" approach based on polynomial acceleration and proved to be much more practical, since in the Krylov scheme only the Fourier coefficients of the function f were used. On this path it was necessary to solve a certain linear equation with the Vandermonde matrix.

We call this approach the Krylov-Eckhoff method (**KE-method**).

When realizing the KE-method, the convergence rate increases significantly, but the solution of Eckhoff equation (for large number of used Fourier coefficients) perceptibly increases of complexity of the coresponding algorithm. In addition, K.Eckhoff used the polynomial acceleration of convergence of the Fourier series. In a number of cases this is reflected already in the efficiency of the KE-method itself. Apparently, this is one of the reasons for the lack of realization of the KE-method proposed in the last century even in such computer systems as *Wolfram Mathematica* and *MatLab*.

1.2

KE-method and other approaches to the acceleration of convergence of Fourier series have been developed in different directions by many researchers (see [3-5], [7-18] and their links). In particular, an algorithm is given in [9] possessing a superconvergency (faster than any power-law convergence) for infinitely smooth functions. An explicit algorithm for the KE-method was

constructed in [13] and the exact asymptotic error of the convergence acceleration was found.

We mention especially the phenomenon discovered numerically in the paper [10]. A combination of KE-method and Pade approximation (applied to the asymptotic expansion of the Fourier coefficients) is implemented there. Some theoretical estimates and explicit formulas related to this approach are given in [11] and [12]. As a result of numerical experiments, it turned out that, as a rule, the corresponding algorithm was "almost exact" on an infinite-dimensional set of exponential functions, although only a finite number of Fourier coefficients was used. A question was raised about the theoretical justification for this unexpected phenomenon.

The acceleration algorithms proposed below are based on explicit formulas. In sections **2** and **3** of the paper traditional "universal" algotithms are presented. The practical application of such an acceleration of convergence of the Fourier series of any dimension is elementary: in classical partial sums of Fourier series, instead of the Fourier exponents, certain predefined functions must be inserted.

In sections **4** the hypothesis of [10] is proved and the corresponding adaptive algorithm is constructed. The effect of this phenomenon (called below "over - convergence") leads to an unprecedented increasing of the accuracy of approximation of given smooth function on the basis of a finite number of its Fourier coefficients.

Numerical results characterizing the proposed algorithms for the acceleration of the convergence of partial sums of Fourier series for smooth functions are given in section **5**.

In section **6** a brief discussion of the results is given.

2 Biorthogonal interpretation of the acceleration of convergence of Fourier series

Take an integer $n \geq 1$ and consider a system of linearly independent smooth functions $\{\Psi_r(x)\}$, $x \in [-1,1]$, $r = 0, \pm 1, \ldots, \pm n$. Denote by $\{\psi_{r,k}\}$ the corresponding Fourier coefficients. Here and below, the Fourier coefficients $\{f_s\}$ of the function $f \in L_2[-1,1]$ are determined by the formula

$$f_s = \frac{1}{2} \int_{-1}^{1} f(t)\, \exp(-i\,\pi\, s\, t)\, dt,\ s = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

We shall construct a biorthogonal system $\{\Xi_k(x), 1/2 \exp(i\pi k x)\}$ in L_2, where the functions $\{\Xi_k\}$ belong to the linear span of the system $\{\Psi_k\}$. It is easy to see that such system exists (and is unique) if the matrix $[\psi_{r,k}]$ $(r, k =$

$0, \pm 1, \ldots, \pm n)$ is invertible. Consider the approximation

$$f(x) \simeq \sum_{r=-n}^{n} f_r \, \Xi_r(x), \; x \in [-1, 1]. \tag{2}$$

Lemma 1 *Approximation (1) is exact for any element of the linear span of the system* $\{\Psi_k\}, k = 0, \pm 1, \ldots, \pm n$.

Proof. It follows that any element $f = \sum_{k=-n}^{n} c_k \, \Psi_k(x)$, $c_k = const$, can be represented in the form $f = \sum_{k=-n}^{n} b_k \, \Xi_k(x)$, $b_k = const$. According to the indicated biorthogonality, $f_r = b_r$, so approximation (2) is exact. □

Remark 1 *This scheme can be rephrased with regard to the "acceleration" of a given orthogonal (or biorthogonal) decomposition in some separable Hilbert space. It is another matter whether such an approach in specific cases will be effective in applying.*

*As for the known methods of accelerating the convergence of the partial sums of the Fourier series created by the KE - method and its modifications (see above **1.1** and **1.2**), it turns out that they are also actually based on the indicated biorthogonalization.*

The most problematic moment in this "acceleration" scheme is the realization of the biorthogonalization process. For large n, the numerical implementation sometimes can be inefficient due to the accumulation of errors. In addition, the study of the properties of approximation (2) is rather difficult (for example, the estimate of its asymptotic error when $n \to \infty$).

In this paper we show that if the system $\{\Psi_r\}$ consists of quasi-polinomials (that is, linear combinations of functions of the form $P(x) \exp(z\, x)$, where $z \in \mathbb{C}$, and P is a polynomial), then such a biorthogonal system can be constructed in explicit, analytical form for any n, and the approximation of the form (2) leads to effective acceleration of convergence of the Fourier series to a function f sufficiently smooth on the whole interval $[-1, 1]$. Such linear algorithms are called "universal" below, since for a fixed n the same system of quasi-polynomials $\{\Xi_k(x)\}$ is used for approximation of any smooth function $f(x) \in L_2[-1, 1]$ (see section **3**).

In section **4** it is proved that the use of a finite number of Fourier coefficients of the function f leads to an exact approximation of f from an infinite-dimensional set of quasi-polynomials. On this basis, an algorithm is proposed that builds its own system $\{\Xi_r\}$ for each function f. Below such an nonlinear algorithm is called "adaptive".

Some of the proposed algorithms are numerically implemented in section **5**. As the results in tables 1 and 2 show, adaptive algorithms have an absolute advantage.

3 Some universal acceleration algorithms

3.1 The acceleration of the one-dimensional Fourier series

For a fixed integer $n \geq 0$ and $r = 0, \pm 1, \ldots, \pm n$, consider the following infinite rational sequence[2]

$$\theta_{r,s} \stackrel{def}{=} (-1)^{s-r} \Big(\prod_{\substack{p=-n \\ p \neq r}}^{n} \frac{s-p}{r-p} \Big) \prod_{q=-n}^{n} \frac{r-\lambda_{r,q}}{s-\lambda_{r,q}}, \; s = 0, \pm 1, \pm 2, \ldots \tag{3}$$

where $\{\lambda_{r,p}\} \subset \mathbb{C}, p = 0, \pm 1, \pm 2, \ldots, \pm n$, is a set of non-integers and s being any integer. Note that $\theta_{r,r} = 1$ and $\theta_{r,s} = 0$ for $s \neq r$, $s = 0, \pm 1, \ldots, \pm n$.

Now consider the following functions represented by their Fourier series.

$$\Theta_r(x) \stackrel{def}{=} \exp(i\,\pi\,r\,x) + \sum_{|s| \geq n+1} \theta_{r,s} \exp(i\,\pi\,s\,x), \; x \in [-1, 1] \tag{4}$$

Theorem 1 *The system* $\{\Theta_r(x), 1/2 \exp(i\,\pi\,r\,x)\}$, $r = 0, \pm 1, \ldots, \pm n$, *is biorthogonal on the segment* $x \in [-1, 1]$. *If* $\lambda_{r,k} - \lambda_{r,p} \neq 0$, $p \neq k$, *the functions* $\{\Theta_r\}$ *are represented in the following explicit form*

$$\Theta_r(x) = \sum_{k=-n}^{n} c_{r,k} \exp(i\,\pi\,\lambda_{r,k}\,x), \;\; r = \pm 1, \pm 2, \pm n, \; x \in [-1, 1], \tag{5}$$

where [3]

$$c_{r,k} = \frac{1}{sinc(\pi(r - \lambda_{r,k}))} \Big(\prod_{\substack{p=-n \\ p \neq k}}^{n} \frac{r-\lambda_{r,p}}{\lambda_{r,k}-\lambda_{r,p}} \Big) \prod_{\substack{q=-n \\ q \neq r}}^{n} \frac{\lambda_{r,k}-q}{r-q}.$$

Proof. We note at first that $\Theta_r \in L_2[-1, 1]$, since evidently $\theta_{r,s} = O(1/s)$ for $s \to \infty$. Biorthogonality follows immediately from (4).

To prove the representation (5), note that for fixed r the function (3) is meromorphic by $s \in \mathbb{C}$ and can be decomposed into simple fractions. Since it has only simple poles, the easiest way to obtain such a decomposition is to use the residues at the points $\{s = \lambda_{r,k}\}$.

[2] To avoid complications in formulas, the dependence of some quantities on n will be omitted below.

[3] $sinc(z) = \sin(z)/z$, $z \in \mathbb{C}$, $sinc(0) = 1$.

From the other hand it is easy to verify that for $\lambda \in \mathbb{C}$ we have

$$\exp(i\,\pi\,\lambda\,x) = \sum_{s=-\infty}^{\infty} \mathrm{sinc}(\pi(s-\lambda))\,\exp(i\,\pi\,s\,x),\ x \in [-1,1]. \tag{6}$$

Therefore, it is easy to verify the validity of formula (5). $\square$

The acceleration of convergence is given by the following biorthogonal approximation

$$f(x) \simeq F_n(x) \stackrel{def}{=} \sum_{r=-n}^{n} f_r \Theta_r(x),\ x \in [-1,1]. \tag{7}$$

According to (4), the error

$$R_n(x) \stackrel{def}{=} f(x) - F_n(x) \tag{8}$$

of this approximation has the form

$$R_n(x) = \sum_{|s|\geq n+1} (f_s - \sum_{r=-n}^{n} f_r \theta_{r,s})\exp(i\,\pi\,s\,x),\ x \in [-1,1] \tag{9}$$

which allows to write down L_2-norm of $R_n(x)$ also in an explicit form.

According to Lemma 1, $R_n(x) \equiv 0$, $x \in [-1,1]$, for any element of the linear span of the system $\{\Theta_r\}, r = 0, \pm 1, \ldots, \pm n$.

Remark 2 *Some of the above restrictions on parameters $\{\lambda_{r,k}\}$ can be removed. Thus, in the representation (3) we can assume, for example, that $\lambda_{r,0} = \lambda_{r,1}$ and then in the proof of Theorem 1 the function, which is meromorphic with respect to s, is equal to $\theta_{r,s}$ has a second-order pole and then the formula (5) have to be corrected. In this case a term of the form $c\,x\exp(i\,\pi\,\lambda_{r,0}x)$ will arise in $\Theta_r(x)$ (c=const.). In the general case $\Theta_r(x)$ will be a quasi-polynomial (see also below **3.3.1**, **3.3.2** and Theorem 3).*

3.2 The acceleration of the multi-dimensional Fourier series

We give one simple generalization of this universal approximation algorithm to the multidimensional case. Let $x = (x_1, x_2, \ldots, x_m), m \geq 2$, be a point of the m-dimensional space, and a function f is given in the domain D_m, defined as the direct product of the segments $-1 \leq x_k \leq 1$, $k = 1, 2, \ldots, m$. Denote by $\{f_r\}$, $r = (r_1, r_2, \ldots, r_m)$ its m-dinensional Fourier coefficients.

For a given vector $n = (n_1, n_2, \ldots, n_m)$ with nonnegative integer components we will say that the corresponding partial sum of Fourier series is approximated ("accelerated") by the following quasi-polynomial F_n on m variables

$$f(x) \stackrel{def}{=} F_n(x) = \sum_{r_1=-n_1}^{n_1} \sum_{r_2=-n_2}^{n_2} \cdots \sum_{r_m=-n_m}^{n_m} f_r \prod_{k=1}^{m} \Theta_{r_k}(x_k), \tag{10}$$

where the functions $\{\Theta_{r_k}(x_k)\}$ are quasi-polynomials, determined on each variable x_k with the own set of parameters $\{\lambda_{r_k,s_k}\}$ (see (4) and Remark 2).

It is not difficult to formulate analogues of considered above results for multi-dimensional case. Thus, obviously the approximation (10) is an expansion by the biorthogonal system

$$\{ \prod_{k=1}^{m} \Theta_{r_k}(x_k),\ \frac{1}{2^m} \prod_{k=1}^{m} \exp(i\,\pi\,k\,x_k)\ \},\ x_k \in [-1,1],\ k = 1, 2, \ldots, m.$$

3.3 Some special cases

For a fixed $n \geq 1$ and r in a one-dimensional sequence (3) there are $(2n+1)^2$ complex parameters $\{\lambda_{r,q}\}$. In the m-dimensional case (see (10)), when $n_k = n, \forall k$, their number is equal to $m\,(2n+1)^2$. As one can see the choice of these parameters is quite wide. Here we shall dwell only on the following two special cases.

3.3.1 Algorithm A

In this case the non-integer parameters $\{\lambda_{r,q}\}$ depend only on the the second index ($\lambda_{r,p} = \lambda_p$, $p = 0, \pm 1, \ldots, n$), so

$$\theta_{r,s} = (-1)^{s-r}\ (\prod_{\substack{p=-n \\ p \neq r}}^{n} \frac{s-p}{r-p}\ )\ \prod_{q=-n}^{n} \frac{r-\lambda_q}{s-\lambda_q},\ s = 0, \pm 1, \pm 2, \ldots \tag{11}$$

If $\lambda_p \neq \lambda_q$, $p \neq q$, the formulas (7) and (10) are approximations of the function $f(x)$ by linear combinations of exponents.

It follows from Lemma 1 that if the function f belongs to the linear span of the system $\{\exp(i\,\pi\,\lambda_p\,x)\}$, $p = 0, \pm 1, \ldots, \pm n$, then the approximations (7) and (10) are exact. If some of the parameters in $\{\lambda_q\}$ coincide, then

$$\theta_{r,s} = (-1)^{s-r}\ (\prod_{\substack{p=-n \\ p \neq r}}^{n} \frac{s-p}{r-p}\ )\ \prod_{q=1}^{m} \left(\frac{r-\lambda_q}{s-\lambda_q}\right)^{n_q},\ \ s = 0, \pm 1, \ldots \tag{12}$$

where $\{n_q\}$, $q = 1, \ldots, m$, are the positive integers and $\sum_{q=1}^{m} n_q = 2n + 1$, $\lambda_k \neq \lambda_p$ if $k \neq p$. In this case, the basis of the acceleration algorithm is the following decomposition into the simple fractions

$$\frac{\prod_{\substack{p=-n \\ p\neq r}}^{n}(s-p)}{\prod_{q=1}^{m}(s-\lambda_q)^{n_q}} = \sum_{k=1}^{m}\sum_{p=1}^{n_q}\frac{a_{k,p}}{(s-\lambda_q)^p},\ a_{k,p} = const.,\ s \in \mathbb{C}. \tag{13}$$

It is not difficult to present here the corresponding system $\{\Theta_r(x)\}$ (see Lemma 1, Remark 2 and Theorem 3 below).

3.3.2 Algorithm B

Here non-integer parameters $\{\lambda_{r,q}\}$ depend only on the first index r ($\lambda_{r,p} = \lambda_r$). Respectively, let us consider for fixed $n \geq 1$ and r $(= 0, \pm 1, \ldots, \pm n)$ the rational with respect to s sequence

$$\theta_{r,s} = (-1)^{s-r}\ \left(\frac{r-\lambda_r}{s-\lambda_r}\right)^{2n+1}\ \prod_{\substack{p=-n \\ p\neq r}}^{n}\frac{s-p}{r-p},\ s = 0, \pm 1, \pm 2, \ldots \tag{14}$$

If $\lambda_r = 0, \forall r$, then the formula formula (7) is an approximation of a function $f(x)$ by polynomials of at most $2n$-th order.

Theorem 2 *If $\{\lambda_r\} \subset \mathbb{C}$, then the system $\{\Theta_r(x)\}$ consists of the following functions*

$$\Theta_r(x) = (r-\lambda_r)\ (\prod_{\substack{p=-n \\ p\neq r}}^{n}\frac{r-\lambda_r}{r-p})\ \sum_{k=0}^{2n} b_{r,2n-k}\,\Lambda_{r,k}(x),\ \ r = 0, \pm 1, \pm 2, \pm n, \tag{15}$$

where the system $\{\Lambda_{r,k}(x)\}$ consists of the following quasi-polynomials

$$\Lambda_{r,k}(x) \stackrel{def}{=} \frac{-\pi}{k!}\,\frac{d^k}{d\,\lambda_r^k}\,\frac{\exp(i\,\pi\,\lambda_r\,x)}{\sin(\pi\lambda_r)}, \tag{16}$$

and

$$b_{r,k} = \frac{1}{k!}\,\frac{d^k}{d\lambda_r^k}\,\prod_{\substack{p=-n \\ p\neq r}}^{n}(\lambda_r - p),\ k = 0, 1, \ldots, 2n,\ r = 0, \pm 1, \pm 2, \pm n.$$

Proof. First we consider the power series at the point $z = \lambda_r$ for the following polynomials of degree $2n$

$$P_r(z) \stackrel{def}{=} \prod_{\substack{p=-n \\ p\neq r}}^{n} (z-p) = \sum_{k=0}^{2n} b_{r,k}(z-\lambda_r)^k,\ b_{r,k} = P_r^{(k)}(\lambda_r)/k!,\ z\in\mathbb{C}.$$

Of course the coefficients $\{b_{r,k}\}$ (also depending on n) can be explicitly expressed with the help of Vieta's formulae.

According to (11) and (16), we have

$$\theta_{r,s} = (r-\lambda_r)\,(\prod_{\substack{p=-n \\ p\neq r}}^{n} \frac{r-\lambda_r}{r-p}\,)\ \sum_{k=0}^{2n} b_{r,k}\,(s-\lambda_r)^{-k-1},\ \ r=0,\pm1,\ldots,\pm n. \quad (17)$$

Denoting now

$$\Lambda_{r,k}(x) = \sum_{s=-\infty}^{\infty} \frac{(-1)^s\,\exp(i\,\pi\,s\,x)}{(s-\lambda_r)^k}, \quad (18)$$

it is not difficult to verify the validity of formula (15). So $\{\Theta_r\}$ is a system of quasi-polynomials. □

4 Adaptive algorithms

In each of the universal algorithms given above, the value of n and parameters $\{\lambda_p\}$ are fixed and in the acceleration process a finite number of Fourier coefficients $\{f_s\}$, of the approximated function f are used.

It is natural to ask whether it is possible to construct an algorithm that, figuratively speaking, pulls out of an additional information from the same Fourier coefficients because of the optimal choice of the parameters $\{\lambda_p\}$.

A positive answer to this question was obtained earlier in [10] (see **1.2** above) at the level of some numerical experiments. Below, we give a rigorous justification for the phenomenon noted there, on the basis of which an algorithm will be implemented a significant additional acceleration of the convergence of Fourier series.

4.1 Some definitions and clarifications

The classical definition of the partial sum of the Fourier series used above is related to traditional numbering of the Fourier system and, apparently, is justified only by the fact that Fourier coefficients tend to zero. The following reasoning is based on the natural definition of the partial sum, which is more flexible than the classical one.

Definition 1 *We call a partial sum of the Fourier series for a function f any sum of the form*

$$\mathfrak{S}_n(x) \stackrel{def}{=} \sum_{k\in\mathfrak{D}_\mathfrak{n}} f_k \exp(i\,\pi\,k\,x),\ x\in[-1,1], \tag{19}$$

where $\{f_k\}$ are Fourier coefficients of $f(x)$, and $\mathfrak{D}_\mathfrak{n} = \{d_k\}, k = 1,\dots,n$, is a set of n different integers ($n \geq 1$). Let's assume that $\mathfrak{D}_\mathfrak{o} = \emptyset$.

Sometimes it is more convenient to use an equivalent recording

$$\mathfrak{S}_n(x) = \sum_{k=1}^{n} f_{d_k} \exp(i\,\pi\,d_k\,x),$$

where the set $\{d_k\}$ is numbered (for instance) in the ascending order.

Consider a set of non-integer numbers $\{\lambda_{r,k}\} \subset \mathbb{C}$, $r,k \in \mathfrak{D}_\mathfrak{n}$. We use (in place of (3)) the following sequence

$$\mathfrak{t}_{r,s} \stackrel{def}{=} (-1)^{s-r} \Big(\prod_{\substack{p\in\mathfrak{D}_\mathfrak{n}\\ p\neq r}} \frac{s-p}{r-p}\Big) \prod_{q\in\mathfrak{D}_\mathfrak{n}} \frac{r-\lambda_{r,q}}{s-\lambda_{r,q}},\ r\in D_n,\ s = 0,\pm1,\dots. \tag{20}$$

Here the numbering of the parameters $\{\lambda_{r,k}\}$ is related to $k \in \mathfrak{D}_\mathfrak{n}$.

By analogy with definitions (4), (7) and (8) we have

$$\mathfrak{T}_r(x) = \exp(i\,\pi\,r\,x) + \sum_{s\notin\mathfrak{D}_\mathfrak{n}} \mathfrak{t}_{r,s} \exp(i\,\pi\,s\,x),\ r\in\mathfrak{D}_\mathfrak{n},$$

$$f(x) \simeq \mathfrak{F}_n(x) \stackrel{def}{=} \sum_{r\in\mathfrak{D}_n} f_r\mathfrak{T}_r(x),\ \mathfrak{R}_n(x) \stackrel{def}{=} f(x) - \mathfrak{F}_n(x),\ x\in[-1,1]. \tag{21}$$

The system $\{\mathfrak{T}_r(x), 1/2\exp(i\,\pi\,r\,x)\}$, $r\in\mathfrak{D}_\mathfrak{n}$, is biorthogonal in $L_2[0,1]$.

The estimation of the L_2-error of the approximation $f(x) \simeq \mathfrak{F}_n(x)$ has (by analogy with (9)) the form

$$||\mathfrak{R}_n||^2 = \sum_{s\notin\mathfrak{D}_\mathfrak{n}} ||f_s - \sum_{r\in\mathfrak{D}_n} f_r\mathfrak{t}_{r,s}||^2 \tag{22}$$

Definition 2 *We say that the definitions (19) - (22) are symbolic if the corresponding parameters $\{\lambda_k\}$ are symbolic.*

Definition 3 *Let $n \geq 1$ be an integer. Consider a system of functions $U_n = \{\exp(i\,\pi\,\mu_k\,x)\}, \mu_k \in \mathbb{C}$, $x\in[-1,1]$, $k = 1,2,\dots,n$, where $\{\mu_k\}$ are arbitrary parameters. Denote by Q_n its linear span ($Q_n = span\,(U_n)$). We call a function $q \in Q_n$ a quasi-polynomial of degree at most n.*

It is easy to see that if $q \in Q_n$, then either $q(x) \equiv 0$ or $q(x) = \sum_k P_{\beta_k}(x)\exp(i\,\pi\,\mu_k\,x)$, where each polynomial $P_{\beta_k}(x) \not\equiv 0$ has exact degree β_k and $\sum_k(1+\beta_k) \le n$ (see **3.3.1**). From the biorthogonality of the system $\{\mathfrak{T}_r(x), 1/2\exp(i\,\pi\,r\,x)\}$, $r \in \mathfrak{D}_{\mathfrak{n}}$, we have (see the proof of Lemma 1) $q(x) = \sum_{r\in\mathfrak{D}_{\mathfrak{n}}} q_r\,\mathfrak{T}_r(x), x \in [-1,1]$, where $\{q_k\}$ are the Fourier coefficients of the function q.

Definition 4 *We denote* $q \in \overset{0}{Q}_n$, *if* $q \in Q_n$ *and*

$$q(x) = \sum_{k\in\mathfrak{D}_{\mathfrak{n}}} P_{\beta_k}(x)\exp(i\,\pi\,\mu_k\,x),$$

where the values of $\{\mu_k\}$ *are not integers.*

The condition $f \in Q_n$ imposes certain restrictions on the total number of non-zero coefficients $\{c_p\}$ and non-zero polynomials $\{P_{\beta_k}(x)\}$ with allowance for their powers $\{\beta_k\}$ (see Definition 3 and the corresponding clarification). We also note that the set $\overset{0}{Q}_n$ is not closed in Q_n.

4.2 A simplest example of the phenomenon

Consider a function $f \in Q_1$, $f(x) \not\equiv 0$ and a simplest partial sum $\mathfrak{S}_1(x) = f_k\exp(i\,\pi\,k\,x)$, $f_k \ne 0$. We have (up to a non-zero constant factor)

$$f_s = \operatorname{sinc}(\pi(s-\mu)),\ \mu \in \mathbb{C},\ s = 0, \pm1, \pm2, \dots$$

.

From (20), we have the following symbolic representation ($\lambda = \lambda_{1,1}$)

$$\mathfrak{t}_{k,s} = \frac{(-1)^{s-k}(k-\lambda)}{(s-\lambda)},\ s = 0, \pm1, \pm2, \dots.$$

Now let the value λ be chosen from the condition

$$f_s - f_k\,\mathfrak{t}_{k,s} = 0,\ s \ne k, \tag{23}$$

for a fixed s.

Suppose we know the values of f_k and f_s but do not know the value of μ. Let's consider the following cases.

1). $\mu \notin \overset{0}{Q}_1$. Then (here $\sin(\pi\,\lambda) \ne 0$)

$$f_s - f_k\,\mathfrak{t}_{k,s} = c_1\left(\frac{(\lambda-\mu)\,(k-s)}{(k-\mu)\,(s-\lambda)\,(\mu-s)}\right) = 0,\ c_1 \ne 0.$$

Then (see (21)) $\lambda = \mu$, and $f(x) = f_k \mathfrak{T}_k(x) = f_k \exp(i\,\pi\,\mu\,x)$. That is, we exactly found the parameter μ (with the function f (see Lemma 1)) only in terms of the two non-zero Fourier coefficients f_k and f_s. Note, that here the set $\{c \exp(i\,\pi\,\mu\,x)\}$, $c = \text{const} \in \mathbb{C}$, $\mu \in \mathbb{C}$, when μ is not integer, **is infinite-dimensional in** $L_2[-1, 1]$.

2). $\mu = k$. Here $f_s = 0$ and $f_k = 1$. It is easy to see that again $\lambda = \mu$ and $f(x) = f_k\,\mathfrak{T}_k(x) = f_k \exp(i\,\pi\,k\,x)$.

3).$\mu \neq k$. Here $f_s \neq 0$ and $f_k = 0$. The equation is $c = 0, c \neq 0$, so there is no solution.

4). If in the previous case we solve Eq.(23) by the least square method, then we obtain $\lambda = 0$ and $f(x) \simeq 0$.

Below we show that the phenomenon noted in **1)** is also observed in the general case, when we use (see **3.3.1**) an A-type algorithm for $n \geq 2$.

4.3 Main theorem

In the previous example, we first used the value of f_k and the symbolic parameter λ. Then an additional coefficient f_s was used, which allowed to find the unknown value $\lambda = \mu$ by solving one linear equation (23). If the function f is not an exponential, then this algorithm could be interpreted as the "separation of the main exponential part" from $f(x)$. The following procedure is based on the realization of such an idea in the general case.

Below we assume that our parameters do not depend on r ($\{\lambda_{r,k}\} = \{\lambda_k\}$, $k \in \mathfrak{D}_n$).

4.3.1 The scheme of the basic algorithm

Consider the following calculation scheme for the realization of an adaptive approximation (21) for a smooth function f.

Let $\mathfrak{S}_n(x)$ be a partial sum of Fourier series (see (19)). We consider the sequence (20) in the symbolic form (see Definition 3) with known coefficients $\{f_k \in \mathfrak{D}_n\}$. Now we choose another set of different integers $\tilde{\mathfrak{D}}_{\mathfrak{n}} = \{\tilde{d}_k\}, k = 1, 2, \ldots, n$, $\mathfrak{D}_n \cap \tilde{\mathfrak{D}}_{\mathfrak{n}} = \emptyset$. To determine the values of parameters $\{\lambda_q\}$, $q \in \mathfrak{D}_{\mathfrak{n}}$, we additionally use Fourier coefficiens $\{f_s\}, s \in \tilde{\mathfrak{D}}_{\mathfrak{n}}$, and solve (similarly to Eq.(23)) the following system of equations

$$f_s - \sum_{r \in \mathfrak{D}_{\mathfrak{n}}} f_r \mathsf{t}_{r,s} = 0,\ s \in \tilde{\mathfrak{D}}_n. \tag{24}$$

Such a choice is natural, since, as a rule, it reduces the approximation error (see (22)). This is our adaptive approach to the choice of the parameters $\{\lambda_q\}$, since they now depend on the Fourier coefficients of the given function f (in contrast to the universal algorithms discussed in **2** above).

If the solution exists, the quasi-polynomials $\{\mathfrak{F}_n\}$ (see (21)) are used to approximate f. According to Definition 1, this scheme relates to the acceleration of the convergence of the following partial sum of a Fourier series

$$\mathfrak{S}_{2n}(x) = \sum_{k\in\mathfrak{D}_\mathfrak{n}\cup\tilde{\mathfrak{D}}_\mathfrak{n}} f_k \exp(i\,\pi\,k\,x),\ x\in[-1,1].$$

Of course, a number of questions arise here. For example, it is unclear how to solve equation (24) (especially for large n) and it is not specified how to act if the solution does not exist or exists, but is not unique. These and other details will be clarified below.

4.3.2 Algorithm $\mathfrak{A}$

Here we propose an implementation of the above scheme.

Let's call Algorithm $\mathfrak{A}$ the realization of the following steps:

Step 1. Reduce each equation of the system (24) to the common denominator and consier the equality to zero of each numerator. Then we have

$$f_s\prod_{q=1}^{n}(s-\lambda_q) = \sum_{r\in\mathfrak{D}_\mathfrak{n}}(-1)^{s-r}f_r\,(\prod_{\substack{p\in\mathfrak{D}_n\\ p\neq r}}\frac{s-p}{r-p}\,)\prod_{q=1}^{n}(r-\lambda_q),\ s\in\tilde{\mathfrak{D}}_n. \qquad (25)$$

Step 2. Using the Vieta's formula decompose the products in Eq.(25) containing parameters from $\{\lambda_k\}$, and denote

$$u_k = (-1)^k\sum_{i_1<i_2<\cdots<i_k}\lambda_{i_1}\lambda_{i_2},\ldots,\lambda_{i_k},\ k=1,2,\ldots,n.$$

It is not difficult to see that now Eq. (25) will take the form of the system of linear equations with respect to the variables $\{u_k\}$.

Step 3. Solve resulting equation by the least square method.

Step 4. Using the soluition, find (again according to the Vieta's formulae) all roots$\{z_k\}$, $k=1,2,\ldots,n$, of the corresponding polynomial $z^n+\sum_{p=1}^{n}u_p\,z^{n-p}$ and put $\{\lambda_k\}=\{z_k\}$.

Step 5. Taking into account the multiplicities of these roots, finally, implement the analog of the Algorithm A (see **3.3.1**) for the sequence (20) when $\{\lambda_{r,q}\}=\{\lambda_q\}$ (see (12)). If the system $\{\mathfrak{F}_n\}$ could not be constructed (for example, if in step **Step 4** a z_k is integer, see (20)), we must return to step **Step 1** and add some new Fourier coefficients to $\mathfrak{D}_n$ and $\tilde{\mathfrak{D}}_\mathfrak{n}$.

Remark 3 *Assume we know a finite set S of Fourier coefficients of the function f, that contains the set $D = \mathfrak{D}_n \bigcup \tilde{\mathfrak{D}}_{\mathfrak{n}}, (D \subset S)$. If $S \setminus D = \emptyset$, then we will apply **Step 3**. Otherwise we choose a set $S_1 \subset (S \setminus D)$, $S_1 \neq \emptyset$, and solve the equation in **Step 2** for $s \in \tilde{\mathfrak{D}}_{\mathfrak{n}} \bigcup S_1$ (instead of $s \in \tilde{\mathfrak{D}}_{\mathfrak{n}}$) by the least square method. In this case f must be approximated by the formula*

$$f(x) \simeq \mathfrak{F}_n(x) \stackrel{def}{=} \sum_{r \in \mathfrak{D}_{\mathfrak{n}}} f_r \mathfrak{T}_r(x) + \sum_{s \in S \setminus \mathfrak{D}_{\mathfrak{n}}} (f_s - \sum_{r \in \mathfrak{D}_{\mathfrak{n}}} f_r \mathfrak{t}_{r,s}) \exp(i\,\pi\, s\, x). \quad (26)$$

This approximation, generally speaking, is more accurate than the approximation from (21).The effectiveness of such an approximations is confirmed by a number of our numerical experiments (see, for example, **5.1.1**, c) and Table 1 below).

Remark 4 *The solution of the system (24) is a set of n parameters $\{\lambda_k\}$, regardless of its numbering. Indeed, a change in the numbering leads only to a permutation of the last product in (20).*

4.3.3 The quasi-polynomial representation

If the parameters $\{\lambda_k\}$ and corresponding multiplicities $\{n_k\}$ are known, then (according to (13)) we can use the representation

$$\mathfrak{t}_{r,s} = (-1)^{s-r} \Big(\prod_{\substack{p \in \mathfrak{D}_{\mathfrak{n}} \\ p \neq r}} \frac{s-p}{r-p}\Big) \prod_{j \in \mathfrak{D}_{\mathfrak{m}}} \left(\frac{r-\lambda_j}{s-\lambda_j}\right)^{n_j}, \; s = 0, \pm 1, \ldots, \quad (27)$$

where $r \in \mathfrak{D}_{\mathfrak{n}}$, $\mathfrak{D}_{\mathfrak{m}} \subset \mathfrak{D}_{\mathfrak{n}}$, $\{n_q\}$ are corresponding positive integers, and $\sum_{j \in \mathfrak{D}_m} n_j = n$, $\lambda_p \neq \lambda_q$ if $p \neq q$.

The following theorem generalizes Theorem 1 for the case $\{\lambda_{r,p}\} = \lambda_p, p \in \mathfrak{D}_m$.

Theorem 3 *Suppose the sequence (27) is given. Then the corresponding functions $\{\mathfrak{T}_r\}$ are quasi-polynomials and have the following explicit form*

$$\mathfrak{T}_r(x) = \sum_{j \in \mathfrak{D}_m} \sum_{k=1}^{n_j} c_{r,j,k}\, \Lambda_{j,k}(x), \;\; r \in \mathfrak{D}_n,\, x \in [-1,1], \quad (28)$$

where (see (16)) the system $\{\Lambda_{r,k}\}$ consists of the following quasi-polynomials

$$\Lambda_{j,k}(x) = \frac{-\pi}{(k-1)!} \frac{d^{k-1}}{d\,\lambda_j^{k-1}} \left(\csc(\pi\lambda_j)\, \exp(i\,\pi\,\lambda_j\, x)\right),$$

and

$$c_{r,j,k}=\frac{(-1)^r\prod_{j\in\mathfrak{D}_m}(r-\lambda_j)^{n_j}}{\prod_{\substack{p\in\mathfrak{D}_{\mathfrak{n}}\\ p\neq r}}(r-p)}\sum_{k=1}^{n_j}\frac{(n_j-1)!}{(n_j-k)!}\frac{d^{n_j-k}}{d\lambda_j^{n_j-k}}\left(\frac{\prod_{\substack{p\in\mathfrak{D}_{\mathfrak{n}}\\ p\neq r}}(\lambda_j-p)}{\prod_{\substack{p\in\mathfrak{D}_{\mathfrak{n}}\\ p\neq j}}(\lambda_j-\lambda_p)^{n_p}}\right)$$

Proof. The function $T_r(s)=\mathfrak{t}_{r,s}$, considered for $s\in\mathbb{C}$, is rational with poles at $s=\lambda_j$ of order n_j, $j\in\mathfrak{D}_m$.

Let $U\subset\mathbb{C}$ be a simply connected open subset containing the points $\{\lambda_j\}$, $j\in\mathfrak{D}_m$, with the positively oriented simple boundary curve $\gamma=\partial U$. We have $T_r(s)=(1/s)$, $s\to\infty$, therefore, according to Cauchy's residue theorem

$$0=\frac{1}{2\,\pi\,i}\int_\gamma\frac{T_r(t)}{t-s}\,dt=T_r(s)+\sum_{j\in\mathfrak{D}_m}\mathrm{Res}_{t=\lambda_j}(\frac{T_r(t)}{t-s}),\ s\in U.$$

Let us show how these residues can be explicitly calculated. For given $r\in\mathfrak{D}_{\mathfrak{n}}$ and $j\in\mathfrak{D}_m$, the problem is reduced (see (27)) to finding the residue at the point $t=\lambda_j$ for the function

$$W_1(t)=\frac{W(t)}{(s-t)(t-\lambda_j)^{n_j}},\ \text{ where }\ W(t)=\frac{\prod_{\substack{p\in\mathfrak{D}_{\mathfrak{n}}\\ p\neq r}}(t-p)}{\prod_{\substack{p\in\mathfrak{D}_{\mathfrak{n}}\\ p\neq j}}(t-\lambda_p)^{n_p}}.$$

From here

$$\mathrm{Res}_{t=\lambda_j}W_1(t)=\frac{d^{n_j-1}}{d\lambda_j^{n_j-1}}(\frac{W(\lambda_j)}{s-\lambda_j})=\sum_{k=1}^{n_j}\frac{(n_j-1)!}{(n_j-k)!\,(s-\lambda_j)^k}\frac{d^{n_j-k}}{d\lambda_j^{n_j-k}}W(\lambda_j).$$

This implies (see (18) and (21)) the formula (29). □

Remark 5 *From Theorem 3 we have*

$$\mathfrak{F}_n(x)=\sum_{j\in\mathfrak{D}_m}\sum_{k=1}^{n_j}a_{j,k}\,\Lambda_{j,k}(x),\tag{29}$$

where $a_{j,k}=\sum_{r\in\mathfrak{D}_n}f_r\,c_{r,j,k}$. *It follows that approximation* $f\simeq\mathfrak{F}_n$ *can be carried out directly by this formula, avoiding the use of the system* $\{\mathfrak{T}_r\}$.

4.3.4 The phenomenon

The following theorem is the main result of this paper.

Theorem 4 {***The phenomenon of the over-convergence***}. *Let* $f \in \overset{0}{Q}_n$ *and the sets* $\mathfrak{D}_n$, $\tilde{\mathfrak{D}}_n$ *and the Fourier coefficients* $\{f_s\}$, $s \in \mathfrak{D}_n \cup \tilde{\mathfrak{D}}_n$, *of the function* f *be given. Then the approximation using Algorithm* $\mathfrak{A}$ *is exact (that is* $f(x) \equiv \mathfrak{F}_n(x)$*).*

Proof. For $f(x) \equiv 0$ this is obvious. Suppose that $f(x) \not\equiv 0$. Then $f(x) = \sum_k P_{m_k}(x)\, \exp(i\,\pi\,\mu_k\,x)$, where each $P_{m_k}(x) \not\equiv 0$ and $\sum_k (1 + m_k) \leq n$.

Turning to the application of Algorithm $\mathfrak{A}$, we emphasize that we know only the values $\{f_s\}$ for $s \in \mathfrak{D}_n \cup \tilde{\mathfrak{D}}_n$ but do not know the values of the parameters $\{\mu_s\}$. The definition of these parameters is our main goal, but first let's analyze the situation using them.

It is not difficult to see that the Fourier coefficients $\{f_s\}$ of the function $f(x)$ have the form

$$f_s = \frac{(-1)^s P_{m-1}(s)}{\prod_{k\in\mathfrak{D}_m}(s-\mu_k)},\ 1 \leq m \leq n,\ s = 0, \pm 1, \pm 2, \ldots \tag{30}$$

where $P_{m-1}(s)$ is a polynomial of degree $m-1$. Here we do not specify the multiplicities of the parameters $\{\mu_k\}$. Without loss of generality, we assume that the fraction on the right hand of (30) is uncancellable.

First we consider the case $m = n$.

Note that this corresponds to the conditions $\sum_{k=1}^{n_j} |a_{j,k}| \neq 0$, $j \in \mathfrak{D}_m$, in the representation (29). From the system (20), which corresponds (instead of $\{\lambda_k\}$) to the parameters $\{\mu_k\}$, we have (see Theorem 1) $f(x) = \sum_{r\in\mathfrak{D}_n} f_r \mathfrak{T}_r(x), x \in [-1, 1]$. In terms of Fourier coefficients, this means that $f_s = \sum_{r\in\mathfrak{D}_n} f_r\, \mathfrak{t}_{r,s}$, $s = 0, \pm 1, \ldots$. In more detail

$$\frac{P_{n-1}(s)}{\prod_{k\in\mathfrak{D}_n}(s-\mu_k)} - \sum_{r\in\mathfrak{D}_n} f_r(-1)^{-r} \Big(\prod_{\substack{p\in\mathfrak{D}_n\\ p\neq r}} \frac{s-p}{r-p}\Big) \prod_{q\in\mathfrak{D}_n} \frac{r-\mu_q}{s-\mu_q} = 0. \tag{31}$$

It is obvious that this equality is true for any $s \in \mathbb{C}, s \notin \{\mu_k\}$. With this in mind, Eq.(24) has the form (see (20))

$$A(s) \overset{def}{=} \frac{P_{n-1}(s)}{\prod_{k=1}^{n}(s-\mu_k)} - \sum_{r\in\mathfrak{D}_n} f_r(-1)^{-r} \Big(\prod_{\substack{p\in\mathfrak{D}_n\\ p\neq r}} \frac{s-p}{r-p}\Big) \prod_{q\in\mathfrak{D}_n} \frac{r-\lambda_q}{s-\lambda_q} = 0,\ s \in \tilde{\mathfrak{D}}_n. \tag{32}$$

Comparing (31) with (30), we note that Eq.(32) has at least a solution $\{\lambda_k\} = \{\mu_k\}$ (see Remark 4).

To show that this solution is unique, suppose that there is an another solution $\{\lambda_k\}$. If $s \in \mathfrak{D}_n$, then the function A is equal to zero (see Lemma 1). On the other hand, $A(s) = 0$ for $s \in \tilde{\mathfrak{D}}_n$. As we see from (31), the numerator of the rational function $A(s)$ is a polynomial of degree at most $2n-1$ which has $2n$ zeros. So $A(s) \equiv 0, s \in \mathbb{C}$.

From (31) we now have

$$P_{n-1}(s) = Q_{n-1}(s) \prod_{p \in S} \frac{s-\mu_p}{s-\lambda_p},\ s \in \mathbb{C},$$

where $Q_{n-1}(s)$ is a polynomial, $S \subset \mathfrak{D}_{\mathfrak{n}}$, $S \not\equiv \emptyset$ and $\lambda_p \neq \mu_q$ for $p \neq q$, $p, q \in S$. Then $P_{n-1}(\mu_p) = 0$, $p \in S$. This contradicts the condition of the irreducibility of the fraction in (30).

Now let $m < n$ in (30).

Then choosing $\{\lambda_j\} = \{\mu_j\}$, $j \in \mathfrak{D}_m$, regardless of the values of parameters $\{\lambda_j\}$, $j \in \mathfrak{D}_n \setminus \mathfrak{D}_m$, we get the same representation (29). Indeed, it is not difficult to see that the values $\lambda_j \in \mathfrak{D}_n \setminus \mathfrak{D}_m$ correspond to the conditions $a_{j,k} = 0$, $k = 1, \ldots, n_j$, in the representation (29), which we use in Algorithm $\mathfrak{A}$. Taking this into account and repeating the above arguments when $m = n$, we conclude that the equation in **Step 3** actually (with accuracy up to values of $\lambda_j \in \mathfrak{D}_n \setminus \mathfrak{D}_m$, that do not affect the expansion (29)) has a unique solution $\{\lambda_j\} = \{\mu_j\}$, $j \in \mathfrak{D}_m$.

Thus, we obtain a constructive proof of the theorem. □

5 Numerical results

The results below are obtained using the symbolic and computational capabilities of *Wolfram Mathematica* computer system (see [19]). When calculating the Fourier coefficients and the L_2 -approximation errors in the one-dimensional case, numerical integrations are performed with increased accuracy. Uniform errors in our experiments were detected graphically.

As an acceleration measure (acceleration rate) for given n, the ratio $ef/e\theta$ is taken, where ef is the error (in L_∞ or L_2) when applying the partial sum of Fourier series, and $e\theta$ is the corresponding error when applying formulas (7), (10) or (21). It is clear that the value of $ef/e\theta > 1$ corresponds to the "acceleration" and $ef/e\theta < 1$ to the "deceleration".

5.1 Comparable algorithms

Given the values of $n = 4, 8$ and 12 in the numerical experiments, we compare the following algorithms (see below Tables 1 and 2).

5.1.1

In the one-dimensional case:

a) Universal Algorithm A (see **3.3.1**). Here we chose

$$\lambda_k = \frac{2\,\arctan\left(\frac{n}{9}+\frac{1}{5}\right)}{\pi}\,k,\quad k=0,\pm1,\ldots,\pm n. \tag{33}$$

As we see, here the real parameters $\{\lambda_k\}$ depend both on n and k, $\lambda_{-k} = -\lambda_k$, $\lambda_k \neq \lambda_p$ for $k \neq p$, $\lambda_n \uparrow n$, when $n \to +\infty$. The approximation $f(x) \simeq F_n(x)$ is made by a linear combination of functions $\{\exp(i\,\pi\,\lambda_k\,x)\}$, $k = 0,\pm1,\ldots,\pm n$ (see Theorem 1) .

b) Universal Algorithm B_0. This is the algorithm B (see **3.3.2**), provided $\lambda_r = 0, \forall r$. For a given n, the approximation $F_n(x) \simeq f(x)$ consists of polynomials of degree at most $2n$.

c) Adaptive Algorithm $\mathfrak{A}_m$, for $m = 2, 4, n/2, n$. In this algorithm, depending on the integer parameter m, $1 \leq m \leq n$, the procedures from **4.3.2** must be changed as follows:

c1) To use the same $(2n+1)$ Fourier coefficients as in the algorithms A and B_0, it is assumed that $\mathfrak{D}_{\mathfrak{m}} = \{k\}$, $k = n-m,\ldots,n-1$, $S = S_1 = \{p\}\bigcup\{n\}$, $p = -n, -n+1,\ldots,-n+m-1$ (see Remark 3).

c2) As the result, in **Step 3** we solve Eq. (24) for additional $s = n$, which consists of $m+1$ equations for m unknown values $\{u_k\}$. It is solved by the least square method and in **Step 4** we determine m parameters $\{\lambda_k\}$.

The above algorithms are implemented in *Wolfram Mathematica* system with an accuracy of 10^{-38}.

5.1.2

In the two-dimensional case $(x,y) \in [-1,1]\times[-1,1]$ we have used (see above **3.2**) only the following universal algorithm (see below Table 2):

Algorithm AA: in definition (9) $n_1 = n_2 = n$, $x_1 = x$, $x_2 = y$ and the functions $\Theta_{r_1}(x), \Theta_{r_2}(y)$ corresponds to parameters (32).

Because of too slow work, this algorithm is implemented in the *Wolfram Mathematica* system with an accuracy of 10^{-7}.

5.2 Test functions

In the one-dimensional case, numerical experiments are presented for the function

$$g_1(x) = (1-i)\,I_{\frac{\pi}{4}}\left((2-4\,i)x + \left(\frac{1}{4}+\frac{5\,i}{2}\right)\right),\ x \in [-1,1], \tag{34}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind.

In the two-dimensional case, we use the function

$$g_2(x,y) = \frac{i \sin\left(-\left(5-\frac{i}{5}\right)x + 3\,y^2 + 1 - i\right)}{x^2 + i\,y + (1-3\,i)} \tag{35}$$

Errors norm $\rightarrow$	L_∞			L_2		
Values of $n \rightarrow$	4	8	12	4	8	12
Fourier Series	2 e+0	2 e+0	2 e+0	5.9 e-1	4.3 e-1	3.5 e-1
Algorithm A	3.8 e-2	2 e-3	2 e-4	7.8 e-3	2.7 e-4	2.2 e-5
Acceleration rate	5.4 e+1	1 e+3	9.6 e+3	7.6 e+1	1.6 e+3	1.6 e+4
Algorithm B_0	3.6 e-2	8.4 e-3	3.9 e-3	6.5 e-3	1.1 e-3	3.7 e-4
Acceleration rate	5.6 e+1	2.3 e+2	5 e+2	9.2 e+1	4 e+2	9.4 e+2
Algorithm $\mathfrak{A}_2$	1.7 e-2	7.7 e-4	4.1 e-5	3.3 e-3	1.1 e-4	4 e-6
Acceleration rate	1.2 e+2	2.6 e+3	4.8 e+4	1.8 e+2	4 e+3	8.7 e+4
Algorithm $\mathfrak{A}_4$	1.1 e-2	3.5 e-4	2.7 e-5	2.7 e-3	4.4 e-5	2.8 e-6
Acceleration rate	1.8 e+2	5.6 e+3	7.2 e+4	2.2 e+2	9.7 e+3	1.2 e+5
Algorithm $\mathfrak{A}_{n/2}$	1.7 e-2	3.5 e-4	1.8 e-6	3.3 e-3	4.4 e-5	2.5 e-7
Acceleration rate	1.2 e+2	5.6 e+3	1.1 e+6	1.8 e+2	9.7 e+3	1.4 e+6
Algorithm $\mathfrak{A}_n$	1.1 e-2	2.8 e-5	3.8 e-8	3.9 e-3	5.6 e-6	1.9 e-8
Acceleration rate	1.8 e+2	7 e+4	5.3 e+7	1.5 e+2	7.6 e+4	1.8 e+7

Table 1. *Errors and corresponding accelerations when the function* $g_1(x)$ *is approximated.*

Errors norm $\rightarrow$	L_∞			L_2		
Values of $n \rightarrow$	4	8	12	4	8	12
Fourier Series	2 e-1	2 e-1	2 e-1	7.5 e-2	5.1 e-2	4.1 e-2
Algorithm AA	6 e-3	3 e-5	8 e-6	1.7 e-3	5.2 e-6	8.2 e-7
Acceleration rate	3.3 e+1	6.7 e+3	2.4 e+5	4.4 e+1	9.8 e+3	5 e+4

Table 2. *Errors and corresponding accelerations when the function* $g_2(x,y)$ *is approximated.*

5.3 Discussion of numerical results

The results presented here are typical for our numerical experiments. Let us dwell on their main features.

First of all, we note (see Table 1) the absolute advantage of adaptive algorithms over universal ones. This advantage is manifested even when using only two symbolic parameters λ_1 and λ_2 for the cases $n = 8$ and 12

(see Algorithm $\mathfrak{A}_2$). The adaptive approximation of a sufficiently smooth function f retains a high accuracy of differentiation. We also note that in practice the case with several roots in **Stage 4** is extremely rare due to the inevitable accumulation of errors.

As for universal algorithms, they can also be considered (in contrast to the traditional partial sums of the Fourier series) to be practically effective. We emphasize that the choice of parameters in Algorithm A by formula (33) was artificial and can not be considered optimal in any sense. In the same time Algorithm A has a noticeable advantage over Algorithm B_0.

In the two-dimensional case (see Table 2), Algorithm AA is also quite effective. According to our experiments, the use of algorithms of the type B_0 in multi-dimensional cases seems to be impractical due to the high level of error accumulation at relatively small values of (n, m) (see **3.2**).

6 Conclusion

The above results show that the formulas (3) and (20) underlying the explicit construction of the biorthogonal system $\{\Theta_r(x), 1/2\exp(i\,\pi\,r\,x)\}$ (see above **3.1** and **4.1**), opens new possibilities in the problem of acceleration of convergence of partial sums of Fourier series.

At the same time, we note that the obtained and verified algorithms do not relate to the case of the dependence of the parameters $\{\lambda_{r,p}\}$ on both r and p (see (3)). The phenomenon of over-convergence is still undetected either in this or in the multidimensional case due to the lack of an analog of step **3** in the implementation **4.3.2**.

We do not touch upon the questions of asymptotic convergence (if $n \to \infty$) for universal algorithms. In the one-dimensional case, the corresponding estimates in terms of the derivatives of the function f in some cases can be obtained by methods used, for example, in [13, 16].

In the multidimensional case, the derivation of asymptotic estimates for universal algorithms (see **3.2** and **5.1.2**) looks problematic. Even more problematic is this question for the above-mentioned adaptive algorithms (e.g. for Algorithm $\mathfrak{A}$). It is not yet clear how to estimate their errors. The only information is given by the formula (see above **4.3.1** and **4.3.2**)

$$||\mathfrak{R}_n||^2 = \sum_{s \notin \mathfrak{D}_n \cup \tilde{\mathfrak{D}}_n} ||f_s - \sum_{r \in \mathfrak{D}_n} f_r \mathfrak{t}_{r,s}||^2,$$

instead of the representation (22) in the case of the corresponding universal algorithm. However, this does not reflect their real enormous practical advantages associated with approximation of function f by the infinite-dimensional set $\overset{0}{Q}_n$.

We note also the special efficiency of the algorithms $\mathfrak{A}_m$ (see **5.1.1**) in the problem of detecting hidden periodicities of a function f, using only a finite set of its Fourier coefficients (for example, when f is an almost periodic function).

As for the phenomenon of the over-convergence itself, Theorem 4 can be considered as a basis for the further construction of corresponding adaptive algorithms for "accelerating the convergence" of the classical Fourier apparatus in the broad sense (sine and cosine series and interpolations, signal and image processing tools etc.).

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Anry Nersessian
Institute of mathematics,
of National Academy of Sciences of Armenia
Bagramian ave. 24B, 0019 Yerevan, Armenia.
nerses@instmath.sci.am

Please, cite to this paper as published in
Armen. J. Math., V. **10**, N. 9(2018), pp. 1–22