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Quasigroups, Units and Belousov's Problem # 18

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Abstract. We investigate {left, right, middle} units of quasigroups and families of identities which might imply their existence. A prominent role is played by the newly introduced notion of derivative operation, generalizing Belousov's notions of left/right derivative operations for quasigroups. Partial solutions of the Belousov's Problem # 18 and its generalizations are obtained. Several related problems remain open.

Key Words: Quasigroup, Loop, Unit, Derivative operation, Isotopy
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Introduction

One of the 20 problems V.D. Belousov posed in his book [3] is the following:

Problem 1 (Belousov's Problem #18) *How to recognize identities which force quasigroups satisfying them to be loops?*

An example of such an identity is the identity of associativity. Even some weak forms of associativity give us solutions of Problem 1, as suggested by the following result [13]:

Theorem 1 *For any quasigroup Q the following properties are equivalent:*

1. *Q is a loop;*
2. *In the quasigroup Q the following equalities are valid:*
   $f_x f_x = f_x\,,\ \ e_x e_x = e\,,\ \ e_x \cdot f_y f_y = e_x f_y \cdot f_y$
   for all $x, y \in Q$, where $e_x (f_x)$ is right (left) local unit for x;
3. *In the quasigroup Q the following equalities are valid:*
   $f_x f_x = f_x\,,\ \ e_x e_x = e\,,\ \ f_x \cdot e_y e_y = f_x e_y \cdot e_y$
   for all $x, y \in Q$, where $e_x (f_x)$ is right (left) local unit for x.

In general, we only have partial answers for Problem 1. If in a quasigroup one of the middle Moufang identities holds, then this quasigroup is a loop [16]. If in a quasigroup any of the Moufang identities holds, then this quasigroup is a Moufang loop [9, 16]. Quasigroups with identities from the Fenyves' list [6] are studied in several articles, in particular [10] and [15].

In [7] N. C. Fiala investigated quasigroup identities related to Belousov's Problem # 18. He obtained the following result: There are exactly 35 quasigroup identities (in the language with the product operation only) with at most six variable occurrences which imply that the quasigroup is necessarily a non-trivial loop or group. Additional information in this direction is given in [8].

We formulate a slight generalization of Belousov's Problem:

Problem 2 *How to recognize identities which force quasigroups satisfying them to have {left, right, middle} unit? (See page 5)*

All of the above examples are also solutions of some versions of Problem 2, but there are new solutions as well. The classic example of left (right) Bol quasigroup, which has right (left) unit, is probably the best known. Several examples can be found in [15] and some are shown in Theorem 4 (page 7) which generalizes the result by J. D. H. Smith [17, Proposition 1.3].

The rest of the paper is dedicated to the construction of the family of identities which gives a partial solution of Problem 2, using so called *derivatives.*

For the most part this paper is about quasigroups. However, there are a few minor and one major exceptions to this. The minor ones are in Sections 1 and 2, where we occasionally consider algebras more general than quasigroups. The major exception concerns derivative operations, where we define derivatives for groupoids (and homotopies), with the consequence that we have new and interesting existence and uniqueness problems for them. We give a few easy results, leaving harder problems for another occasion. Here we concentrate on quasigroups as an appropriate and more useful case for our present purpose.

1 Preliminaries

Quasigroups may be defined in several ways. We give two definitions which are the most common. One is to consider them as special groupoids $(Q; A)$ in which linear equations $A(x, a) = b$ and $A(a, y) = b$ are uniquely solvable for x, y (for any given a, b). The other way is to define quasigroups as algebras with three binary operations $A, A^{-1}, {}^{-1}A$.

Definition 1 *A* quasigroup *is an algebra* $(Q; A, A^{-1}, {}^{-1}A)$ *which satisfies the following identities:*

$$\begin{array}{rclcrcl} A^{-1}(x, Ax, y)) & = & y & \quad & {}^{-1}A(A(x, y), y) & = & x \\ A(x, A^{-1}(x, y)) & = & y & & A({}^{-1}A(x, y), y) & = & x \end{array}$$

The later type is sometimes called primitive, equational or equasigroup, but because there is a bijection between the two kinds of quasigroups, we shall call them just quasigroups. However, it is important to note that the two kinds of quasigroups have different properties. For example, homomorphic images of quasigroups as groupoids need not be quasigroups themselves, while with 3–operation quasigroups this is always the case. These and other basic facts about quasigroups can be found in [3, 5, 14].

Definition 2 *Parastrophes* A^σ $(\sigma \in S_3)$ *of a binary operation* A *are defined by*

$$A^\sigma(x_{\sigma(1)}, x_{\sigma(2)}) = x_{\sigma(3)} \text{ iff } A(x_1, x_2) = x_3$$

By default: $A^\varepsilon = A$. Usually, the operations $A^{(12)}, A^{(13)}, A^{(23)}$ are denoted by $A^*, {}^{-1}A$ and A^{-1} respectively, as in the above quasigroup axioms.

It is easy to see that any groupoid $(Q; A)$ has the parastrophe $(Q; A^*)$, so called *dual* of $(Q; A)$. The other parastrophes of an arbitrary operation need not exist. For the existence of $A^{(13)}$ $(A^{(23)})$, the operation A needs to be right (respectively left) quasigroup, and then $A^{(13)}$ $(A^{(23)})$ is also a right (left) quasigroup. Consequently, every parastrophe of a quasigroup is also a quasigroup.

When we use multiplicative notation, i.e. when $\cdot$ is a binary infix operation symbol, we also use the following customary symbols for it's parastrophes:

$$* = \cdot^{(12)}, \qquad / = \cdot^{(13)}, \qquad \backslash = \cdot^{(23)}, \qquad /\!\!/ = \cdot^{(123)}, \qquad \backslash\!\!\backslash = \cdot^{(132)}$$

or explicitly:

$$\begin{array}{cccccc} x \cdot y = z & \text{iff} & x \backslash z = y & \text{iff} & z/y = x & \text{iff} \\ y * x = z & \text{iff} & z \backslash\!\!\backslash x = y & \text{iff} & y /\!\!/ z = x & \end{array}$$

In this case the quasigroup axioms take the more familiar form:

$$\begin{array}{rclcrcl} x\backslash xy & = & y, & \quad & xy/y & = & x, \\ x(x\backslash y) & = & y, & & (x/y)y & = & x. \end{array}$$

It is less well known, although equally obvious, that the dual algebra $(Q; *, /\!\!/, \backslash\!\!\backslash)$ of the quasigroup $(Q; \cdot, \backslash, /)$ is also a quasigroup.

Definition 3 *Let* $(Q;\cdot)$ *be a groupoid, and let* a *be a fixed element in* Q. Left (right) translation L_a (R_a) *is defined by* $L_a x = a \cdot x$ $(R_a x = x \cdot a)$ *for all* $x \in Q$.

For quasigroups it is possible to define a third kind of translation, namely, middle translations: P_a is a *middle translation* of a quasigroup $(Q;\cdot)$ iff $x \cdot P_a x = a$ for all $x \in Q$ (see [4]).

It follows that:

$$P_a x = x \backslash a, \qquad\qquad P_a^{-1} x = a/x.$$

The relationship between (inverses of) translations of A and of parastrophes of A is shown in the Table 1.

	ε	(12)	(13)	(23)	(123)	(132)
R	R	L	R^{-1}	P	P^{-1}	L^{-1}
L	L	R	P^{-1}	L^{-1}	R^{-1}	P
P	P	P^{-1}	L^{-1}	R	L	R^{-1}
R^{-1}	R^{-1}	L^{-1}	R	P^{-1}	P	L
L^{-1}	L^{-1}	R^{-1}	P	L	R	P^{-1}
P^{-1}	P^{-1}	P	L	R^{-1}	L^{-1}	R

Table 1: Translations of quasigroup parastrophes

Consequently:

Theorem 2 *Every (inverse of)* $\{$*left, right, middle*$\}$ *translation of* A *is the left translation of some parastrophe of* A.

Using a metavariable T for translations (including ε, which may, but need not be actual translation), we can reformulate Theorem 2; thus:
For any translation T_a *there is a parastrophe* $\circ$ *of* $\cdot$ *such that* $T_a^{\pm 1} = L_a^{\circ}$.
Therefore, when we write 'translation' this includes inverses of translations as well. Also note that we abbreviated translations T_a in Table 1 to T as $a \in Q$ is fixed in this case.

Definition 4 *Let groupoids* $(Q;\circ)$ *and* $(Q;\cdot)$ *be given. A triple* (α, β, γ) *of mappings* $\alpha, \beta, \gamma : Q \longrightarrow Q$ *is a* homotopy *of* $(Q;\circ)$ *into* $(Q;\cdot)$ *if* $\gamma(x \circ y) = \alpha x \cdot \beta y$. *We say that* $(Q;\circ)$ *is a* homotopic pre–image *of* $(Q;\cdot)$.

A homotopy (α, β, γ) *is an* isotopy *if* α, β, γ *are all bijections i.e. permutations of* Q.

Definition 5 *A homotopy* (α, β, γ) *is* derivative *if one of* α, β, γ *is the identity map* ε. *A derivative homotopy is* $\{$left, right, middle$\}$ *if* $\{\alpha = \varepsilon, \beta = \varepsilon, \gamma = \varepsilon\}$.

If the remaining two maps are also bijections, we say that (α, β, γ) *is* (left, right, middle) derivative isotopy. *If* γ *is a bijection, we write* $(Q; \circ) = (Q; \cdot)(\alpha, \beta, \gamma)$ *or somewhat shorter:* $\circ = \cdot(\alpha, \beta, \gamma)$.

2 Units

This Section serves as a reminder of properties of quasigroup units.

Definition 6

- *An element* $i \in Q$ *is an* idempotent *of* $(Q; \cdot)$ *iff* $i \cdot i = i$.
- *An element* $e_\ell \in Q$ $(e_r \in Q)$ *is a* left (right) unit *of* $(Q; \cdot)$ *iff* $e_\ell \cdot x = x$ $(x \cdot e_r = x)$ *for all* $x \in Q$.
- *An element* $e \in Q$ *is a* (two–sided) unit *of* $(Q; \cdot)$ *iff it is both left and right unit.*
- *An element* $e_m \in Q$ *is a* middle unit *of* $(Q; \cdot)$ *iff* $x \cdot x = e_m$ *for all* $x \in Q$.

Theorem 3 *A {left, right, middle} unit is the unique idempotent in a quasigroup. On the other hand, a quasigroup need not have idempotents or it may have several of them. But, even if it has a unique idempotent, it need not be a unit – left, right or middle.*

Proof. The first statement is well known (a two-sided or middle unit is necessarily unique, even in groupoid case). The next example confirms the second statement. □

Example 1 *The quasigroup given by the Cayley table:*

$\circ$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	3	2	1	0
3	2	3	0	1

has the unique idempotent 0 *which is neither left, right nor middle unit.*

Definition 7 *A quasigroup* $(Q; \cdot)$ *is:*

- *A* left (right) loop *if* $(Q; \cdot)$ *has a left (right) unit.*
- *An* unipotent quasigroup *if* $(Q; \cdot)$ *has a middle unit.*
- *A* loop *if it has both left and right units.*

- *An* unipotent left (right) loop *if it has left (right) and middle unit.*
- *An* unipotent loop *if it has left, right and middle unit.*

Lemma 1 *A quasigroup $(Q;\cdot,\backslash,/)$ is a left (right) loop iff $x/x = y/y$ $(x\backslash x = y\backslash y)$ for all $x, y \in Q$.*

A quasigroup $(Q,\cdot,\backslash,/)$ is a loop iff $x/x = y\backslash y$ for all $x, y \in Q$.

A quasigroup $(Q,\cdot,\backslash,/)$ is unipotent iff $xx = yy$ for all $x, y \in Q$.

A quasigroup $(Q,\cdot,\backslash,/)$ is unipotent left (right) loop iff $x/x = yy$ $(x\backslash x = yy)$ for all $x, y \in Q$.

A quasigroup $(Q,\cdot,\backslash,/)$ is unipotent loop iff $x\backslash x = y/y = zz$ for all $x, y, z \in Q$.

Unit	Symbol	Identities (1)	Formulas (3)
none	(Q)	$x = x$	$x = x$
idempotent	(i)	$ee = e$	$\exists x(xx = x)$
left	(eQ)	$ex = x$	$x/x = y/y$
right	(Qe)	$xe = x$	$x\backslash x = y\backslash y$
middle	(U)	$xx = e$	$xx = yy$
ℓ + r	(Q1)	$ex = x, xe = x$	$x/x = y\backslash y$
ℓ + m	(eU)	$ex = x, xx = e$	$x/x = yy$
r + m	(Ue)	$xe = x, xx = e$	$x\backslash x = yy$
ℓ + r + m	(U1)	$ex = x, xe = x, xx = e$	$x/x = y\backslash y = zz$

Table 2: Units in quasigroups

Identities which define various units and their equivalents from Lemma 1 are collected in Table 2 for easy reference. Note that we are implicitly dealing with four different types of algebras: 1–operation quasigroups, 1–operation quasigroups with a constant, 3–operation quasigroups, and 3–operation quasigroups with a constant. All lattices of classes of quasigroups, defined above in one of the four languages mentioned, are isomorphic to the *generic lattice*, given in Figure 1.

From Lemma 1 it follows that an operation $\cdot$ has a unit iff $\backslash$ and $/$ are both unipotent i.e. they have the (common) middle unit. Similar connections between different kinds of units in various parastrophes of a quasigroup $(Q;\cdot)$ are given in Table 3.

We see that every type of quasigroup from Lemma 1 may be defined by a single identity. The only exceptions are unipotent loops which require two identities. This suggests the following:

Problem 3 *Is there a single identity (in the language $\{\cdot,\backslash,/\}$ of quasigroups) which defines unipotent loops among them?*

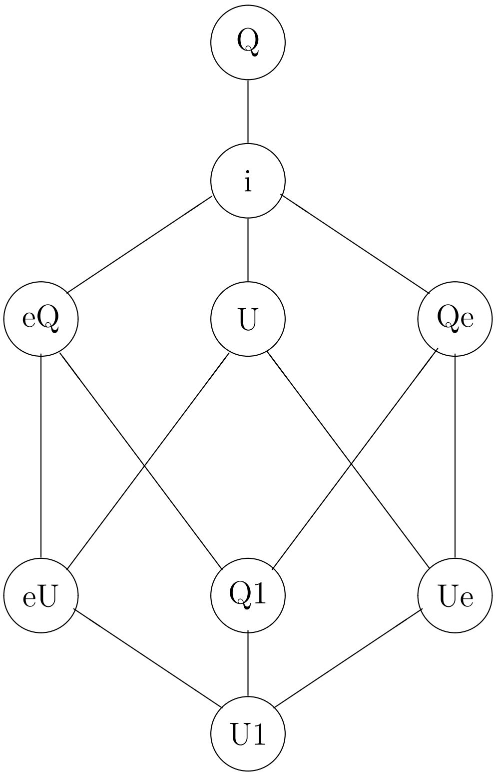

Figure 1: Generic lattice of classes of quasigroups

* * *

The following Theorem generalizes [17, Proposition 1.3], giving a type of result we are after – a family of identities which forces the existence of various units, giving partial solutions to Problem 2.

Theorem 4 *A nonempty quasigroup* $(Q, \cdot, \backslash, /)$ *is a loop iff any of the conditions hold:*

$$x(y/y)\cdot z = x\cdot(y/y)z \quad (1)$$

$$x(y\backslash y)\cdot z = x\cdot(y\backslash y)z. \quad (2)$$

A nonempty quasigroup $(Q, \cdot, \backslash, /)$ *is an unipotent left loop iff any of the conditions hold:*

$$(x\backslash yy)\backslash z = x\backslash(yy\backslash z) \quad (3)$$

$$(x\backslash(y/y))\backslash z = x\backslash((y/y)\backslash z). \quad (4)$$

	ε	(12)	(13)	(23)	(123)	(132)
i	i	i	i	i	i	i
e_ℓ	e_ℓ	e_r	e_m	e_ℓ	e_r	e_m
e_r	e_r	e_ℓ	e_r	e_m	e_m	e_ℓ
e_m	e_m	e_m	e_ℓ	e_r	e_ℓ	e_r

Table 3: Units of parastrophic quasigroups.

A nonempty quasigroup $(Q, \cdot, \backslash, /)$ *is an unipotent right loop iff any of the conditions hold:*

$$(x/yy)/z = x/(yy/z) \tag{5}$$
$$(x/(y\backslash y))/z = x/((y\backslash y)/z. \tag{6}$$

Proof. The proofs of (2)–(6) are analogous to the original one for (1), relying on the internal symmetry of quasigroups. As an example, we prove just (5).

(a) If $(Q; \cdot, \backslash, /)$ is unipotent right loop with the middle unit e, then $xe = x$ and $yy = e$; therefore, $x/e = x$ and $e/y = y$. Replacing in (5), we get $(x/yy)/z = (x/e)/z = x/z = x/(e/z) = x/(yy/z)$, which means that (5) is identically true.

(b) Assume (5) and replace z by y. We get $(x/yy)/y = x/(yy/y) = x/y$. Cancellation from the right gives us $x/yy = x$ i.e. $x = x \cdot yy$ and finally $x\backslash x = yy$, which, by Lemma 1, forces quasigroup to be a unipotent right loop. □

3 Derivatives

We start with Belousov's derivatives – the right derivative $(A_a\ )$ and the left derivative $({}_aA)$, which he introduced in [2] and used in the investigation of G-loops in [1, 3, 5]. We end up with the general notion of derivative $D = A(\alpha, \beta, \gamma)$. Somewhere in between is the notion of the inner derivative $D_a = A(T_1, T_2, T)$ which depends on two translations by the given $a \in Q$. We discuss the relationship between various types of derivatives and, in the case of (general) derivatives, their existence and uniqueness.

3.1 Quasigroup derivatives – the classical case

It is clear that the identity of associativity is not true in all quasigroups. But we can replace it by the following equality which is true in all quasigroups

$(Q;A)$:

$$A(A(a,x),y) = A(a, A_a(x,y)), \tag{7}$$

where $a,x,y \in Q$, and A_a is some binary operation which depends on the element a.

Definition 8 *The operation A_a is called* Belousov's right derivative operation *of A relative to the element a.*

Obviously, $A_a(x,y) = L_a^{-1}A(L_a x, y)$ i.e. $A_a = A(L, \varepsilon, L)$. Dually,

Definition 9 *The operation ${}_aA$, defined by the equation*

$$A(x, A(y,a)) = A({}_aA(x,y), a) \tag{8}$$

is Belousov's left derivative operation *of A relative to the element a.*

Consequently, ${}_aA(x,y) = R_a^{-1}A(x, R_a y)$ i.e. ${}_aA = A(\varepsilon, R, R)$.

Lemma 2 ([5]) *1. Any right derivative $(Q;A_a)$ of a quasigroup $(Q;A)$ has the left identity element, i.e., $(Q;A_a)$ is a left loop.*

2. Any left derivative $(Q;{}_aA)$ of a quasigroup $(Q;A)$ has the right identity element, i.e., $(Q;{}_aA)$ is a right loop.

Proof. Case 1. As it was noticed above, a right derivative $(Q;A_a)$ of a quasigroup $(Q;A)$ is also a quasigroup. In the equality (7) we put $x = e$, where e is the right local unit for a (i.e. $A(a,e) = a$). We obtain $A(A(a,e),y) = A(a, A_a(e,y))$, $A(a,y) = A(a, A_a(e,y))$, and $y = A_a(e,y)$ for all $y \in Q$. So $(Q;A_a)$ is a left loop with the left unit e.

Case 2. Follows by duality. $\square$

From Lemma 2 it follows that any quasigroup has left (right) derivatives, and all these derivatives are right (left) loops.

Theorem 5 *For every quasigroup A, ${}_aA = ((A^*)_a)^*$.*

Proof. From the definition of the left derivative ${}_aA$, we get

$$A^*(A^*(a,y),x) = A^*(a, ({}_aA)^*(y,x)),$$

i.e. $({}_aA)^* = (A^*)_a$. Obviously, ${}_aA = ((A^*)_a)^*$. $\square$

Inspired by Belousov's derivatives, we define the middle derivative operation:

Definition 10 *The operation $\underset{a}{A}$, defined by*

$$\underset{a}{A}(x,y) = P_a x \cdot P_a y \tag{9}$$

is called Belousov's middle derivative operation *of A relative to the element a.*

Obviously, $\underset{a}{A} = A(P,P,\varepsilon)$. Belousov's middle derivative need not have any of the units. See below.

3.2 Quasigroup derivatives – the general case

Definition 11 *An operation D on Q is called a* derivative operation *of A if $(Q;D)$ is a homotopic pre–image of $(Q;A)$ i.e. if*

$$\gamma D(x,y) = A(\alpha x, \beta y), \tag{10}$$

where (α,β,γ) is a derivative homotopy.

Operation D is called $\{$left, right, middle$\}$ derivative iff $\{\alpha = \varepsilon, \beta = \varepsilon, \gamma = \varepsilon\}$.

If (and *only* if) γ is a bijection, we write: $D = A(\alpha,\beta,\gamma)$.

Definition 12 *An operation D_a on Q is called an* inner derivative operation *of A (*relative to the element $a \in Q$*) if D_a is a derivative operation of A, where one of the maps α,β,γ is ε while the other two are some translations by a.*

As above, if γ is a bijection, we write $D_a = A(\alpha,\beta,\gamma)$. But when we want to emphasize that two of α,β,γ are translations of a (and assuming that T_a is a bijection), we write $D_a = A(T'_a, T''_a, T_a)$.

Example 2 *Let $(Q;A)$ be a quasigroup. Belousov's right derivative A_a (resp. Belousov's left derivative ${}_aA$) of A is a right (resp. left) inner derivative of A relative to a in the sense of Definition* 12.

Just as in the case of Belousov's derivative operations, we have:

Theorem 6 *An operation D is a right derivative of an operation A iff D^* is a left derivative of the operation A^*.*

In particular, $D_a(T_1,\varepsilon,T_2)$ is an inner right derivative of the operation A iff $D^*_a(\varepsilon,T^*_3,T^*_4)$ is an inner left derivative of the operation A^*.

Note, T^*_3 and T^*_4 are $*$–translations by a, determined by T_1 and T_2 as in the Table 1.

Proof. Hint: if D is the right derivative of the operation A, then there are functions α,γ such that $(\alpha,\varepsilon,\gamma)$ is a right derivative homotopy. □

Analogously, for the middle derivatives we have:

Theorem 7 *$D = A(\alpha,\beta,\varepsilon)$ iff $D^* = A^*(\beta,\alpha,\varepsilon)$.*

In particular, $D_a = A(T_1,T_2,\varepsilon)$ iff $D^*_a = A^*(T^*_3,T^*_4,\varepsilon)$ for the appropriate T^*_3, T^*_4.

Corollary 1 *The duals of left derivatives are right derivatives and vice versa.*

The duals of middle derivatives are also middle derivatives.

Example 3 *Let A be an arbitrary binary operation on Q and let (α, β, γ) be a derivative isotopy on Q. A derivative operation $D(x, y) = \gamma^{-1}A(\alpha x, \beta y)$ is unique but need not be a quasigroup.*

Example 4 *Let $A(x, y) = \gamma x = a$ for some $a \in Q$. Then* (10) *is true for* any *D. Therefore, any binary operation D on Q is a derivative of A but is certainly not unique.*

Example 5 *Let $A(x, y) = a$ and $\gamma x = b$ for some $a, b \in Q$; $a \neq b$. The identity* (10) *reduces to $a = b$, which is false. Therefore, no D exists in this case.*

The previous example shows that derivative need not exist in general. This rises several related problems:

Problem 4 *Find conditions for the existence of the derivative operation D of the operation A on Q.*

Problem 5 *Find conditions for the uniqueness of D.*

Problem 6 *Find conditions for D to be {left, right} quasigroup.*

Problem 7 *Find conditions for D to be a quasigroup with {left, right, middle} unit.*

This is just a special case of Problem 2.

Problem 8 *Find conditions for the existence of all D_a $(a \in Q)$.*

Problem 9 *Find conditions, forcing all D_a $(a \in Q)$ to be equal (and consequently independent of a).*

We try to answer some of the above-mentioned questions in the following theorems.

Theorem 8 *The operation D exists if the mapping γ is injective. The operation D is unique iff γ is bijective.*

In particular, the unique middle derivative exists for every A, α and β.

Proof. Obvious. □

Theorem 9 *Let α, β, γ be a triple of bijections. Then the operation D is a {left, right} quasigroup iff A is {left, right} quasigroup.*

4 Units from derivatives

We are returning back to Problem 2, trying to find out which identities: $D_a = \cdot$ (in a (pointed) quasigroup $(Q; \cdot, \backslash, /, a)$) imply the existence of a $\{left, right, middle\}$ unit. There are 108 inner derivatives $D_a = \cdot(T', T'', T)$, and consequently there are 108 candidate identities to check

$$\cdot(T', T'', T) = \cdot \qquad (T'T''T)$$

We investigate these cases mostly using programs `'Prover9'` and `'Mace4'` by McCune [12, 11]. Obtained results are presented in Tables 4 and 5. In Table 4 identities are collected by the type of unit whose existence they imply. It is assumed that the particular identity does not imply the existence of units which are not emphasized in the title of its group.

We give here human readable forms of proofs found by `Prover9`. Occasionally, we present more general proof – an instance of some identity

$$\cdot(\alpha, \beta, \gamma) = \cdot \qquad (\alpha\beta\gamma)$$

and use it to pack several identities and related proofs together.

We denote identities from Table 4 using three-letter words (as above) so that the identity $\cdot(\alpha, \beta, \gamma) = \cdot$ is denoted by $(\alpha\beta\gamma)$. (Warning – to save some space, we use $\bar{T} = T^{-1}$, but only in Table 4. Also, in the rest of the paper we drop the subscript a). For example, the identity $ax \cdot (a\backslash y) = x \cdot y$ is denoted by $(L\bar{L}\varepsilon)$. Also note that a notation which contains the letter T and/or one of $L^{\pm1}, R^{\pm1}, P^{\pm1}$ actually represents a family of identities. The results from Table 4 can be summarized, thus:

Theorem 10 *1) Quasigroups satisfying any of the 24 identities:* $(\varepsilon L^{\pm1} L^{\pm1})$, $(\varepsilon R^{\pm1} P^{\pm1})$, $(L^{\pm1} \varepsilon P^{\pm1})$, $(L^{\pm1} R^{\pm1} \varepsilon)$, $(R^{\pm1} \varepsilon R^{\pm1})$, $(P^{\pm1} P^{\pm1} \varepsilon)$ *need not have any kind of unit.*

2) Quasigroups satisfying any of the 32 identities: $(T\varepsilon L^{\pm1})$, $(TL^{\pm1}\varepsilon)$, $(L^{\pm1}\varepsilon R^{\pm1})$, $(L^{\pm1} P^{\pm1} \varepsilon)$ *are left loops.*

a) Quasigroups satisfying any of the four of these identities: $(R^{\pm1} L^{\pm1} \varepsilon)$*, are loops.*

b) Quasigroups satisfying one of the other four identities: $(P^{\pm1} \varepsilon L^{\pm1})$*, are unipotent left loops.*

c) Quasigroups satisfying any of the remaining 24 identities need not have right and/or middle unit.

3) Quasigroups satisfying any of the 32 identities: $(\varepsilon T R^{\pm1})$, $(R^{\pm1} T \varepsilon)$, $(\varepsilon R^{\pm1} L^{\pm1})$, $(P^{\pm1} R^{\pm1} \varepsilon)$ *are right loops.*

a) The four identities $(R^{\pm1} L^{\pm1} \varepsilon)$ *were discussed in 2a).*

No units					
$x \cdot ay = a \cdot xy$	(εLL)	$ax \cdot y = xy\backslash a$	$(L\varepsilon P)$	$xa \cdot y = xy \cdot a$	$(R\varepsilon R)$
$x \cdot ay = a\backslash xy$	$(\varepsilon L\bar{L})$	$ax \cdot y = a/xy$	$(L\varepsilon\bar{P})$	$xa \cdot y = xy/a$	$(R\varepsilon\bar{R})$
$x(a\backslash y) = a \cdot xy$	$(\varepsilon\bar{L}L)$	$(a\backslash x)y = xy\backslash a$	$(\bar{L}\varepsilon P)$	$(x/a)y = xy \cdot a$	$(\bar{R}\varepsilon R)$
$x(a\backslash y) = a\backslash xy$	$(\varepsilon\bar{L}\bar{L})$	$(a\backslash x)y = a/xy$	$(\bar{L}\varepsilon\bar{P})$	$(x/a)y = xy/a$	$(\bar{R}\varepsilon\bar{R})$
$x \cdot ya = xy\backslash a$	(εRP)	$ax \cdot ya = xy$	$(LR\varepsilon)$	$(x\backslash a)(y\backslash a) = xy$	$(PP\varepsilon)$
$x \cdot ya = a/xy$	$(\varepsilon R\bar{P})$	$ax \cdot (y/a) = xy$	$(L\bar{R}\varepsilon)$	$(x\backslash a)(a/y) = xy$	$(P\bar{P}\varepsilon)$
$x(y/a) = xy\backslash a$	$(\varepsilon\bar{R}P)$	$(a\backslash x) \cdot ya = xy$	$(\bar{L}R\varepsilon)$	$(a/x)(y\backslash a) = xy$	$(\bar{P}P\varepsilon)$
$x(y/a) = a/xy$	$(\varepsilon\bar{R}\bar{P})$	$(a\backslash x)(y/a) = xy$	$(\bar{L}\bar{L}\varepsilon)$	$(a/x)(a/y) = xy$	$(\bar{P}\bar{P}\varepsilon)$

Left unit		Middle unit		Right unit	
$ax \cdot y = a \cdot xy$	$(L\varepsilon L)$	$x(y\backslash a) = a \cdot xy$	(εPL)	$x \cdot ay = xy \cdot a$	(εLR)
$ax \cdot y = a\backslash xy$	$(L\varepsilon\bar{L})$	$x(y\backslash a) = a\backslash xy$	$(\varepsilon P\bar{L})$	$x \cdot ay = xy/a$	$(\varepsilon L\bar{R})$
$(a\backslash x)y = a \cdot xy$	$(\bar{L}\varepsilon L)$	$x(a/y) = a \cdot xy$	$(\varepsilon\bar{P}L)$	$x(a\backslash y) = xy \cdot a$	$(\varepsilon\bar{L}R)$
$(a\backslash x)y = a\backslash xy$	$(\bar{L}\varepsilon\bar{L})$	$x(a/y) = a\backslash xy$	$(\varepsilon\bar{P}\bar{L})$	$x(a\backslash y) = xy/a$	$(\varepsilon\bar{L}\bar{R})$
$xa \cdot y = a \cdot xy$	$(R\varepsilon L)$	$x(y\backslash a) = xy\backslash a$	(εPP)	$x \cdot ya = xy \cdot a$	(εRR)
$xa \cdot y = a\backslash xy$	$(R\varepsilon\bar{L})$	$x(y\backslash a) = a/xy$	$(\varepsilon P\bar{P})$	$x \cdot ya = xy/a$	$(\varepsilon R\bar{R})$
$(x/a)y = a \cdot xy$	$(\bar{R}\varepsilon L)$	$x(a/y) = xy\backslash a$	$(\varepsilon\bar{P}P)$	$x(y/a) = xy \cdot a$	$(\varepsilon\bar{R}R)$
$(x/a)y = a\backslash xy$	$(\bar{R}\varepsilon\bar{L})$	$x(a/y) = a/xy$	$(\varepsilon\bar{P}\bar{P})$	$x(y/a) = xy/a$	$(\varepsilon\bar{R}\bar{R})$
$ax \cdot ay = xy$	$(LL\varepsilon)$	$(x\backslash a)y = xy \cdot a$	$(P\varepsilon R)$	$xa \cdot ya = xy$	$(RR\varepsilon)$
$ax \cdot (a\backslash y) = xy$	$(L\bar{L}\varepsilon)$	$(x\backslash a)y = xy/a$	$(P\varepsilon\bar{R})$	$xa \cdot (y/a) = xy$	$(R\bar{R}\varepsilon)$
$(a\backslash x) \cdot ay = xy$	$(\bar{L}L\varepsilon)$	$(a/x)y = xy \cdot a$	$(\bar{P}\varepsilon R)$	$(x/a) \cdot ya = xy$	$(\bar{R}R\varepsilon)$
$(a\backslash x)(a\backslash y) = xy$	$(\bar{L}\bar{L}\varepsilon)$	$(a/x)y = xy/a$	$(\bar{P}\varepsilon\bar{R})$	$(x/a)(y/a) = xy$	$(\bar{R}\bar{R}\varepsilon)$
$(x\backslash a) \cdot ay = xy$	$(PL\varepsilon)$	$(x\backslash a)y = xy\backslash a$	$(P\varepsilon P)$	$xa \cdot (y\backslash a) = xy$	$(RP\varepsilon)$
$(x\backslash a)(a\backslash y) = xy$	$(P\bar{L}\varepsilon)$	$(x\backslash a)y = a/xy$	$(P\varepsilon\bar{P})$	$xa \cdot (a/y) = xy$	$(R\bar{P}\varepsilon)$
$(a/x) \cdot ay = xy$	$(\bar{P}L\varepsilon)$	$(a/x)y = xy\backslash a$	$(\bar{P}\varepsilon P)$	$(x/a)(y\backslash a) = xy$	$(\bar{R}P\varepsilon)$
$(a/x)(a\backslash y) = xy$	$(\bar{P}\bar{L}\varepsilon)$	$(a/x)y = a/xy$	$(\bar{P}\varepsilon\bar{P})$	$(x/a)(a/y) = xy$	$(\bar{R}\bar{P}\varepsilon)$
$ax \cdot y = xy \cdot a$	$(L\varepsilon R)$	$x \cdot ay = xy\backslash a$	(εLP)	$x \cdot ya = a \cdot xy$	(εRL)
$ax \cdot y = xy/a$	$(L\varepsilon\bar{R})$	$x \cdot ay = a/xy$	$(\varepsilon L\bar{P})$	$x \cdot ya = a\backslash xy$	$(\varepsilon R\bar{L})$
$(a\backslash x)y = xy \cdot a$	$(\bar{L}\varepsilon R)$	$x(a\backslash y) = xy\backslash a$	$(\varepsilon\bar{L}P)$	$x(y/a) = a \cdot xy$	$(\varepsilon\bar{R}L)$
$(a\backslash x)y = xy/a$	$(\bar{L}\varepsilon\bar{R})$	$x(a\backslash y) = a/xy$	$(\varepsilon\bar{L}\bar{P})$	$x(y/a) = a\backslash xy$	$(\varepsilon\bar{R}\bar{L})$
$ax \cdot (y\backslash a) = xy$	$(LP\varepsilon)$	$xa \cdot y = xy\backslash a$	$(R\varepsilon P)$	$(x\backslash a) \cdot ya = xy$	$(PR\varepsilon)$
$ax \cdot (a/y) = xy$	$(L\bar{P}\varepsilon)$	$xa \cdot y = a/xy$	$(R\varepsilon\bar{P})$	$(x\backslash a)(y/a) = xy$	$(P\bar{R}\varepsilon)$
$(a\backslash x)(y\backslash a) = xy$	$(\bar{L}P\varepsilon)$	$(x/a)y = xy\backslash a$	$(\bar{R}\varepsilon P)$	$(a/x) \cdot ya = xy$	$(\bar{P}R\varepsilon)$
$(a\backslash x)(a/y) = xy$	$(\bar{L}\bar{P}\varepsilon)$	$(x/a)y = a/xy$	$(\bar{R}\varepsilon\bar{P})$	$(a/x)(y/a) = xy$	$(\bar{P}\bar{R}\varepsilon)$

Left and middle unit		Left and right unit		Right and middle unit	
$(x\backslash a)y = a \cdot xy$	$(P\varepsilon L)$	$xa \cdot ay = xy$	$(RL\varepsilon)$	$x(y\backslash a) = xy \cdot a$	(εPR)
$(x\backslash a)y = a\backslash xy$	$(P\varepsilon\bar{L})$	$xa \cdot (a\backslash y) = xy$	$(R\bar{L}\varepsilon)$	$x(y\backslash a) = xy/a$	$(\varepsilon P\bar{R})$
$(a/x)y = a \cdot xy$	$(\bar{P}\varepsilon L)$	$(x/a) \cdot ay = xy$	$(\bar{R}L\varepsilon)$	$x(a/y) = xy \cdot a$	$(\varepsilon\bar{P}R)$
$(a/x)y = a\backslash xy$	$(\bar{P}\varepsilon\bar{L})$	$(x/a)(a\backslash y) = xy$	$(\bar{R}\bar{L}\varepsilon)$	$x(a/y) = xy/a$	$(\varepsilon\bar{P}\bar{R})$

Table 4: Classification of 108 identities

   b) *Quasigroups satisfying any of the other four identities:* $(\varepsilon P^{\pm1}R^{\pm1})$ *are unipotent right loops.*
   c) *Quasigroups satisfying any of the remaining 24 identities need not have left and/or middle unit.*

4) *Quasigroups satisfying any of the 32 identities:* $(\varepsilon P^{\pm1}T), (P^{\pm1}\varepsilon T)$, $(\varepsilon L^{\pm1}P^{\pm1}), (R^{\pm1}\varepsilon P^{\pm1})$ *are unipotent.*
   a) *The four identities* $(P^{\pm1}\varepsilon L^{\pm1})$ *were discussed in 2b).*
   b) *The four identities* $(\varepsilon P^{\pm1}R^{\pm1})$ *were discussed in 3b).*
   c) *Quasigroups satisfying any of the remaining 24 identities need not have left and/or right unit.*

5) *Among 108 given identities, there is no identity which forces a quasigroup satisfying it to have all three types of units.*

In the rest of this section we give the proof of Theorem 10.

4.1 Positive results

4.1.1 Identities implying the existence of the left unit

Theorem 11 *A sufficient condition for a derivative* $D = \cdot(\alpha, \beta, \gamma)$ *of a quasigroup* $\cdot$ *to have left unit is:* $\gamma = L_a\beta$. *Moreover:*

- *If* D *is right derivative, then* $\gamma = L_a$ *is sufficient.*
- *If* D *is middle derivative, then* $\beta = L_a^{-1}$ *is sufficient.*

Proof. Assume $D(x,y) = \gamma^{-1}(\alpha x \cdot \beta y)$. Replace x by $\alpha^{-1}a$. We get $D(\alpha^{-1}a, y) = \gamma^{-1}(\alpha\alpha^{-1}a \cdot \beta y) = \gamma^{-1}(a \cdot \beta y) = \gamma^{-1}L_a\beta y$. If $\gamma = L_a\beta y$ then $D(\alpha^{-1}a, y) = y$.

The rest of the statement is obvious. □

Corollary 2 $(T\varepsilon L) \Rightarrow$ (eQ).

Corollary 3 $(TL^{-1}\varepsilon) \Rightarrow$ (eQ).

Theorem 12 *The identity* $(\alpha\varepsilon L^{-1})$ *implies* (eQ).

Proof. Assume $x \cdot y = L(\alpha x \cdot y)$. For $x = a$ we get $a \cdot y = a \cdot (\alpha a \cdot y)$. Cancelling from the left yields $y = \alpha a \cdot y$, i.e. αa is the left unit. □

Corollary 4 $(T\varepsilon L^{-1}) \Rightarrow$ (eQ).

Theorem 13 *The identity* $(\alpha L\varepsilon)$ *implies* (eQ).

Proof. Assume $x\cdot y=\alpha x\cdot Ly$. For $x=a$ we get $a\cdot y=\alpha a\cdot(a\cdot y)$. Replacing $a\cdot y$ by z yields $z=\alpha a\cdot z$, i.e. αa is the left unit. □

Corollary 5 $(TL\varepsilon)\Rightarrow$ (eQ).

Lemma 3 $(L\varepsilon R)\Rightarrow$ (eQ).

Proof. Assume $x\cdot y=R^{-1}(L_a x\cdot y)$. Then:

$$R(x\cdot y)=Lx\cdot y. \tag{11}$$

Replacing y by a in (11), we get $R(xa)=RLx$. Cancelling yields

$$Rx=Lx. \tag{12}$$

Replacing (12) in (11) provides $L(xy)=Lx\cdot y$, i.e. identity $(L\varepsilon L)$. By Corollary 2, (eQ) follows. □

Lemma 4 $(L\varepsilon R^{-1})\Rightarrow$ (eQ).

Proof. Assume $x\cdot y=R(Lx\cdot y)$. Then:

$$R^{-1}(x\cdot y)=Lx\cdot y. \tag{13}$$

Replacing y by a in (13), we get $R^{-1}(xa)=RLx$. Cancelling yields $x=RLx$, i.e.

$$R^{-1}x=Lx. \tag{14}$$

Replacing (14) in (13) provides $L(xy)=Lx\cdot y$, i.e. identity $(L\varepsilon L)$. By Corollary 2, (eQ) follows. □

Lemma 5 $(LP\varepsilon)\Rightarrow$ (eQ).

Proof. Assume $x\cdot y=Lx\cdot Py$. Then

$$x\cdot y=ax\cdot(y\backslash a). \tag{15}$$

Replacing y by ax in (15), we get $x\cdot ax=ax\cdot(ax\backslash a)=a$. Consequently

$$ax=x\backslash a$$

Using this in (15), we get $xy=Lx\cdot Ly$, i.e. identity $(LL\varepsilon)$. By Corollary 5, (eQ) follows. □

Lemma 6 $(LP^{-1}\varepsilon)\Rightarrow$ (eQ).

Proof. Assume $x \cdot y = Lx \cdot P^{-1}y$. Then

$$x \cdot y = ax \cdot (a/y). \tag{16}$$

Replacing y by Px in (16), we get $x(x\backslash a) = ax \cdot P^{-1}Px$. Consequently

$$ax = a/x$$

Using this in (16), we get $xy = Lx \cdot Ly$, i.e. identity $(LL\varepsilon)$. By Corollary 5, (eQ) follows. □

Lemma 7 $(L^{-1}\varepsilon R) \Rightarrow$ (eQ).

Proof. Assume $xy = R^{-1}(L^{-1}x \cdot y)$, i.e.

$$R(xy) = L^{-1}x \cdot y. \tag{17}$$

Replacing y by a, we get $R(xa) = RL^{-1}x$. After cancellation we obtain

$$Rx = L^{-1}x.$$

Using this in (17), we get $L^{-1}(xy) = L^{-1}x \cdot y$, which is identity $(L^{-1}\varepsilon L^{-1})$. By Corollary 4, (eQ) follows. □

Lemma 8 $(L^{-1}\varepsilon R^{-1}) \Rightarrow$ (eQ).

Proof. Assume $xy = R(L^{-1}x \cdot y)$, i.e.

$$R^{-1}(xy) = L^{-1}x \cdot y. \tag{18}$$

Replacing y by a, we get $R^{-1}Rx = RL^{-1}x$. After cancellation we obtain

$$R^{-1}x = L^{-1}x.$$

Using this in (18), we get $L^{-1}(xy) = L^{-1}x \cdot y$, which is identity $(L^{-1}\varepsilon L^{-1})$. By Corollary 4, (eQ) follows. □

Lemma 9 $(L^{-1}P\varepsilon) \Rightarrow$ (eQ).

Proof. Assume $xy = L^{-1}x \cdot Py$, i.e.

$$xy = L^{-1}x \cdot (y\backslash a). \tag{19}$$

Replacing y by $L^{-1}x$, we get $x(a\backslash x) = (L^{-1}x) \cdot ((L^{-1}x)\backslash a) = a$ i.e.

$$L^{-1}x = a\backslash x = x\backslash a = Px.$$

Using this in (19), we get $xy = L^{-1}x \cdot L^{-1}y$, which is identity $(L^{-1}L^{-1}\varepsilon)$. By Corollary 3, (eQ) follows. □

Lemma 10 $(L^{-1}P^{-1}\varepsilon) \Rightarrow$ (eQ).

Proof. Assume

$$xy = L^{-1}x \cdot P^{-1}y. \tag{20}$$

Let $u = L^{-1}x$ and $v = P^{-1}y$. Identity (20) becomes $au \cdot (v\backslash a) = uv$. Replacing v by au, we get $u \cdot au = au \cdot (au\backslash a) = a$. From $Pu = u\backslash a = au = Lu$ we get $P^{-1} = L^{-1}$. This, used in (20), yields $xy = L^{-1}x \cdot L^{-1}y$, which is identity $(L^{-1}L^{-1}\varepsilon)$. By Corollary 3, (eQ) follows. □

4.1.2 Identities implying the existence of the right unit

Theorem 14 *A sufficient condition for a derivative $D = \cdot(\alpha, \beta, \gamma)$ of a quasigroup $\cdot$ to have right unit is: $\gamma = R\alpha$. Moreover:*

- *If D is left derivative, then $\gamma = R$ is sufficient.*
- *If D is middle derivative, then $\alpha = R^{-1}$ is sufficient.*

Proof. Let $D(x,y) = \gamma^{-1}(\alpha x \cdot \beta y)$. By Theorem 6 $D^*(y,x) = \gamma^{-1}(\beta y * \alpha x)$, i.e. D^* is a derivative operation of the operation $*$. As a parastrophe of the quasigroup $\cdot$, operation $*$ is also a quasigroup, and we can apply Theorem 11 to it. It means that there is a left unit e for $*$ if $\gamma = L^*\alpha$. Therefore $\gamma = R\alpha$ implies $e * x = x$ i.e. $x \cdot e = x$.

The rest of the statement is obvious. □

Analogously, using duality of quasigroups as above, we can prove:

Theorem 15

- *The identity* $(\varepsilon\beta R^{-1})$ *implies* (Qe).
- *The identity* $(R\beta\varepsilon)$ *implies* (Qe).

as well as:

Lemma 11 *Any of the identities:* $(\varepsilon TR), (\varepsilon TR^{-1}), (RT\varepsilon), (R^{-1}T\varepsilon), (\varepsilon RL), (\varepsilon RL^{-1}), (PR\varepsilon), (P^{-1}R\varepsilon), (\varepsilon R^{-1}L), (\varepsilon R^{-1}L^{-1}), (PR^{-1}\varepsilon), (P^{-1}R^{-1}\varepsilon)$ *implies* (Qe).

4.1.3 Identities implying the existence of the middle unit

Theorem 16 *A sufficient condition for a derivative $D = \cdot(\alpha, \beta, \gamma)$ of a quasigroup $\cdot$ to have middle unit is: $\beta = P\alpha$. Moreover:*

- *If D is left derivative, then $\beta = P$ is sufficient.*
- *If D is right derivative, then $\alpha = P^{-1}$ suffice.*

Proof. Assume $D(x,y) = \gamma^{-1}(\alpha x \cdot \beta y)$. Replace β by $P\alpha$. Then $D(x,y) = \gamma^{-1}(\alpha x \cdot P\alpha y)$ and $D(x,x) = \gamma^{-1}(\alpha x \cdot P\alpha x) = \gamma^{-1}(\alpha x \cdot (\alpha x \backslash a)) = \gamma^{-1}a$ i.e. $\gamma^{-1}a$ is the middle unit.

The rest of the statement is obvious. □

Corollary 6 *Every one of the identities:* $(\varepsilon PT), (P^{-1}\varepsilon T)$ *implies* (U).

Theorem 17 *The identity* $(\varepsilon P^{-1}\gamma)$ *implies* (U).

Proof. Assume $x \cdot y = \gamma^{-1}(x \cdot P^{-1}y)$. Then

$$\gamma(xy) = x(a/y). \tag{21}$$

For $y = x\backslash a$ we get $\gamma(x(x\backslash a)) = x \cdot (a/(x\backslash a))$. Therefore $\gamma a = xx$, i.e. γa is the middle unit. □

Corollary 7 *$(\varepsilon P^{-1}T) \Rightarrow$ (U).*

Theorem 18 *The identity $(P\varepsilon\gamma)$ implies (U).*

Proof. From Theorem 17, by duality. □

Corollary 8 *$(P\varepsilon T) \Rightarrow$ (U)*

Lemma 12 *$(\varepsilon LP) \Rightarrow$ (U).*

Proof. Assume $P(xy) = x \cdot Ly$ i.e.

$$(xy)\backslash a = x \cdot ay. \tag{22}$$

In this identity we replace x by z and then y by $z\backslash x$. We get $z(z\backslash x)\backslash a = z \cdot a(z\backslash x)$ i.e.

$$x\backslash a = z \cdot a(z\backslash x) \tag{23}$$

If we put $z = a$ in the equality (23), we get $x\backslash a = a \cdot a(a\backslash x) = ax$ so $Px = Lx$. Using this in (22), we get:

$$P(xy) = x \cdot Py,$$

which is (εPP). By Corollary 6, (U) follows. □

Lemma 13 *$(\varepsilon LP^{-1}) \Rightarrow$ (U).*

Proof. Assume $P^{-1}(xy) = x \cdot Ly$ i.e.

$$a/xy = x \cdot ay. \tag{24}$$

Replacing x by z and then y by $z\backslash x$, we get

$$a/x = z \cdot a(z\backslash x).$$

If we replace z by a in this identity, we get $a/x = a \cdot a(a\backslash x) = ax$ i.e. $P^{-1}x = Lx$. Using this in (24), we get $P^{-1}(xy) = x \cdot P^{-1}y$, which is $(\varepsilon P^{-1}P^{-1})$. By Corollary 7, (U) follows. □

Lemma 14 *$(\varepsilon L^{-1}P) \Rightarrow$ (U).*

Proof. Assume

$$xy = P^{-1}(x \cdot L^{-1}y). \tag{25}$$

It follows that $xy\backslash a = x(a\backslash y)$ and $a = xy \cdot x(a\backslash y)$. Replacing x by a, we get $a = ay \cdot a(a\backslash y)$, which is equivalent to $a = ay \cdot y$. Then $P^{-1}y = a/y = ay = Ly$ and $P = L^{-1}$. Replacing this in (25), we get $xy = P^{-1}(x \cdot Py)$, which is the identity (εPP). Therefore, (U) follows by Corollary 6. □

Lemma 15 $(\varepsilon L^{-1}P^{-1}) \Rightarrow$ (U).

Proof. Assume $P^{-1}(xy) = x \cdot L^{-1}y$ i.e.

$$a/xy = x(a\backslash y). \tag{26}$$

Replacing x by a, we get $a/ay = a(a\backslash y) = y$, which implies $a = y \cdot ay$ i.e.

$$y\backslash a = ay.$$

This means that $Py = Ly$ and consequently $P^{-1}y = L^{-1}y$. Replacing this in (26), we get $P^{-1}(xy) = x \cdot P^{-1}y$ which is $(\varepsilon P^{-1}P^{-1})$. By Corollary 7, (U) follows. □

Lemma 16 *Any of the identities:* $(R\varepsilon P), (R\varepsilon P^{-1}), (R^{-1}\varepsilon P), (R^{-1}\varepsilon P^{-1})$ *implies* (U).

Proof. The above identities are dual to the identities $(\varepsilon LP^{-1}), (\varepsilon LP)$, $(\varepsilon L^{-1}P^{-1}), (\varepsilon L^{-1}P)$ respectively. Therefore, the Lemma is the consequence of Lemmas 13, 12, 15, 14 respectively. □

In some sense, there exists a theorem that is the converse of the previous theorems.

Theorem 19 *If a quasigroup* $(Q;\cdot)$ *has a left (right, middle) unit, then* $(Q;\cdot)$ *has an autotopy.*

Proof. If the quasigroup $(Q;\cdot)$ has a left unit f, then the following equality $f \cdot xy = fx \cdot y$ is true, and $(Q;\cdot)$ has autotopy (L_f, ε, L_f).

If the quasigroup $(Q;\cdot)$ has a right unit e, then the following equality $xy \cdot e = x \cdot ye$ is true, and $(Q;\cdot)$ has autotopy (ε, R_e, R_e).

If the quasigroup $(Q;\cdot)$ has a middle unit, then it is unipotent, and the following equalities are true $xy \cdot xy = y \cdot y = a$ for all $x, y \in Q$ and some fixed $a \in Q$.

From these equalities we have the following $(xy)\backslash a = xy$, $y\backslash a = y$. Further we have $(xy)\backslash a = x \cdot (y\backslash a)$. The last means that the quasigroup $(Q;\cdot)$ has the following autotopy (ε, P_a, P_a) (see Table 1). □

4.2 Negative results

In order to prove the negative results of Theorem 10, we use the following quasigroups:

Q_1:

$\bullet$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Q_2:

$\bullet$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Q_2^*:

$\bullet$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Q_3:

$\bullet$	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

Q_3^*:

$\bullet$	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

Q_4:

$\bullet$	0	1	2
0	1	2	0
1	2	0	1
2	0	1	2

Q_5:

$\bullet$	0	1	2	3	4	5
0	1	0	2	3	5	4
1	0	1	4	5	3	2
2	4	3	1	2	0	5
3	5	2	3	1	4	0
4	2	5	0	4	1	3
5	3	4	5	0	2	1

Q_6:

$\bullet$	0	1	2	3	4	5
0	1	0	2	3	5	4
1	0	1	5	4	2	3
2	4	2	3	1	0	5
3	5	3	1	2	4	0
4	2	4	0	5	3	1
5	3	5	4	0	1	2

Q_7:

$\bullet$	0	1	2	3	4	5
0	1	0	2	4	3	5
1	0	1	3	5	2	4
2	2	4	5	1	0	3
3	3	5	1	2	4	0
4	4	2	0	3	5	1
5	5	3	4	0	1	2

Q_8:

$\bullet$	0	1	2	3	4	5
0	1	0	2	4	3	5
1	0	1	4	2	5	3
2	4	3	0	5	1	2
3	5	2	3	0	4	1
4	2	5	1	3	0	4
5	3	4	5	1	2	0

Q_9:

$\bullet$	0	1	2	3	4	5
0	1	0	4	2	3	5
1	0	1	5	4	2	3
2	4	3	2	1	5	0
3	5	2	1	3	0	4
4	2	5	3	0	4	1
5	3	4	0	5	1	2

Q_6^*:

$\bullet$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	2	3	4	5
2	2	5	3	1	0	4
3	3	4	1	2	5	0
4	5	2	0	4	3	1
5	4	3	5	0	1	2

Q_{10}:

$\bullet$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	2	3	4	5
2	4	3	5	2	0	1
3	5	2	1	0	3	4
4	2	5	3	4	1	0
5	3	4	0	1	5	2

Q_{11}:

$\bullet$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	3	2	5	4
2	2	4	0	3	1	5
3	3	5	1	4	0	2
4	4	2	5	0	3	1
5	5	3	2	1	4	0

Q_5^*:

$\bullet$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	3	2	5	4
2	2	4	1	3	0	5
3	3	5	2	1	4	0
4	5	3	0	4	1	2
5	4	2	5	0	3	1

Q_{10}^*:

$\bullet$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	3	2	5	4
2	4	2	5	1	3	0
3	5	3	2	0	4	1
4	2	4	0	3	1	5
5	3	5	1	4	0	2

Q_{12}:

$\bullet$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	3	2	5	4
2	4	5	2	1	3	0
3	2	4	1	3	0	5
4	3	2	5	0	4	1
5	5	3	0	4	1	2

Q_{13}:

$\bullet$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	3	2	5	4
2	5	3	2	1	4	0
3	4	2	1	3	0	5
4	3	5	0	4	1	2
5	2	4	5	0	3	1

Note that quasigroups Q_2^* (resp. $Q_3^*, Q_5^*, Q_6^*, Q_{10}^*$) are dual to Q_2 (resp. Q_3, Q_5, Q_6, Q_{10}).

The negative results are collected in Table 5. It consists of entries of the form:

Q_i

Text1 Text2

List of identities
$(T_1 T_2 T)$,

where:

- Q_i is either one of the above quasigroups or a Cartesian product of two of them;
- "Text1" is either "$a = 0$" or "$a = (0,0)$" so that $a \in Q_i$;
- T_1, T_2 and T are translations of Q_i by the element a such that all formulas $(T_1 T_2 T)$ in the List are true in Q_i;
- "Text2" is either "no units" (when Q_i has no units) or the equality "$e = p$" ("$e = (p, q)$"), claiming that $p \in Q_i$ ($(p, q) \in Q_i$) is the unit of Q_i of the type indicated by the title of the group to which Q_i belongs.

Quasigroups		
Q_{11}	Q_{12}	Q_8
$a=0$ no units $(\varepsilon L^{\pm1}L^{\pm1})$	$a=0$ no units $(L^{\pm1}\varepsilon P^{\pm1})$	$a=0$ no units $(R^{\pm1}\varepsilon R^{\pm1})$
Q_9	Q_{13}	Q_7
$a=0$ no units $(\varepsilon R^{\pm1}P^{\pm1})$	$a=0$ no units $(L^{\pm1}R^{\pm1}\varepsilon)$	$a=0$ no units $(P^{\pm1}P^{\pm1}\varepsilon)$
Left loops	**Unipotent quasigroups**	**Right loops**
$Q_2\times Q_4$	Q_5^*	Q_{10}^*
$a=(0,0)$ $e=(0,2)$ $(L\varepsilon L),(L^{-1}\varepsilon L^{-1}),$ $(LL^{-1}\varepsilon),(L^{-1}L\varepsilon)$	$a=0$ $e=1$ $(\varepsilon P^{\pm1}L^{\pm1}),(\varepsilon L^{\pm1}P^{\pm1})$	$a=0$ $e=1$ $(\varepsilon L^{\pm1}R^{\pm1}),(\varepsilon R^{\pm1}L^{\pm1})$
$Q_1\times Q_3^*$	$Q_2^*\times Q_3^*$	$Q_2^*\times Q_4$
$a=(0,0)$ $e=(0,1)$ $(L\varepsilon L^{-1}),(L^{-1}\varepsilon L),$ $(LL\varepsilon),(L^{-1}L^{-1}\varepsilon)$	$a=(0,0)$ $e=(0,1)$ $(\varepsilon PP),(\varepsilon P^{-1}P^{-1}),$ $(P\varepsilon P^{-1}),(P^{-1}\varepsilon P)$	$a=(0,0)$ $e=(0,2)$ $(\varepsilon RR),(\varepsilon R^{-1}R^{-1}),$ $(RR^{-1}\varepsilon),(R^{-1}R\varepsilon)$
Q_{10}	$Q_2\times Q_3$	$Q_1\times Q_3$
$a=0$ $e=1$ $(R^{\pm1}\varepsilon L^{\pm1}),(L^{\pm1}\varepsilon R^{\pm1})$	$a=(0,0)$ $e=(0,1)$ $(\varepsilon PP^{-1}),(\varepsilon P^{-1}P),$ $(P\varepsilon P),(P^{-1}\varepsilon P^{-1})$	$a=(0,0)$ $e=(0,1)$ $(\varepsilon RR^{-1}),(\varepsilon R^{-1}R),$ $(RR\varepsilon),(R^{-1}R^{-1}\varepsilon)$
Q_6^*	Q_5	Q_6
$a=0$ $e=1$ $(P^{\pm1}L^{\pm1}\varepsilon),(L^{\pm1}P^{\pm1}\varepsilon)$	$a=0$ $e=1$ $(P^{\pm1}\varepsilon R^{\pm1}),(R^{\pm1}\varepsilon P^{\pm1})$	$a=0$ $e=1$ $(R^{\pm1}P^{\pm1}\varepsilon),(P^{\pm1}R^{\pm1}\varepsilon)$
Unipotent left loops	**Loops**	**Unipotent right loops**
Q_3^*	Q_1	Q_2^*
$a=0$ $e=1$ $(P\varepsilon L),(P^{-1}\varepsilon L^{-1})$	$a=0$ $e=0$ $(RL\varepsilon),(R^{-1}L^{-1}\varepsilon)$	$a=0$ $e=0$ $(\varepsilon PR),(\varepsilon P^{-1}R^{-1})$
Q_2	Q_4	Q_3
$a=0$ $e=0$ $(P\varepsilon L^{-1}),(P^{-1}\varepsilon L)$	$a=0$ $e=2$ $(RL^{-1}\varepsilon),(R^{-1}L\varepsilon)$	$a=0$ $e=1$ $(\varepsilon PR^{-1}),(\varepsilon P^{-1}R)$

Table 5: Countermodels

However, e is not a unit of any other type, and this is the essence of our negative result.

Let us see how it works in some examples.

Example 6 *In the group 'Unipotent left loops' there is an entry:*

$$Q_2$$

$$a = 0 \qquad\qquad e = 0$$

$$(P\varepsilon L^{-1}), (P^{-1}\varepsilon L),$$

It means that the quasigroup Q_2 satisfies identities $(P_0\varepsilon L_0^{-1})$ and $(P_0^{-1}\varepsilon L_0)$. Moreover, the element 0 is both left and middle unit but not right unit. Although this is not written explicitly, Q_2 also satisfies $(\varepsilon P_0 P_0^{-1}), (\varepsilon P_0^{-1} P_0)$, $(P_0\varepsilon P_0), (P_0^{-1}\varepsilon P_0^{-1})$, namely:

$$x(y\backslash 0) = 0/xy, \tag{27}$$

$$x(0/y) = xy\backslash 0, \tag{28}$$

$$(x\backslash 0)y = xy\backslash 0, \tag{29}$$

$$(0/x)y = 0/xy. \tag{30}$$

Example 7 *Analogously, the quasigroup Q_3 from the group 'Unipotent right loops' satisfies $(\varepsilon P_0 R_0^{-1})$ and $(\varepsilon P_0^{-1} R_0)$. Moreover, the element 1 is both right and middle unit but not left unit. Q_3 also satisfies $(\varepsilon P_0 P_0^{-1}), (\varepsilon P_0^{-1} P_0)$, $(P_0\varepsilon P_0), (P_0^{-1}\varepsilon P_0^{-1})$ i.e. (27)–(30).*

Example 8 *From Examples 6 and 7 it follows that the quasigroup $Q_2 \times Q_3$ also satisfies $(\varepsilon P_a P_a^{-1}), (\varepsilon P_a^{-1} P_a)$, $(P_a\varepsilon P_a), (P_a^{-1}\varepsilon P_a^{-1})$ but for $a = (0, 0)$. Likewise, the element $e = (0, 1)$ is the middle unit but neither left nor right unit.*

This completes the justification related to the entry:

$$Q_2 \times Q_3$$

$$a = (0, 0) \qquad\qquad e = (0, 1)$$

$$\begin{gathered}(\varepsilon P P^{-1}), (\varepsilon P^{-1} P),\\ (P\varepsilon P), (P^{-1}\varepsilon P^{-1})\end{gathered}$$

in the group 'Unipotent quasigroups'.

5 The case of idempotents

There is a close connection between units and idempotents in a quasigroup. Recall Theorem 3 and Example 1. We can consider several aspects of idempotents:

1. The existence of an idempotent.
2. The uniqueness of an existing idempotent.
3. The closedness of the set $E(Q) \neq \phi$ of all idempotents of Q.
4. The universality of idempotence in Q.

The formulas which axiomatize these properties are collected in Table 6.

Property	Symbol	with e	without e
none	(Q)	$x = x$	
existence	(i)	$ee = e$	$\exists x(xx = x)$
uniqueness	(j)	$xx = x \Rightarrow x = e$	$(xx = x \wedge yy = y) \Rightarrow x = y$
$E(Q)$ closed	(k)	$(xx = x \wedge yy = y) \Rightarrow xy \cdot xy = xy$	
universal	(I)	$xx = x$	
trivial	(T)	$x = y$	

Table 6: Idempotents in quasigroups

If a quasigroup has one of the properties 1–4 above relative to the operation $\cdot$, then it has the same properties relative to operations $\backslash, /$. The converse is also true.

These properties are closely related and mutually dependent. For example, we have:

Theorem 20 $((\mathrm{I}) \wedge (\mathrm{j})) \Rightarrow (\mathrm{T})$.

The lattice of classes of quasigroups with various idempotence properties is given in Figure 2.

We can pose the following problems, analogous to Belousov's Problem # 18:

Problem 10 *Find identites which force quasigroups satisfying them to have an idempotent.*

Problem 11 *Find identites which force quasigroups satisfying them to have a unique idempotent.*

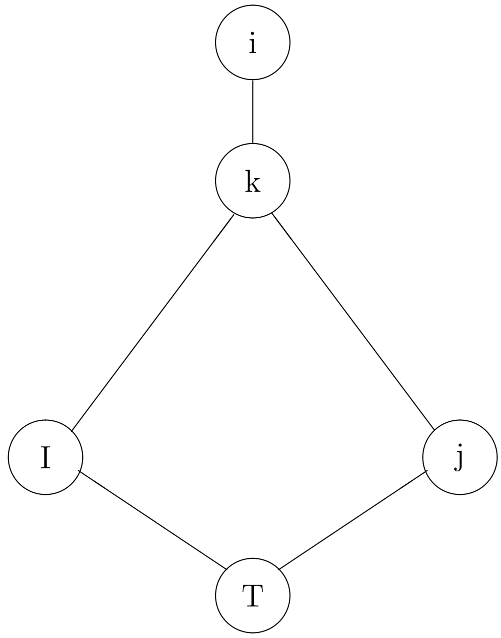

Figure 2: Lattice of classes of quasigroups related to idempotency

Problem 12 *Find identites which force quasigroups satisfying them to have closed* $E(Q)$*.*

Problem 13 *Find identites which force quasigroups satisfying them to be idempotent.*

For example:

Example 9 *Any identity which implies the existence of* {*left, right, middle*} *unit is a solution of Problems* (10)–(12) *(see Theorem* 3*).*

But we are more interested in identities which yield idempotents which are not necessarily units. One such example is:

Example 10 *Every quasigroup which satisfies* (i) *and the following weak medial identity:* $xy \cdot xy = xx \cdot yy$*, also satisfies* (k)*.*

Proof. *Easy.* □

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