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Fractional maximal and integral operators in variable exponent Morrey spaces

Panwang Wang and Zongguang Liu

Abstract. In this paper, we study the boundedness of the fractional maximal operator and fractional integral operator on the variable exponent Morrey spaces defined over spaces (X, d, μ) of homogeneous type.

Key Words: fractional integral operator, Morrey spaces, spaces of Homogeneous type
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Introduction

A quasi-metric d on a set X is a function $d : X \times X \to [0, \infty]$ satisfying

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) There exists a constant $A < \infty$ such that $d(x, y) < A(d(y, z) + d(x, z))$ for $x, y, z \in X$.

The space of homogeneous type (X, d, μ) in the sense of Coifman and Weiss [4] is a topological space X defined by d with nonnegative measure μ which is defined on the σ-algebra generated by quasi-metric balls and open sets such that $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and arbitrary $r > 0$, and so that there exists a constant $b > 0$ such that

$$\mu(B(x, 2r)) \leqslant b\mu(B(x, r)) < \infty, \tag{1}$$

where $B(x, r)$ is the ball centred at x with radius r. Iterating (1) we obtain that there exists a positive constant C_μ such that for all $x \in X$, $0 < r < R$ and $y \in B(x, R)$,

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geqslant C_\mu \Big(\frac{r}{R}\Big)^{\log_2 b}. \tag{2}$$

If b is the smallest constant for the measure μ satisfying (2), we call the number $Q = \log_2 b$ the doubling order of μ. Obviously, in the case of R^n with the Lebesgue measure, $Q = n$. In addition, we say (X, d, μ) is a reverse doubling spaces if there exists a constant γ, $0 < \gamma < 1$ such that for every $x \in X$ and $r > 0$ such that $B(x, r) \subset X$,

$$\mu(B(x, r/2)) \leqslant \gamma\mu(B(x, r)).$$

For any spaces of homogeneous type (X, d, μ), Macías and Segovia [8] prove that there exists an equivalent quasi-metric ρ such that all balls with respect to ρ are open in the topology induced by ρ. As in [12], the definition of the reverse doubling condition would need to be changed slightly: there exist constants C and $\gamma, 0 < \gamma < 1$ such that for any ball $B(x, r) \subset X$ and any $i \geqslant 1$

$$\mu(B(x, 2^{-i}r)) \leqslant C\gamma^i\mu(B(x, r)).$$

For more details on this perspective, see [5]. From [10], we know that any doubling measure on any metric space which is connected is reverse doubling. It is valid on any space of homogeneous type that satisfies a non-empty annuli condition, the details to see [14]. The similar conclusions on space of homogeneous type can be seen in [15] and [4].

Let $p : X \to [1, \infty)$ be a measurable function. We suppose that

$$1 < p_- \leqslant p(\cdot) \leqslant p_+ < \infty, \tag{3}$$

where $p_- = \operatorname{ess\,inf}_{x\in X} p(x)$, $p_+ = \operatorname{ess\,sup}_{x\in X} p(x)$. We let $L^{p(\cdot)}(X)$ be the set of functions f such that

$$\rho_{p(\cdot)}(f) = \int_X |f(x)|^{p(x)}dx < \infty.$$

It is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\right\}.$$

We denote the conjugate exponent by $p'(x) = \dfrac{p(x)}{p(x) - 1}$ for $x \in X$. The Hölder inequality is valid in the form

$$\int_X |f(x)g(x)|dx \leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

The variable Morrey spaces over a bounded open set $\Omega \subset \mathbf{R}^n$ were introduced in [10]. In [11] the authors introduced the following variable Morrey spaces on the space of homogeneous type with $\operatorname{diam}(X) < \infty$.

Definition 1 *Let $1 < q_- \leqslant q(\cdot) \leqslant p(\cdot) \leqslant p_+ < \infty$. We say that a measurable locally integrable function f on X belongs to the class $M^{p(x)}_{q(x)}$ if*

$$\|f\|_{M^{p(x)}_{q(x)}(X)} = \sup_B (\mu(B(x,r)))^{1/p(x)-1/q(x)} \|f\|_{L^{q(\cdot)}(B(x,r))}.$$

It is obvious that $M^{p(\cdot)}_{q(\cdot)} = L^{p(\cdot)}$ when $p = q$; when p, q are constants, the space $M^{p(\cdot)}_{q(\cdot)}$ coincides with the classical Morrey space M^p_q. The definition and some properties of $M^{p(\cdot)}_{q(\cdot)}$ we can see from [16, 1, 9] and so on.

As in the Euclidean case we know the log-Hölder continuity condition has play an important role, for details see [17]. On unbounded spaces, it was used in [3, 12] the similar condition to control the continuity of $p(\cdot)$ locally and at infinity.

Definition 2 *Given a function $r(\cdot) : X \to [0, \infty)$, we say that $r(\cdot)$ satisfies the local log-Hölder condition, and denote this by $r(\cdot) \in LH_0$, if there exists a constant C_0 such that for all $x, y \in X$, $d(x, y) < 1/2$,*

$$|r(x) - r(y)| \leqslant \frac{C_0}{-\log d(x, y)}.$$

The constant C_0 is called the LH_0 constant of $r(\cdot)$.

Definition 3 *Given a function $r(\cdot) : X \to [0, \infty)$, we say that $r(\cdot)$ satisfies the log-Hölder condition with respect to a base point $x_0 \in X$, and denote this by $r(\cdot) \in LH_\infty$, if there exist constants C_∞, r_∞ such that for all $x \in X$*

$$|r(x) - r_\infty| \leqslant \frac{C_\infty}{\log(e + d(x, x_0))}.$$

The constant C_∞ is called the LH_∞ constant of $r(\cdot)$.

When $p(\cdot) \in LH = LH_0 \cap LH_\infty$ we say $p(\cdot)$ satisfies the global log-Hölder condition.

For η, $0 \leqslant \eta < 1$, the fractional maximal operator is given by

$$M_\eta f(x) = \sup_{x \in B} \frac{1}{\mu(B)^{1-\eta}} \int_B |f| d\mu.$$

When $\eta = 0$ this reduces to the Hardy-Littlewood maximal operator, denoted by M. On the classical Morrey spaces over $\mathbb{R}^n$, the weighted norm inequalities for M_η were proved in [18]. For variable spaces over spaces of homogeneous type, norm inequalities was proved in [12] extending the work on the Hardy-Littlewood maximal operator. We state our result on M_η on the variable exponent Morrey spaces defined over spaces of homegeneous type as following.

Theorem 1 *Let (X, d, μ) be a space of homogeneous type. For η, $0 \leqslant \eta < 1$, let $p_1(\cdot) : X \to [1, \infty)$, $q_1(\cdot) : X \to [1, \infty)$ be such that $1/p_1(\cdot), 1/q_1(\cdot) \in LH$, $1 < q_{1-} \leqslant p_1(\cdot) \leqslant p_{1+} < 1/\eta$ and $q_1(\cdot) \leqslant p_1(\cdot)$. For each $x \in X$, define $1/p_1(x) - 1/p_2(x) = \eta$ and $1/q_1(x) - 1/q_2(x) = \eta$. Then M_η is bounded from $M^{p_1(x)}_{q_1(x)}$ to $M^{p_2(x)}_{q_2(x)}$. Moreover, if $\mu(X) < \infty$, then we can replace the hypothesis $1/p_1(\cdot), 1/q_1(\cdot) \in LH$ with $1/p_1(\cdot), 1/q_1(\cdot) \in LH_0$.*

The definition of the fractional integral operator is

$$I_\eta f(x) = \int_X \frac{f(y)}{(\mu(B(x, d(x, y))))^{1-\eta}} d\mu(y).$$

The results about fractional integrals defined on quasi-metric measure space but without doubling condition discussed in the monograph [24]. When $\mu(X) < \infty$ these operators were considered in [19, 20] on spaces of homogeneous type. In [12] the boundedness of them were discussed when $\mu(X) = \infty$ on reverse doubling space of homogeneous type. We have the following result on the variable exponent Morrey spaces defined over spaces of homogeneous type.

Theorem 2 *Let (X, d, μ) be a reverse doubling space of homogeneous type. For η, $0 \leqslant \eta < 1$, let $p_1(\cdot) : X \to [1, \infty)$, $q_1(\cdot) : X \to [1, \infty)$ be such that $p_1(\cdot), q_1(\cdot) \in LH$, $1 < p_{1-} \leqslant p_1(\cdot) \leqslant p_{1+} < 1/\eta$ and $q_1(\cdot) \leqslant p_1(\cdot)$. For each $x \in X$, define $1/p_1(x) - 1/p_2(x) = \eta$ and $1/q_1(x) - 1/q_2(x) = \eta$. Then I_η is bounded from $M^{p_1(x)}_{q_1(x)}$ to $M^{p_2(x)}_{q_2(x)}$. Moreover, if $\mu(X) < \infty$, then we can replace the hypothesis $1/p_1(\cdot), 1/q_1(\cdot) \in LH$ with $1/p_1(\cdot), 1/q_1(\cdot) \in LH_0$.*

We say (X, d, μ) is Ahlfors$-Q$ regular if there exist constants C_1, C_2 such that

$$C_1 r^Q \leqslant \mu(B(x, r)) \leqslant C_2 r^Q$$

for $x \in X$ and $r > 0$.

Next we introduce two classes of operators which were applied to study the Sobolev and Poincaré inequalities over metric spaces, for details see [21, 22]. Given $0 < \alpha < Q$, define the operators

$$I^*_\alpha f(x) = \int_X \frac{f(y)}{d(x, y)^{Q-\alpha}} d\mu(y),$$

$$I^{**}_\alpha f(x) = \int_X \frac{f(y) d(x, y)^\alpha}{\mu(B(x, d(x, y)))} d\mu(y).$$

It is immediate that these operators are pointwise equivalent to I_η with $\eta = \alpha/Q$, if (X, d, μ) be an Ahlfors regular space of homogeneous type. Based on Theorem 1 and Theorem 2, we obtain the following conclusion at once.

Corollary 1 *Let (X, d, μ) be an Ahlfors regular space of homogeneous type. For α, $0 < \alpha < Q$, let $p_1(\cdot) : X \to [1, \infty], q_1(\cdot) : X \to [1, \infty]$ be such that $p_1(\cdot), q_1(\cdot) \in LH$, $1 < p_{1-} \leqslant p_1(\cdot) \leqslant p_{1+} < Q/\alpha$ and $q_1(\cdot) \leqslant p_1(\cdot)$. For each $x \in X$, define $1/p_1(x) - 1/p_2(x) = \alpha/Q$ and $1/q_1(x) - 1/q_2(x) = \alpha/Q$. Then I_α^{**} is bounded from $M_{q_1(x)}^{p_1(x)}$ to $M_{q_2(x)}^{p_2(x)}$, and I_α^{**} is also bounded from $M_{q_1(x)}^{p_1(x)}$ to $M_{q_2(x)}^{p_2(x)}$.*

1 Preliminaries

Lemma 1 [7] *Assume $\mu(X) < \infty$ and let $p \in LH_0$. Then there exists a positive constant C such that for all balls $B \subset X$ the inequality*

$$\mu(B)^{p_-(B) - p_+(B)} \leqslant C$$

holds.

There seems to require a stronger hypothesis in more general setting of spaces of homogeneous type.

Lemma 2 [12] *Given a space of homogeneous type (X, d, μ), let $p(\cdot) : X \to [1, \infty)$ be such that $1/p(\cdot) \in LH$. Then there is a positive constant C such that for any ball B*

(i) $\mu(B)^{1/p_-(B) - 1/p_+(B)} \leqslant C$;

(ii) for all $x \in B$, $\mu(B)^{1/p(x) - 1/p_-(B)} \leqslant C$ and $\mu(B)^{1/p_+(B) - 1/p(x)} \leqslant C$.

If we assume $\mathrm{diam}(X) < \infty$ and $\mu\{x\} = 0$, we have the following satement.

Lemma 3 [11] *Let β be a measurable function on X satisfying $\beta(x) < -1$ for all X. Suppose that r is a small positive number. Then there exists a positive constant C independent of r and x such that*

$$A(x, r) = \int_{X \setminus B(x,r)} (\mu(B_{xy}))^{\beta(x)} d\mu(y) \leqslant C \frac{\beta(x) + 1}{\beta(x)} (\mu(x, d(x, y)))^{\beta(x)+1},$$

where

$$B_{xy} = \mu(x, d(x, y)).$$

Lemma 4 [12] *If (X, d, μ) is a reverse doubling spaces, then for all $x \in X$, $\mu\{x\} = 0$.*

Comparing Lemmas 3 and 4 we have the following conclusion.

Lemma 5 *Assume (X, d, μ) is a reverse doubling spaces and $\mu(X) < \infty$. Let β be a measurable function on X satisfying $\beta(x) < -1$ for all X. Suppose that r is a small positive number. Then there exists a positive constant C independent of r and x such that*

$$A(x,r) = \int_{X\setminus B(x,r)} (\mu(B_{xy}))^{\beta(x)} d\mu(y) \leqslant C\frac{\beta(x)+1}{\beta(x)}(\mu(x, d(x,y)))^{\beta(x)+1},$$

where

$$B_{xy} = \mu(x, d(x,y)).$$

Lemma 6 *Assume (X, d, μ) is a reverse doubling spaces. Let β be a measurable function on X satisfying $\beta(x) < -1$ for all X. Suppose that r is a small positive number. Then there exists a positive constant $C \geqslant C_\mu$ independent of r and x such that*

$$A(x,r) = \int_{X\setminus B(x,r)} (\mu(B_{xy}))^{\beta(x)} d\mu(y) \leqslant \frac{C^{-2\beta(x)-1}}{1-\gamma^{-\beta(x)-1}}(\mu(x, d(x,y)))^{\beta(x)+1},$$

where

$$B_{xy} = \mu(x, d(x,y)).$$

Proof: For $i \geqslant 1$ we define

$$R_i = \{y \in X : 2^{i-1}r \leqslant d(x,y) < 2^i r\}.$$

Since the measure μ is both doubling and reverse doubling, we get

$$\begin{aligned}
& \int_{X\setminus B(x,r)} (\mu(B_{xy}))^{\beta(x)} d\mu(y) \\
\leqslant\ & \sum_{i\geqslant 1} \int_{R_i} (\mu(B_{xy}))^{\beta(x)} d\mu(y) \\
\leqslant\ & \sum_{i\geqslant 1} \mu(B(x, 2^{i-1}r))^{\beta(x)} \mu(B(x, 2^i r)) \\
=\ & \sum_{i\geqslant 1} \Big(\frac{\mu(B(x,2^i r))}{\mu(B(x,2^{i-1}r))}\Big)^{-\beta(x)} \mu(B(x,2^i r))^{\beta(x)+1} \\
\leqslant\ & \sum_{i\geqslant 1} C_\mu^{-\beta(x)} C^{-\beta(x)-1} (\gamma^i)^{-\beta(x)-1} \mu(B(x,r))^{\beta(x)+1} \\
\leqslant\ & \frac{C^{-2\beta(x)-1}}{1-\gamma^{-\beta(x)-1}} \mu(B(x,r))^{\beta(x)+1}.
\end{aligned}$$

Lemma 7 [13] *Let f be a measurable function on X and let E be a measurable subset of X. Then the following inequalities hold:*

$$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)} \leqslant \rho_p(f\chi_E) \leqslant \|f\|^{p_-(E)}_{L^{p(\cdot)}(E)},\ \|f\|_{L^{p(\cdot)}(E)} \leqslant 1;$$

$$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)} \leqslant \rho_p(f\chi_E) \leqslant \|f\|^{p_+(E)}_{L^{p(\cdot)}(E)},\ \|f\|_{L^{p(\cdot)}(E)} \geqslant 1.$$

Lemma 8 [12] *Let (X, d, μ) be a space of homogeneous type. For η, $0 \leqslant \eta < 1$, let $p(\cdot) : X \to [1,\infty)$ be such that $1/p(\cdot) \in LH$, $1 < p_- \leqslant p_+ < 1/\eta$. For each $x \in X$, define $1/p(x) - 1/q(x) = \eta$. Then M_η is bounded from $L^{p(x)}(X)$ to $L^{q(x)}(X)$.*

Lemma 9 [12] *Let (X, d, μ) be a space of homogeneous type. For η, $0 \leqslant \eta < 1$, let $p(\cdot) : X \to [1,\infty)$ be such that $p(\cdot) \in LH$, $1 < p_- \leqslant p_+ < 1/\eta$. For each $x \in X$, define $1/p(x) - 1/q(x) = \eta$. Then I_η is bounded from $L^{p(x)}(X)$ to $L^{q(x)}(X)$.*

2 Proof of the Main Results

Proof of Theorem 1 : Let r be a small positive number. Decompose f as follows:$f = f_1 + f_2$, where $f_1 = f\chi_{B(x,A(2A+1)r)}$, $f_2 = f - f_1$. We get

$$\begin{aligned} &(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|M_\eta f\|_{L^{q_2(x)}(B(x,r))} \\ \leqslant\ &(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|M_\eta f_1\|_{L^{q_2(x)}(B(x,r))} \\ &+(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|M_\eta f_2\|_{L^{q_2(x)}(B(x,r))} \\ =\ &I_1 + I_2. \end{aligned}$$

By Lemma 8 and doubling condition we have

$$\begin{aligned} I_1 &\leqslant C(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|f_1\|_{L^{q_1(x)}(B(x,r))} \\ &\leqslant C\|f\|_{M^{p_1(x)}_{q_1(x)}(X)}. \end{aligned}$$

Since $B(x,r) \subset B(y, 2Ar) \subset B(x, A(2A+1)r)$, we have

$$M_\eta f_2(y) \leqslant \sup_{B\supset B(x,r)} \frac{1}{(\mu(B))^{1-\eta}} \int_B |f|d\mu$$

for $y \in B(x,r)$.

Hence by Lemma 2, the Hölder inequality, Lemma 7 we obtain

$$
\begin{aligned}
I_2 &\leqslant C(B(x,r))^{\frac{1}{p_2(x)}-\frac{1}{q_2(x)}}\left[\sup_{B\supset B(x,r)}\frac{1}{(\mu(B))^{1-\eta}}\int_B|f|d\mu\right]\|\chi_{B(x,r)}(\cdot)\|_{L^{q_2(\cdot)}(X)}\\
&\leqslant C(\mu(B(x,r))^{1/p_2(\cdot)}\sup_{B\supset B(x,r)}\frac{1}{(\mu(B))^{1-\eta}}\|f\|_{L^{q_1(\cdot)}(B)}\|\chi_B\|_{L^{q_1'(\cdot)}(B)}\\
&\leqslant C(\mu(B(x,r))^{1/p_2(\cdot)}\sup_{B\supset B(x,r)}\frac{1}{(\mu(B))^{1-\eta}}\|f\|_{L^{q_1(\cdot)}(B)}(\mu(B))^{1/q_{1-}'(B)}\\
&= C(\mu(B(x,r))^{1/p_2(\cdot)}\sup_{B\supset B(x,r)}(\mu(B))^{\frac{1}{p_1(x)}-\frac{1}{p_2(x)}}\|f\|_{L^{q_1(\cdot)}(B)}(\mu(B))^{-1/q_{1-}(B)}\\
&\leqslant C\sup_{B\supset B(x,r)}(\mu(B))^{\frac{1}{p_1(x)}-\frac{1}{q_1(x)}}\|f\|_{L^{q_1(\cdot)}(B)}\\
&\leqslant C\|f\|_{M^{p_1(x)}_{q_1(x)}(X)}.
\end{aligned}
$$

Proof of Theorem 2 : Let r be a small positive number. Decompose f as follows:$f=f_1+f_2$, where $f_1=f\chi_{B(x,2Ar)}$, $f_2=f-f_1$. For $y\in B(x,r)$ and $z\in X\setminus B(x,2Ar)$, we have

$$d(x,z)\leqslant A(d(x,y)+d(y,z))\leqslant Ar+Ad(y,z)\leqslant d(x,z)/2+Ad(y,z).$$

So we obtain

$$\mu(B(x,d(x,z)))\leqslant C\mu(B(x,d(y,z))).$$

In addition, for $t\in B(x,d(y,z))$, we get

$$
\begin{aligned}
d(y,t) &\leqslant A(d(y,z))+Ad(z,t)\\
&\leqslant Ad(y,z)+Ad(z,x)+Ad(x,t)\\
&\leqslant Ad(y,z)+2A^2d(y,z)+Ad(y,z)\\
&= 2A(1+A)d(y,z).
\end{aligned}
$$

Hence, $\mu(B(x,d(y,z)))\leqslant C\mu(B(y,d(y,z)))$. Finally, we obtain

$$\mu(B(x,d(x,z)))\leqslant C\mu(B(x,d(y,z)))\leqslant C\mu(B(y,d(y,z))).$$

Since the reverse doubling condition, we can take integer m and $<\theta<1$ so that θ^m (if $\mathrm{diam}(X)<\infty$, we take $\theta^m\mathrm{diam}(X)$) is sufficiently small. For

$y \in B(x,r)$ we have

$$
\begin{aligned}
&|I_\eta f_2(y)| \\
&\leqslant \int_{X\setminus B(x,2Ar)} \frac{|f(z)|}{\mu(B(x,d(x,z)))^{1-\eta}} d\mu(z) \\
&\leqslant \int_{X\setminus B(x,2Ar)} \int_{B(x,\theta^{m-2}d(x,z))\setminus B(x,\theta^{m-1}d(x,z))} \frac{|f(z)|}{\mu(B(x,d(x,t)))^{2-\eta}} d\mu(t)d\mu(z) \\
&\leqslant \int_{X\setminus B(x,2Ar)} \int_{B(x,\theta^{m-2}d(x,z))\setminus B(x,\theta^{m-1}d(x,z))} \frac{|f(z)|}{\mu(B(x,d(x,t)))^{2-\eta}} d\mu(t)d\mu(z) \\
&\leqslant \int_{X\setminus B(x,2\theta^{m-1}Ar)} \frac{1}{\mu(B(x,d(x,t)))^{2-\eta}} \int_{B(x,\theta^{1-m}d(x,z))} |f(z)|d\mu(z)d\mu(t) \\
&\leqslant \int_{X\setminus B(x,2\theta^{m-1}Ar)} \frac{\bar{f}(x,t)}{\mu(B(x,d(x,t)))} d\mu(t),
\end{aligned}
$$

where

$$
\bar{f}(x,t) = \frac{1}{(\mu(B(x,\theta^{1-m}d(x,t)))^{1-\eta}} \int_{B(x,\theta^{1-m}d(x,t))} |f(z)|d\mu(z).
$$

By Lemma 2 and doubling condition, we get

$$
\begin{aligned}
\bar{f}(x,t) &\leqslant \frac{\|f\|_{L^{q_1(\cdot)}(B(x,\theta^{1-m}d(x,t)))}\|\chi_{B(x,\theta^{1-m}d(x,t))}\|_{L^{q_1'(\cdot)}}}{(\mu(B(x,\theta^{1-m}d(x,t)))^{1-\eta}} \\
&\leqslant \|f\|_{M^{p_1(x)}_{q_1(x)}(X)} (\mu(B(x,\theta^{1-m}d(x,t)))^{-1/q_1'(x)-1/p_1(x)+1/q_1'(x)+\eta} \\
&\leqslant C\|f\|_{M^{p_1(x)}_{q_1(x)}(X)} (\mu(B(x,d(x,t)))^{-1/p_2(x)}.
\end{aligned}
$$

Since $1+p_2(x) > 1$, by Lemma 6 we have

$$
\begin{aligned}
|I_\eta f_2(y)| &\leqslant C\|f\|_{M^{p_1(x)}_{q_1(x)}(X)} \int_{X\setminus B(x,2A\theta^{1-m}r)} \frac{1}{(\mu(B(x,d(x,t)))^{1+1/p_2(x)}} d\mu(t) \\
&\leqslant C\|f\|_{M^{p_1(x)}_{q_1(x)}(X)} (\mu(B(x,r))^{-1/p_2(x)}.
\end{aligned}
$$

Then by Lemma 2 and Lemma 9, we get

$$
\begin{aligned}
&(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|I_\eta f\|_{L^{q_2(x)}(B(x,r))} \\
\leqslant\ &(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|I_\eta f_1\|_{L^{q_2(x)}(B(x,r))} \\
&+(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|I_\eta f_2\|_{L^{q_2(x)}(B(x,r))} \\
\leqslant\ &C(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|f_1\|_{L^{q_1(x)}(B(x,2Ar))} \\
&+(\mu(B(x,r)))^{1/p_2(x)-1/q_2(x)}\|I_\eta f_2\|_{L^{q_2(x)}(B(x,r))} \\
\leqslant\ &C\|f\|_{M^{p_1(x)}_{q_1(x)}(X)} + C(\mu B(x,r))^{-1/q_2(x)}\|\chi_{B(x,r)}\|_{L^{q_2(\cdot)(X)}}\|f\|_{M^{p_1(x)}_{q_1(x)}(X)} \\
\leqslant\ &C\|f\|_{M^{p_1(x)}_{q_1(x)}(X)}.
\end{aligned}
$$

3 Application

Let $\Gamma = \{t \in \mathbb{C} : t = t(s), 0 \leqslant s \leqslant \ell \leqslant \infty\}$ be a connected rectifiable curve and and let ν be an arc-length measure on Γ, that is $\nu(t) = s$. We denote

$$\Gamma(t,r) = \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z - t| < r\}$ is a disc in $\mathbb{C}$ with center t and radius r. Γ is called a Carleson curve (regular curve), if there exists a constant C not depending on t and r, such that

$$\nu(\Gamma(t,r)) \leqslant Cr.$$

If we equip Γ with the measure ν and the Euclidean metric, the regular curve becomes a space of homogeneous type.

The maximal operator is

$$M_\Gamma f(t) = \sup_{r>0} \frac{1}{\nu(\Gamma(t,r))} \int_{\Gamma(t,r)} f(\tau) d\nu(\tau)$$

and the potential type operator is

$$I_\alpha f(t) = \int_\Gamma \frac{f(\tau)}{|t-\tau|^{1-\alpha}} d\nu(\tau)$$

for $0 < \alpha < 1$.

The Cauchy integral is

$$S_\Gamma f(t) = \int_\Gamma \frac{f(\tau)}{t-\tau} d\nu(\tau).$$

The associated kernel is

$$k(z,\omega) = \frac{1}{z-\omega}$$

and it is a Calderón-Zygmund kernel in the case of regular curves.

Definition 4 *Let $1 < q_- \leqslant q(\cdot) \leqslant p(\cdot) \leqslant p_+ < \infty$. We say that a measurable locally integrable function f on Γ belongs to the class $M^{p(\cdot)}_{q(\cdot)}(\Gamma)$ if*

$$\|f\|_{M^{p(\cdot)}_{q(\cdot)}(\Gamma)} = \sup_{t\in\Gamma,r} (\nu(\Gamma(t,r)))^{1/p(t)-1/q(t)} \|f\|_{L^{q(\cdot)}(B(t,r))}.$$

Theorem 1 implies the following statement.

Theorem 3 *Let $1 < q_- \leqslant q(\cdot) \leqslant p(\cdot) \leqslant p_+ < \infty$. Suppose that $p, q \in HL$. Then M_Γ is bounded in $M^{p(x)}_{q(x)}(\Gamma)$.*

Theorem 2 implies the following statement.

Proposition 1 *Let Γ be a regular curve. Suppose $1 < q_- \leqslant q(t) \leqslant p(t) \leqslant p_+ < \infty$ for all $t \in \Gamma$. such that $1/p_1(\cdot), 1/q_1(\cdot) \in LH$, $1 < q_{1-} \leqslant p_1(\cdot) \leqslant p_{1+} < 1/\alpha$ and $q_1(\cdot) \leqslant p_1(\cdot)$. For each $x \in X$, define $1/p_1(x) - 1/p_2(x) = \alpha$ and $1/q_1(x) - 1/q_2(x) = \alpha$. Then I_α is bounded from $M^{p_1(x)}_{q_1(x)}(\Gamma)$ to $M^{p_2(x)}_{q_2(x)}(\Gamma)$.*

Let $K : X \times X \setminus \{(x, x) : x \in X\} \to \mathbb{R}$ be a measurable function satisfying the condition:

(i) $K(x, y) \leqslant \dfrac{C}{\mu(B(x, d(x, y)))}$, $x, y \in X$, $x \neq y$;

(ii) $|K(x_1, y) - K(x_2, y)| + |K(y, x_1) - K(y, x_2)| \leqslant \dfrac{C\omega\left(\frac{d(x_1,x_2)}{d(x_2,y)}\right)}{\mu(B(x_2, d(x_2, y)))}$

for all x_1, x_2 and y with $d(x_2, y) > d(x, y)$, where ω is a positive, non-decreasing function on $(0, \infty)$ satisfying $\omega(2t) < \omega(t)$ for $t > 0$ and the Dini condition $\int_0^1 \omega(t)/t\, dt < \infty$. The definition of Calderrón-Zygmund operator T is

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{X \setminus B(x,\varepsilon)} K(x, y) f(y) d\mu(y)$$

and for some p_0, $1 < p_0 < \infty$ and all $f \in L^{p_0}(X)$ the limit exists almost everywhere on X and T is bounded at $f \in L^{p_0}(X)$.

If we assume $\mathrm{diam}(X) < \infty$ and $\mu\{x\} = 0$, we know the following statement from [11].

Theorem 4 [11] *Let $1 < q_- \leqslant q(\cdot) \leqslant p(\cdot) \leqslant p_+ < \infty$. Suppose that $p, q \in HL_0$. Then T is bounded in $M^{p(x)}_{q(x)}(X)$.*

Assume (X, d, μ) is a reverse doubling spaces and $\mu(X) < \infty$, by lemma 4 and theorem 4 we have

Theorem 5 *Let $1 < q_- \leqslant q(\cdot) \leqslant p(\cdot) \leqslant p_+ < \infty$. Suppose that $p, q \in HL_0$. Then T is bounded in $M^{p(x)}_{q(x)}(X)$.*

By theorem 1 we have

Theorem 6 *Let $1 < q_- \leqslant q(\cdot) \leqslant p(\cdot) \leqslant p_+ < \infty$. Suppose that $p, q \in HL_0$. Then M_Γ is bounded in $M^{p(x)}_{q(x)}(\Gamma)$.*

Let Γ be a subset of $\mathbf{R}^n$ which is an s-set $(0 \leqslant s \leqslant n)$ in the sense that there is a Borel measure μ in $\mathbf{R}^n$ such that

(i) $\operatorname{supp} \mu = \Gamma$;

(ii) there are positive constants C_1 and C_2 such that for all $z \in \Gamma$ and all $r \in (0,1)$,

$$C_1 r^s \leqslant \mu(B(x,r) \cap \Gamma) \leqslant C_2 r^s.$$

More detailed properties of s-sets one can see in [23]. By Theorem 2 we have

Proposition 2 *Suppose I_α is the fractional operator on an s-set Γ. Suppose $1 < q_- \leqslant q(t) \leqslant p(t) \leqslant p_+ < \infty$ for all $t \in \Gamma$, such that $1/p_1(\cdot), 1/q_1(\cdot) \in LH$, $1 < q_{1-} \leqslant p_1(\cdot) \leqslant p_{1+} < 1/\alpha$ and $q_1(\cdot) \leqslant p_1(\cdot)$. For each $x \in X$, define $1/p_1(x) - 1/p_2(x) = \alpha$ and $1/q_1(x) - 1/q_2(x) = \alpha$. Then I_α is bounded from $M^{p_1(x)}_{q_1(x)}(\Gamma)$ to $M^{p_2(x)}_{q_2(x)}(\Gamma)$.*

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Panwang Wang
Department of Mathematics and Physics,
Shijiazhuang Tiedao University
Shijiazhuang, 050043, P. R. China.
panwangw@gmail.com

Zongguang Liu
Department of Mathematics,
China University of Mining and Technology
Beijing 100083, P. R. China.
liuzg@cumtb.edu.cn

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