

Generalized Gibbs Formula and the Duality of Energy and Probability in Models of Statistical Physics

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Abstract. In this paper, we define a generalized Gibbs formula based on the concept of transition energy. This formula establishes a duality between energy and probability. Using this duality, we derive properties of transition energy from known properties of probability.

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Introduction

Duality is a principle that establishes a correspondence between concepts and laws (or other true statements) of one theory and those of its dual theory. Duality appears in different fields of science, particularly in mathematics. For example, in projective geometry, there is a duality between the concepts of a point and a line; in set theory, the operations of union and intersection can be interchanged to obtain equivalent statements. Duality also plays an important role in physics. Well-known examples include wave–particle duality, position–momentum duality, and electric–magnetic duality. Duality takes on a deeper and more fundamental character when it arises between two concepts of different natures, a phenomenon that may be referred to as interdisciplinary duality.

Statistical physics postulates that the possible positions (configurations) of particles of a physical system corresponding to its equilibrium state are random, and the probabilities of such configurations are calculated using the Gibbs formula. This formula establishes the probability–energy relationship and completely determines the statistical behaviour of a physical system. It is the basis (according to Feynman [5], the summit) of statistical physics.

At the same time, the Gibbs formula gave a powerful impetus to the development of both corresponding theories: probability theory was enriched by the theory of Gibbs random fields and the problem of mathematizing thermodynamics was further developed in the form of mathematical statistical physics.

Using the Gibbs formula, one can compute the probability of a state of a system defined in a finite volume knowing the Hamiltonian (potential energy) of this state. Representation of the Hamiltonian as a sum of interaction potential is an essential requirement within the Gibbs approach. It is considered that such a representation is quite justified since an extensive and deep theory has been constructed on this basis.

In its classical form, the Gibbs formula cannot be inverted because the Hamiltonian in this formula is defined up to an additive constant. However, if the classical Gibbs formula could be generalized in such a way as to make it invertible, it would allow one to establish a duality between energy and probability within the framework of mathematical statistical physics.

In this paper, we define a generalized Gibbs formula (see also [7]) based on the concept of transition energy introduced in [2]. This formula establishes a symbiosis, or duality, between the seemingly alien concepts of energy and probability. This duality can be used to obtain results in probability theory based on properties of energy. For example, it can be used to solve Dobrushin's problem of describing a specification by one-point consistent systems of probability distributions parameterized by infinite boundary conditions (see [2]), as well as the problem of describing a finite random field by a consistent system of conditional distributions (see [7,8]). In the present paper, we consider the converse problem: how to derive properties of transition energy from known properties of probability.

The duality between energy and probability is inherent in both finite and infinite systems of mathematical statistical physics. However, in the case of infinite systems, the duality is proved not between two functions (transition energy and probability distribution) but between two systems of finite-volume functions parameterized by infinite boundary conditions: transition energy field and specification. First this connection was obtained in [2]. In the present paper, we will show how to make the passage from finite- to infinite-volume systems based on the Markov–Dobrushin scheme of describing probability distribution by the system of conditional distributions and on the additivity property of the transition energy established in [8].

1 Generalized Gibbs formula

We start with some necessary notations. Let Λ be a finite subset of the integer lattice $\mathbb{Z}^d$, $d \geq 1$. Following physical terminology, Λ is sometimes

called the finite volume. For one-point sets $\{t\}$, $t \in \Lambda$, braces will be omitted.

Let each point $t \in \Lambda$ be associated with a set X^t , which is a copy of some finite set X , $1 < |X| < \infty$, called single-spin space. Further, we define the set of system states as a set X^Λ of configurations on Λ , that is, the set of functions on Λ taking values in X : $X^\Lambda = \{x = (x_t, t \in \Lambda) : x_t \in X^t, t \in \Lambda\}$. By $x_V = (x_t, t \in V)$ we denote the restriction of configuration $x \in X^\Lambda$ on $V \subset \Lambda$.

When considering a system in a finite volume Λ , both probability and energy are functions on X^Λ . Unlike energy, the probability distribution P_Λ is defined by its very simple and well-known properties — it is a non-negative function, sum of whose values is equal to one. In what follows, we will consider only (strictly) positive probability distributions, that is, we always assume that

$$P_\Lambda(x) > 0, x \in X^\Lambda, \quad \sum_{x \in X^\Lambda} P_\Lambda(x) = 1. \quad (1)$$

The Gibbs formula (distribution), which relates the energy $H_\Lambda(x)$ of the state $x \in X^\Lambda$ of a physical system to the probability $P_\Lambda(x)$ of the system being in this state, is the cornerstone of statistical physics and has the following form:

$$P_\Lambda(x) = \frac{\exp\{-H_\Lambda(x)\}}{\sum_{z \in X^\Lambda} \exp\{-H_\Lambda(z)\}}, \quad x \in X^\Lambda. \quad (2)$$

According to physical concepts, the main contribution to potential energy is made by the interaction of microscopic components of the system, and the Hamiltonian in its simplest form is the sum of pairwise interactions of the components

$$H_\Lambda(x) = \sum_{t,s \in V} \Phi(x_t x_s), \quad x \in X^\Lambda, \quad (3)$$

where $\Phi(x_t x_s)$ is the interaction potential of particles located at points t and s in states x_t and x_s , respectively, $t, s \in \Lambda$. In the general case, the interactions of all possible finite groups of components can be taken into account in the Hamiltonian. In this case, it is assumed

$$H_\Lambda(x) = \sum_{\emptyset \neq V \subset \Lambda} \Phi_V(x_V), \quad x \in X^\Lambda.$$

The following natural questions arise:

1. Why probability and energy must be connected through the Gibbs formula?
2. Why function H_Λ defined by (3) is a Hamiltonian? Is there a general (axiomatic) definition of a Hamiltonian?

As Georgii [4, p. 4] mentions, “The rigorous justification of the [Gibbs formula] is a long story which is still far from being finished.” After that, he

adds: “In the meantime, the [Gibbs formula] may be regarded as a postulate which is justified by its stupendous success.” Below we show that it is possible to give purely mathematical justification of the Gibbs formula. This is possible thanks to the concepts of transition energy and generalized Gibbs formula.

First, let us note that Feynman described energy not as a physical substance, but as an abstract numerical quantity that remains constant through all physical changes. We define the transition energy as a function of two variables satisfying a certain property.

A real-valued function Δ_Λ on $X^\Lambda \times X^\Lambda$ is called a *transition energy in* (finite volume) Λ if it satisfies the following condition:

$$\Delta_\Lambda(x, u) = \Delta_\Lambda(x, z) + \Delta_\Lambda(z, u), \quad x, u, z \in X^\Lambda. \quad (4)$$

Such a name is physically valid since relation (4) is a direct consequence of the energy conservation law.

Using the concept of transition energy, we can now define the generalized Gibbs formula connecting probability distribution and transition energy.

Generalized Gibbs formula is a probability distribution on X^Λ represented in the following form:

$$P_\Lambda(x) = \frac{\exp\{\Delta_\Lambda(x, u)\}}{\sum_{z \in X^\Lambda} \exp\{\Delta_\Lambda(z, u)\}}, \quad x, u \in X^\Lambda, \quad (5)$$

where Δ_Λ is a transition energy in Λ .

Note that formula (5) is correct since, due to (4), values of P_Λ do not depend on the choice of configuration $u \in X^\Lambda$. Further, unlike (2), formula (5) is invertible:

$$\Delta_\Lambda(x, u) = \ln \frac{P_\Lambda(x)}{P_\Lambda(u)}, \quad x, u \in X^\Lambda. \quad (6)$$

Formula (6) can also be inverted. Indeed, according to (6), for all $x, u \in X^\Lambda$, we have

$$P_\Lambda(u) = e^{\Delta_\Lambda(u, x)} P_\Lambda(x),$$

and it remains to take the sum of both parts of the obtained expression over all $u \in X^\Lambda$.

Thus, we see that functions (5) and (6) are mutually inverse if and only if they satisfy conditions (1) and (4), respectively. This allows us to formulate the next fundamental result (see [2] as well as [8]).

Theorem 1 [2, Proposition 2.1] *For a function P_Λ on X^Λ to be a (positive) probability distribution, it is necessary and sufficient that it has form (5), where Δ_Λ is the transition energy in Λ . In this case, the transition energy Δ_Λ is uniquely determined by formula (6).*

It is important to note that since the generalized Gibbs formula is invertible, we are now able to derive the characteristic properties of energy based on the known properties of probability. In particular, based on the generalized Gibbs formula, one can understand what the Hamiltonian is without resorting to the notion of potential.

Indeed, since transition energy satisfies (4), there exists a function $H_\Lambda : X^\Lambda \rightarrow \mathbb{R}$ such that

$$\Delta_\Lambda(x, u) = H_\Lambda(u) - H_\Lambda(x), \quad x, u \in X^\Lambda, \quad (7)$$

and H_Λ can be interpreted as a Hamiltonian of a physical system. Thus, in the case of finite volume Λ , Hamiltonian is any function on the set of configurations X^Λ . Thanks to this fact, there are numerous applications of the Gibbs distribution outside statistical physics (pattern recognition, optimization problems, econometrics, etc.). Particularly, if we take H_Λ defined by formula (3), we obtain the classical Gibbs formula. Such a representation of Hamiltonian is very convenient for constructing models (probability distributions) defined using potentials.

Previous reasoning shows that it is more convenient and correct to use the concept of transition energy for constructing a general theory of Gibbs distribution in a finite volume. First of all, as we have seen, there is a one-to-one correspondence between the probability distribution and the transition energy. In addition, unlike the Hamiltonian, the transition energy possess the additivity property (see [8]).

Remark 1 *The function on the right-hand side of (6) has been considered by many authors (see, for example, [9, 10]). However, it was always considered in the form*

$$H_\Lambda(x) = \ln \frac{P_\Lambda(x)}{P_\Lambda(u_0)}, \quad x \in X^\Lambda,$$

where $u_0 \in X^\Lambda$ is some fixed configuration, that is, as a function of one variable. In our approach, we consider the logarithm of the ratio of probabilities as a function of two variables.

Remark 2 *The difference (7) between Hamiltonian values (the relative Hamiltonian) has also been considered in earlier works. For instance, it was used as a technical tool in Pirogov–Sinai theory (see, for example, [10]). However, although this difference was defined for any pair of configurations, in applications, one of the configurations was always fixed as the ground state.*

The main difference between the relative Hamiltonian and the transition energy is as follows. To define the relative Hamiltonian, one first specifies the Hamiltonian as a sum of interactions and then calculates the difference between the values of this function. In contrast, the transition energy is defined axiomatically — as a function satisfying the defining property (4).

2 Duality of energy and probability in finite-volume models of statistical physics

Let $\mathcal{P} = \{P_\Lambda\}$ be the set of all probability distributions on X^Λ and let $\mathcal{D} = \{\Delta_\Lambda\}$ be the set of all transition energies in Λ . Consider the operator $T : \mathcal{P} \rightarrow \mathcal{D}$ which maps an element from $\mathcal{P}$ to an element from $\mathcal{D}$ according to the formula:

$$(TP_\Lambda)(x, u) = \ln \frac{P_\Lambda(x)}{P_\Lambda(u)}, \quad x, u \in X^\Lambda,$$

and the operator $T^{-1} : \mathcal{D} \rightarrow \mathcal{P}$ which maps an element from $\mathcal{D}$ to an element from $\mathcal{P}$ by the formula

$$(T^{-1}\Delta_\Lambda)(x) = \frac{\exp\{\Delta_\Lambda(x, u)\}}{\sum_{z \in X^\Lambda} \exp\{\Delta_\Lambda(z, u)\}}, \quad x \in X^\Lambda,$$

where $u \in X^\Lambda$ is an arbitrary configuration. Due to condition (4), the operator T^{-1} is correctly defined (it does not depend on the choice of u). It is clear that both operators T and T^{-1} depend on Λ , but to simplify the notations, we do not directly specify this dependence.

The following statement holds true.

Theorem 2 [7, Proposition] *Operators T and T^{-1} are mutually inverse, that is, for all $P_\Lambda \in \mathcal{P}$ and $\Delta_\Lambda \in \mathcal{D}$, it holds*

$$T^{-1}TP_\Lambda = P_\Lambda, \quad TT^{-1}\Delta_\Lambda = \Delta_\Lambda.$$

It is easy to see that for any $P_\Lambda \in \mathcal{P}$, function TP_Λ satisfies the characteristic property (4) of transition energies, while for any $\Delta_\Lambda \in \mathcal{D}$, function $T^{-1}\Delta_\Lambda$ satisfies (1), which characterizes a probability distribution. Therefore, any statement about probability P_Λ can be formulated in terms of corresponding transition energy Δ_Λ , and vice versa.

Our next goal is to establish energy-probability duality in the case of infinite-volume systems, that is, to consider the whole integer lattice $\mathbb{Z}^d$ instead of a finite set $\Lambda \subset \mathbb{Z}^d$. We start with some heuristic remarks.

First note that the transition energy for such systems cannot be defined on all pairs of configurations, because on pairs of configurations that differ in an infinite number of points, it will, as a rule, take infinite value. In this connection, already Sinai [10] noted that for infinite-volume systems, the Hamiltonian cannot be defined as a function on $X^{\mathbb{Z}^d}$, where $X^{\mathbb{Z}^d} = \{\bar{x} = (\bar{x}_t, t \in \mathbb{Z}^d) : \bar{x}_t \in X^t, t \in \mathbb{Z}^d\}$. However, it makes sense to consider the “differences of the Hamiltonians” for configurations $\bar{x}, \bar{u} \in X^{\mathbb{Z}^d}$ that differ

in a finite number of points. Let us denote the set of all pairs of such configurations by T .

Further, note that when a system moves from one configuration (state) to another, the energy is spent only on changing the state of the system in the volume in which these configurations are different, while the state of the system in the volume in which they coincide plays the role of a boundary condition. Suppose now that we deal with a transition energy as a function on pairs of configurations from T . If we consider the unchangeable part of the configuration as a fixed (frozen) boundary condition, we obtain a transition energy in a finite volume, which, as we have seen, is dual to a probability distribution, but in this case, this probability distribution is also parameterised by the fixed boundary condition. In the case of the finite volume, such probability distribution parameterised by boundary conditions turn out to be conditional probabilities of some (uniquely determined) joint probability distribution (see [7, 8]).

To make these considerations rigor, let us first consider the Markov–Dobrushin scheme of describing joint probability distribution by a system of consistent probability distribution parameterised by boundary conditions.

3 Markov–Dobrushin scheme

By the Markov–Dobrushin scheme, we mean the approach for describing a joint probability distribution using a family of consistent probability distributions parameterized by boundary conditions. This idea goes back to the concept of the finite Markov chain, and Dobrushin [3] was the first to consider the problem of characterization of a random field on an infinite integer lattice by specification, a system of consistent probability distributions parameterized by infinite boundary conditions.

It should be noted, that the methods used to solve the problem of characterization of a random field by a consistent system of probability distributions parameterized by boundary conditions in finite and infinite volumes are fundamentally different. For example, in the case of finite volumes, the system $\mathbf{P}(P_\Lambda) = \{P_V, V \subsetneq \Lambda\}$ of restrictions $P_V = (P_\Lambda)_V$, $V \subsetneq \Lambda$, (marginal distributions) of the probability measure P_Λ does not restore P_Λ .

In [6], simple necessary and sufficient conditions were obtained under which a given system of distributions is a one-point conditional distribution of a finite random field P_Λ , and an explicit formula was also given expressing P_Λ through the elements of this system. To obtain this result, we used the approach developed in [1], where a solution to Dobrushin’s problem of describing a random field (in infinite volume) by a consistent system of one-point distributions was obtained. Below we provide the solution to the problem of describing finite random fields in a more general settings, namely,

by considering systems of distributions corresponding to some partition of Λ .

Before proceeding to a rigorous formulation of the aforementioned problems, let us introduce some notations. We assume that $X^\emptyset = \{\emptyset\}$ where $\emptyset$ denotes an empty configuration. For any $V, I \subset \Lambda$, $V \cap I = \emptyset$, and any $x \in X^V$, $y \in X^I$, by xy we denote the concatenation of x and y , that is, the configuration on $V \cup I$ equal to x on V and to y on I . In particular, for $y = \emptyset$, we put $x\emptyset = x$.

Let P_Λ be a finite random field, that is, a probability distribution on X^Λ . For any nonempty $V \subsetneq \Lambda$ and $z \in X^{\Lambda \setminus V}$, the conditional probability on X^V under condition z is defined as

$$Q_V^z(x) = \frac{P_\Lambda(xz)}{\sum_{u \in X^V} P_\Lambda(uz)}, \quad x \in X^V.$$

In mathematical statistical physics, the parameter z is usually called the boundary condition. In the future, we will adhere to this terminology.

Note that since we consider positive finite random fields only, all conditional distributions are also strictly positive.

In the case $V = \emptyset$, we can put $Q_\emptyset^z(\emptyset) = 1$ for any $z \in X^\Lambda$, while for $V = \Lambda$, we have

$$Q_\Lambda^\emptyset(x) = \frac{P_\Lambda(x\emptyset)}{\sum_{u \in X^\Lambda} P_\Lambda(u\emptyset)} = \frac{P_\Lambda(x)}{\sum_{u \in X^\Lambda} P_\Lambda(u)} = P_\Lambda(x), \quad x \in X^\Lambda. \quad (8)$$

The family $\mathbf{Q}(P_\Lambda) = \{Q_V^z, z \in X^{\Lambda \setminus V}, V \subsetneq \Lambda\}$ is called the *conditional distribution* of the random field P_Λ .

Let $\Pi = \{V_1, V_2, \dots, V_n\}$ be a partition of Λ , that is, $V_i \cap V_j = \emptyset$, $i \neq j$, and $\bigcup_{i=1}^n V_i = \Lambda$. Denote by $\mathscr{W}_\Pi$ the family of all sets obtained as unions of the elements of Π . Consider families $\mathbf{Q}_\Pi(P_\Lambda) = \{Q_V^z, z \in X^{\Lambda \setminus V}, V \in \Pi\}$ and $\mathbf{Q}_{\mathscr{W}}(P_\Lambda) = \{Q_V^z, z \in X^{\Lambda \setminus V}, V \in \mathscr{W}_\Pi\}$ of conditional probabilities of P_Λ corresponding to the elements of Π and $\mathscr{W}_\Pi$, respectively.

Theorem 3 *For any partition Π and any finite random field P_Λ , system $\mathbf{Q}_\Pi(P_\Lambda)$ restores system $\mathbf{Q}_{\mathscr{W}}(P_\Lambda)$*

Proof. It is sufficient to note that for any $W_1, W_2 \in \mathscr{W}_\Pi$ and $\bar{x} \in X^{\Lambda \setminus (W_1 \cup W_2)}$, we have

$$\frac{Q_{W_1 \cup W_2}^{\bar{x}}(x_1 x_2)}{Q_{W_1 \cup W_2}^{\bar{x}}(u_1 u_2)} = \frac{P_\Lambda(x_1 x_2 \bar{x})}{P_\Lambda(u_1 x_2 \bar{x})} \cdot \frac{P_\Lambda(u_1 x_2 \bar{x})}{P_\Lambda(u_1 u_2 \bar{x})} = \frac{Q_{W_1}^{\bar{x} x_2}(x_1)}{Q_{W_1}^{\bar{x} x_2}(u_1)} \cdot \frac{Q_{W_2}^{\bar{x} u_1}(x_2)}{Q_{W_2}^{\bar{x} u_1}(u_2)},$$

$x_i, u_i \in X^{W_i}$, $i = 1, 2$. Taking into account that

$$\sum_{x_1 \in X^{W_1}, x_2 \in X^{W_2}} Q_{W_1 \cup W_2}^{\bar{x}}(x_1 x_2) = 1,$$

we obtain

$$Q_{W_1 \cup W_2}^{\bar{x}}(x_1 x_2) = \frac{Q_{W_1}^{\bar{x}x_2}(x_1)}{Q_{W_1}^{\bar{x}x_2}(u_1)} \cdot \frac{Q_{W_2}^{\bar{x}u_1}(x_2)}{Q_{W_2}^{\bar{x}u_1}(u_2)} \left(\sum_{\substack{z_1 \in X^{W_1}, \\ z_2 \in X^{W_2}}} \frac{Q_{W_1}^{\bar{x}z_2}(z_1)}{Q_{W_1}^{\bar{x}z_2}(u_1)} \cdot \frac{Q_{W_2}^{\bar{x}u_1}(z_2)}{Q_{W_2}^{\bar{x}u_1}(u_2)} \right)^{-1}.$$

□

Since $\Lambda = \bigcup_{i=1}^n V_i \in \mathscr{V}_\Pi$, from the theorem above we immediately obtain the following result.

Corollary 1 *For any partition Π , the family of conditional probabilities $\mathbf{Q}_\Pi(P_\Lambda)$ restores its generating finite random field P_Λ .*

The following natural question arises: does the system $\mathbf{Q}_\Pi(P_\Lambda)$ possess such properties (consistency conditions) which allow restoring the random field P_Λ without taking into account the fact that the elements of $\mathbf{Q}_\Pi(P_\Lambda)$ are generated by P_Λ ? If such characterizing properties are found, one can expect that for an *a priori* given system $\mathbf{Q}_\Pi$ of distributions satisfying these properties, there exists the *compatible* with it finite random field, that is, such random field P_Λ that $\mathbf{Q}_\Pi(P_\Lambda) = \mathbf{Q}_\Pi$.

Below we present the solution to this problem in terms of systems $\mathbf{Q}_\Pi$ and $\mathbf{Q}_{\mathscr{W}}$.

The system $\mathbf{Q}_\Pi = \{q_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \in \Pi\}$ of (strictly) positive probability distributions $q_V^{\bar{x}}$ on X^V parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus V}$, $V \in \Pi$, will be called Π -*specification* if its elements are consistent in the following sense: for any $V_1, V_2 \in \Pi$ and $\bar{x} \in X^{\Lambda \setminus (V_1 \cup V_2)}$, it holds

$$q_{V_1}^{\bar{x}x_2}(x_1)q_{V_2}^{\bar{x}x_1}(u_2)q_{V_1}^{\bar{x}u_2}(u_1)q_{V_2}^{\bar{x}u_1}(x_2) = q_{V_1}^{\bar{x}x_2}(u_1)q_{V_2}^{\bar{x}u_1}(u_2)q_{V_1}^{\bar{x}u_2}(x_1)q_{V_2}^{\bar{x}x_1}(x_2), \quad (9)$$

$x_i, u_i \in X^{V_i}$, $i = 1, 2$.

The system $\mathbf{Q}_{\mathscr{W}} = \{q_W^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathscr{W}_\Pi\}$ of (strictly) positive probability distributions $q_W^{\bar{x}}$ on X^W parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus W}$, $W \in \mathscr{W}_\Pi$, will be called $\mathscr{W}$ -*specification* if its elements are consistent in the following sense: for any disjoint $W_1, W_2 \in \mathscr{W}_\Pi$ and $\bar{x} \in X^{\Lambda \setminus (W_1 \cup W_2)}$, it holds

$$q_{W_1 \cup W_2}^{\bar{x}}(xy)q_{W_1}^{\bar{x}y}(u) = q_{W_1 \cup W_2}^{\bar{x}}(uy)q_{W_1}^{\bar{x}y}(x), \quad (10)$$

$x, u \in X^{W_1}$, $y \in X^{W_2}$.

Note that the notion of Π -specification generalizes the notion of 1-specification, while $\mathscr{W}$ -specification corresponds to specification.

Theorem 4 *Let $\mathbf{Q}_\Pi$ be a Π -specification. Then there exists a unique $\mathscr{W}$ -specification containing $\mathbf{Q}_\Pi$.*

Proof. The proof is similar to the proof of the corresponding result for 1-specification and specification (see Theorem 19 in [1]). We briefly mention the main idea of the proof.

For any $W \in \mathcal{W}_\Pi$ and $\bar{x} \in X^{\Lambda \setminus W}$, put

$$q_W^{\bar{x}}(x) = \prod_{j=1}^k \frac{q_{V_j}^{\bar{x}(xu)_j}(x_j)}{q_{V_j}^{\bar{x}(xu)_j}(u_j)} \cdot \left(\sum_{z \in X^W} \prod_{j=1}^k \frac{q_{V_j}^{\bar{x}(zu)_j}(z_j)}{q_{V_j}^{\bar{x}(zu)_j}(u_j)} \right)^{-1},$$

where $V_1, V_2, \dots, V_k$ is some enumeration of those elements of the partition Π for which $W = \bigcup_{j=1}^k V_j$, and

$$(xu)_j = x_1 \dots x_{j-1} u_{j+1} \dots u_k, \quad 1 < j < k,$$

$$(xu)_1 = u_2 u_3 \dots u_k, \quad (xu)_k = x_1 x_2 \dots x_{k-1},$$

$x_j = x_{t_j}$, $u_j = u_{t_j}$, $1 \leq j \leq k$, $x, u \in X^W$. Due to the consistency conditions (9), function $q_W^{\bar{x}}$ is correctly defined, that is, its values do not depend on the way of enumeration of sets $V_1, V_2, \dots, V_k \subset W$ and on the choice of configuration u . It is not difficult to see that $q_W^{\bar{x}}$ is a (strictly) positive probability distribution on X^W .

Consider now the system $\mathbf{Q}_{\mathcal{W}} = \{q_W^z, z \in X^{\Lambda \setminus W}, W \in \mathcal{W}_\Pi\}$ of the constructed probability distributions. Direct computations show that its elements are consistent in the sense (10), and therefore, $\mathbf{Q}_{\mathcal{W}}$ is the unique $\mathcal{W}$ -specification containing $\mathbf{Q}_\Pi$. $\square$

Noting again that $\Lambda = \bigcup_{i=1}^n V_i \in \mathcal{W}_\Pi$, we obtain the following result.

Corollary 2 *Let $\mathbf{Q}_\Pi$ be a Π -specification. Then there exists a unique finite random field P_Λ such that $\mathbf{Q}_\Pi(P_\Lambda) = \mathbf{Q}_\Pi$.*

In particular, one can consider one-point partition $\Pi = \{t_1, t_2, \dots, t_N\}$ where $t_1, t_2, \dots, t_N$ is some enumeration of points in Λ , $N = |\Lambda|$. To it corresponds finite-volume 1-specification $\mathbf{Q}_1 = \{q_t^z, z \in X^{\Lambda \setminus t}, t \in \Lambda\}$, that is, a system of positive probability distributions parameterized by boundary conditions and consistent in the following sense: for all $t, s \in \Lambda$ and $x, u \in X^t$, $y, v \in X^s$, $\bar{x} \in X^{\Lambda \setminus \{t, s\}}$, it holds

$$q_t^{zy}(x) q_s^{zx}(v) q_t^{zv}(u) q_s^{zu}(y) = q_t^{zy}(u) q_s^{zu}(v) q_t^{zv}(x) q_s^{zx}(y).$$

By Theorem 4 it uniquely determines the finite-volume analogue of specification $\mathbf{Q} = \{q_W^z, z \in X^{\Lambda \setminus W}, W \subsetneq \Lambda\}$, which is the system of positive probability distributions parameterized by boundary conditions and consistent in the sense (10) for all disjoint sets $W_1, W_2 \subsetneq \Lambda$. Moreover, by Corollary 2,

we conclude that $\mathbf{Q}_1$ uniquely determines compatible with it finite random field P_Λ :

$$P_\Lambda(x) = \prod_{j=1}^n \frac{q_{t_j}^{(xu)_j}(x_j)}{q_{t_j}^{(xu)_j}(u_j)} \cdot \left(\sum_{z \in X^\Lambda} \prod_{j=1}^n \frac{q_{t_j}^{(zu)_j}(z_j)}{q_{t_j}^{(zu)_j}(u_j)} \right)^{-1},$$

where $u \in X^\Lambda$ and $\Lambda = \{t_1, t_2, \dots, t_N\}$ is some enumeration of points in Λ (cf. Theorem 1 in [6]).

4 Transition energy field in finite volume

We have seen that, in the case of finite-volume systems, there is a one-to-one correspondence between a random field and a specification. Therefore, the concept of transition energy, dual to the concept of probability distribution, should have a corresponding analogue dual to the concept of specification. This is so-called transition energy field which was introduced in [2] in the framework of infinite-volume models of statistical physics (see also [6] for the finite-volume case).

Let P_Λ be a probability distribution on X^Λ and Δ_Λ be the corresponding transition energy in Λ . Consider conditional distribution of P_Λ , $\mathbf{Q}(P_\Lambda) = \{Q_{\bar{V}}, \bar{x} \in X^{\Lambda \setminus V}, V \subsetneq \Lambda\}$. For any $V \subset \Lambda$ and $\bar{x} \in X^{\Lambda \setminus V}$, choosing some $u \in X^V$, we can write

$$Q_{\bar{V}}^{\bar{x}}(x) = \frac{P_\Lambda(x\bar{x})}{\sum_{z \in X^V} P_\Lambda(z\bar{x})} = \frac{\exp\{\Delta_\Lambda(x\bar{x}, u\bar{x})\}}{\sum_{z \in X^V} \exp\{\Delta_\Lambda(z\bar{x}, u\bar{x})\}} = \frac{\exp\{\Delta_{\bar{V}}^{\bar{x}}(x, u)\}}{\sum_{z \in X^V} \exp\{\Delta_{\bar{V}}^{\bar{x}}(z, u)\}},$$

$x \in X^V$, where

$$\Delta_{\bar{V}}^{\bar{x}}(x, u) = \Delta_\Lambda(x\bar{x}, u\bar{x}), \quad x, u \in X^V. \quad (11)$$

In the case $V = \emptyset$, we assume that $\Delta_{\bar{V}}^{\bar{x}}(\emptyset, \emptyset) = 0$ for any $\bar{x} \in X^\Lambda$. Note also, that if we take $V = \Lambda$ in (11), we will have

$$\Delta_\Lambda^\emptyset(x, u) = \Delta_\Lambda(x\emptyset, u\emptyset) = \Delta_\Lambda(x, u), \quad x, u \in X^\Lambda,$$

which is in full correspondence with (8).

Formula for $Q_{\bar{V}}^{\bar{x}}$ can be inverted, and we obtain

$$\Delta_{\bar{V}}^{\bar{x}}(x, u) = \ln \frac{Q_{\bar{V}}^{\bar{x}}(x)}{Q_{\bar{V}}^{\bar{x}}(u)}, \quad x, u \in X^V.$$

Hence, for each $\bar{x} \in X^{\Lambda \setminus V}$, function $\Delta_{\bar{V}}^{\bar{x}}$ satisfies

$$\Delta_{\bar{V}}^{\bar{x}}(x, u) = \Delta_{\bar{V}}^{\bar{x}}(x, y) + \Delta_{\bar{V}}^{\bar{x}}(y, u), \quad x, u, y \in X^V.$$

The value $\Delta_V^{\bar{x}}(x, u)$ corresponds to the amount of energy required to transfer the system from state x to state u in part V of volume Λ , considering the unchangeable part $\bar{x}$ of the configuration of the system as a fixed boundary condition (environment). The function $\Delta_V^{\bar{x}}$ is called a *transition energy in V with boundary condition $\bar{x} \in X^{\Lambda \setminus V}$* .

Thus, for any transition energy Δ_Λ in Λ , it corresponds the system $\mathbf{D}(\Delta_\Lambda) = \{\Delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \subsetneq \Lambda\}$ of transition energies in sub-volumes $V \subsetneq \Lambda$ with boundary conditions $\bar{x} \in X^{\Lambda \setminus V}$.

Now, let $\Pi = \{V_1, V_2, \dots, V_n\}$ be some partition of Λ and $\mathbf{Q}_\Pi = \{q_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \in \Pi\}$ be a Π -specification. According to Theorem 4, $\mathbf{Q}_\Pi$ uniquely determines $\mathscr{W}$ -specification $\mathbf{Q}_{\mathscr{W}} = \{q_W^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathscr{W}_\Pi\}$ containing $\mathbf{Q}_\Pi$. Since for any fixed $W \in \mathscr{W}_\Pi$ and $\bar{x} \in X^{\Lambda \setminus W}$, function $q_W^{\bar{x}}$ is a positive probability distribution on X^W , the function

$$\delta_W^{\bar{x}}(x, u) = \ln \frac{q_W^{\bar{x}}(x)}{q_W^{\bar{x}}(u)}, \quad x, u \in X^W, \quad (12)$$

is properly defined and satisfies

$$\delta_W^{\bar{x}}(x, u) = \delta_W^{\bar{x}}(x, y) + \delta_W^{\bar{x}}(y, u), \quad x, u, y \in X^W. \quad (13)$$

Hence, $\delta_W^{\bar{x}}$ is the transition energy in W with boundary condition $\bar{x} \in X^{\Lambda \setminus W}$.

Consider now the system $\mathbf{D}_{\mathscr{W}}(\mathbf{Q}_{\mathscr{W}}) = \{\delta_W^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathscr{W}_\Pi\}$ of transition energies with boundary conditions defined by (12). Due to consistency conditions (10), we conclude that the elements of $\mathbf{D}_{\mathscr{W}}(\mathbf{Q}_{\mathscr{W}})$ satisfy

$$\delta_{W_1 \cup W_2}^{\bar{x}}(xy, uy) = \delta_{W_1}^{\bar{x}y}(u), \quad x, u \in X^{W_1}, y \in X^{W_2}, \quad (14)$$

for all $W_1, W_2 \in \mathscr{W}_\Pi$ and $\bar{x} \in X^{\Lambda \setminus (W_1 \cup W_2)}$.

The family $\mathbf{D}_{\mathscr{W}} = \{\delta_W^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathscr{W}_\Pi\}$ of transition energies parameterized by boundary conditions, that is, functions $\delta_W^{\bar{x}}$ on $X^W \times X^W$, $W \in \mathscr{W}_\Pi$, satisfying (13), is called a *transition energy field in Λ under partition Π* if its elements are consistent in the sense (14).

Now, for a given transition energy field $\mathbf{D}_{\mathscr{W}}$ in Λ under partition Π , define the system $\mathbf{Q}_{\mathscr{W}}(\mathbf{D}_{\mathscr{W}}) = \{q_W^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathscr{W}_\Pi\}$ of functions

$$q_W^{\bar{x}}(x) = \frac{\exp\{\delta_V^{\bar{x}}(x, u)\}}{\sum_{z \in X^V} \exp\{\delta_V^{\bar{x}}(z, u)\}}, \quad x \in X^V, \quad (15)$$

where $u \in X^V$. These functions are properly defined (their values do not depend on the choice of configuration u), for a fixed $W \in \mathscr{W}_\Pi$ and $\bar{x} \in X^{\Lambda \setminus W}$, $q_W^{\bar{x}}$ is a (strictly) positive probability distribution on X^W , and the elements of $\mathbf{Q}_{\mathscr{W}}(\mathbf{D}_{\mathscr{W}})$ are consistent in the sense (10). Hence, $\mathbf{Q}_{\mathscr{W}}(\mathbf{D}_{\mathscr{W}})$ forms a $\mathscr{W}$ -specification.

The above discussion shows that the following result holds true.

Theorem 5 *The system $\mathbf{Q}_{\mathcal{W}} = \{q_{\bar{W}}^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathcal{W}_{\Pi}\}$ of positive functions parameterized by boundary conditions is a $\mathcal{W}$ -specification if and only if its elements have the Gibbs form (15), where $\mathbf{D}_{\mathcal{W}} = \{\delta_{\bar{W}}^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathcal{W}_{\Pi}\}$ is a transition energy field in Λ under partition Π .*

The similar considerations remain true if we consider a subsystem of $\mathbf{Q}_{\mathcal{W}}$, namely, Π -specification $\mathbf{Q}_{\Pi}$. To it corresponds the subsystem of transition energy field $\mathbf{D}_{\mathcal{W}}$, which we denote $\mathbf{D}_{\Pi} = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \in \Pi\}$. Due to the duality between the elements of $\mathbf{Q}_{\mathcal{W}}$ and $\mathbf{D}_{\mathcal{W}}$, we conclude that for all $V \in \Pi$ and $\bar{x} \in X^{\Lambda \setminus V}$, it holds

$$q_V^{\bar{x}}(x) = \frac{\exp\{\delta_V^{\bar{x}}(x, u)\}}{\sum_{z \in X^V} \exp\{\delta_V^{\bar{x}}(z, u)\}}, \quad \delta_V^{\bar{x}}(x, u) = \ln \frac{q_V^{\bar{x}}(x)}{q_V^{\bar{x}}(u)}, \quad (16)$$

for all $x, u \in X^V$. Moreover, since the elements of $\mathbf{Q}_{\Pi}$ satisfy consistency conditions (9), the elements of $\mathbf{D}_{\Pi}$ must be consistent in the following sense: for any $V_1, V_2 \in \Pi$ and $\bar{x} \in X^{\Lambda \setminus (V_1 \cup V_2)}$, it holds

$$\delta_{V_1}^{\bar{x}x_2}(x_1, u_1) + \delta_{V_2}^{\bar{x}u_1}(x_2, u_2) = \delta_{V_2}^{\bar{x}x_1}(x_2, u_2) + \delta_{V_1}^{\bar{x}u_2}(x_1, u_1), \quad (17)$$

$x_i, u_i \in X^{V_i}$, $i = 1, 2$.

The family $\mathbf{D}_{\Pi} = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \in \Pi\}$ of transition energies in the elements of Π parameterized by boundary conditions and consistent in the sense (17) is called an *elementary transition energy field in Λ under partition Π* .

Theorem 4, stating connection between Π - and $\mathcal{W}$ -specifications, allows us to prove the following important statement: transition energies in finite volumes with boundary conditions are additive functions. That is, to compute the transition energy in a finite volume, one should add the transition energies of the sub-volumes forming a partition of the volume, with an appropriate change in the boundary conditions. For another proof of this statement, see Theorem 3 in [8].

Theorem 6 *Let Π be a partition of Λ and $\mathbf{D}_{\Pi}$ be the corresponding elementary transition energy field. Then there exists the unique transition energy field $\mathbf{D}_{\mathcal{W}} = \{\delta_W^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus W}, W \in \mathcal{W}_{\Pi}\}$ in Λ under partition Π such that for any $W = \bigcup_{j=1}^k V_{i_j}$, $V_{i_j} \in \Pi$, $\bar{x} \in X^{\Lambda \setminus W}$ and $x, u \in X^W$, it holds*

$$\delta_W^{\bar{x}}(x, u) = \delta_{V_{i_1}}^{\bar{x}x_2 \dots x_k}(x_1, u_1) + \delta_{V_{i_2}}^{\bar{x}u_1 x_3 \dots x_k}(x_2, u_2) + \dots + \delta_{V_{i_k}}^{\bar{x}u_1 \dots u_{k-1}}(x_k, u_k),$$

and therefore, $\mathbf{D}_{\Pi} \subset \mathbf{D}_{\mathcal{W}}$. Here z_j denotes the restriction of configuration $z \in X^W$ on V_{i_j} , $1 \leq j \leq k$.

Proof. Let $\mathbf{Q}_\Pi$ be the Π -specification corresponding to $\mathbf{D}_\Pi$. It uniquely determines $\mathscr{W}$ -specification $\mathbf{Q}_{\mathscr{W}}$, which, in its turn, is dual to the transition energy field $\mathbf{D}_{\mathscr{W}}$.

Further, let $W \in \mathscr{W}_\Pi$ and let $V_1, V_2, \dots, V_k$ be some enumeration of the elements of partition Π for which $W = \bigcup_{j=1}^k V_j$. Then for any $\bar{x} \in X^{\Lambda \setminus W}$ and $x, u \in X^W$, denoting for brevity $x_j = x_{V_j}$, $u_j = u_{V_j}$, $1 \leq j \leq k$, we can write

$$\begin{aligned} \delta_W^{\bar{x}}(x, u) &= \ln \frac{q_W^{\bar{x}}(x)}{q_W^{\bar{x}}(u)} = \ln \frac{q_W^{\bar{x}}(x_1 x_2 \dots x_k)}{q_W^{\bar{x}}(u_1 x_2 \dots x_k)} + \ln \frac{q_W^{\bar{x}}(u_1 x_2 x_3 \dots x_k)}{q_W^{\bar{x}}(u_1 u_2 x_3 \dots x_k)} + \dots \\ &\quad + \ln \frac{q_W^{\bar{x}}(u_1 u_2 \dots u_{k-1} x_k)}{q_W^{\bar{x}}(u_1 u_2 \dots u_{k-1} u_k)} = \\ &= \ln \frac{q_{V_1}^{\bar{x} x_2 \dots x_k}(x_1)}{q_{V_1}^{\bar{x} x_2 \dots x_k}(u_1)} + \ln \frac{q_{V_2}^{\bar{x} u_1 x_3 \dots x_k}(x_2)}{q_{V_2}^{\bar{x} u_1 x_3 \dots x_k}(u_1)} + \dots + \ln \frac{q_{V_k}^{\bar{x} u_1 \dots u_{k-1}}(x_k)}{q_{V_k}^{\bar{x} u_1 \dots u_{k-1}}(u_k)} = \\ &= \delta_{V_1}^{\bar{x} x_2 \dots x_k}(x_1, u_1) + \delta_{V_2}^{\bar{x} u_1 x_3 \dots x_k}(x_2, u_2) + \dots + \delta_{V_k}^{\bar{x} u_1 \dots u_{k-1}}(x_k, u_k), \end{aligned}$$

where we sequentially used (12), (10) and (16). $\square$

In particular, since for any partition $\Pi = \{V_1, V_2, \dots, V_n\}$ of Λ , we have $\Lambda = \bigcup_{j=1}^n V_j \in \mathscr{W}_\Pi$, the system $\mathbf{D}_{\mathscr{W}}$ contains the transition energy $\Delta_\Lambda^\emptyset = \Delta_\Lambda$ in Λ . Therefore, the following result holds true.

Corollary 3 [8, Corollary 2] *Any elementary transition energy field $\mathbf{D}_\Pi$ corresponding to some partition Π of Λ uniquely determines transition energy Δ_Λ in Λ such that for any $V \in \Pi$ and $\bar{x} \in X^{\Lambda \setminus V}$, it holds*

$$\Delta_\Lambda(x\bar{x}, u\bar{x}) = \delta_V^{\bar{x}}(x, u), \quad x, u \in X^V.$$

In this case, for all $x, u \in X^\Lambda$,

$$\Delta_\Lambda^{\bar{x}}(x, u) = \delta_{V_1}^{\bar{x} x_{\Lambda \setminus V_1}}(x_{V_1}, u_{V_1}) + \delta_{V_2}^{\bar{x} u_{V_1} x_{\Lambda \setminus (V_1 \cup V_2)}}(x_{V_2}, u_{V_2}) + \dots + \delta_{V_n}^{\bar{x} u_{\Lambda \setminus V_n}}(x_{V_n}, u_{V_n}).$$

The case when the elements of Π are one-point subsets of Λ is of a special interest since it allows constructing transition energies in finite volumes starting from transition energies in points parameterized by boundary conditions.

5 Energy-probability duality in the case of infinite-volume systems

We now consider the case of infinite volume, that is, instead of the finite set Λ , we will consider $\mathbb{Z}^d$.

In his seminal paper [3], Dobrushin considered the problem of describing random fields (probability measures on the infinite space of configurations $X^{\mathbb{Z}^d}$) by a system of consistent finite-dimensional distributions parameterized by infinite boundary conditions. Later, such systems were called specifications.

Specification is a system $\mathbf{Q} = \{q_{\Lambda}^{\bar{x}}, \bar{x} \in X^{\Lambda^c}, \Lambda \Subset \mathbb{Z}^d\}$ of probability distributions $q_{\Lambda}^{\bar{x}}$ on X^{Λ} parameterized by infinite boundary conditions $\bar{x} \in X^{\Lambda^c}$, $\Lambda \Subset \mathbb{Z}^d$, and consistent in the following sense: for any $I \subset \Lambda \Subset \mathbb{Z}^d$ and $\bar{x} \in X^{\Lambda^c}$, it holds

$$q_{\Lambda}^{\bar{x}}(xy) = q_I^{\bar{x}y}(x) \sum_{u \in X^I} q_{\Lambda}^{\bar{x}}(uy), \quad x \in X^I, y \in X^{\Lambda \setminus I}. \quad (18)$$

Here $\Lambda \Subset \mathbb{Z}^d$ means that Λ is a finite subset of $\mathbb{Z}^d$ and $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$.

It is not difficult to see that consistency conditions (18) can be written in the following equivalent form: for any $I \subset \Lambda \Subset \mathbb{Z}^d$, $\bar{x} \in X^{\Lambda^c}$ and $x, u \in X^I$, $y \in X^{\Lambda \setminus I}$, it holds

$$q_{\Lambda}^{\bar{x}}(xy)q_I^{\bar{x}y}(u) = q_{\Lambda}^{\bar{x}}(uy)q_I^{\bar{x}y}(x). \quad (19)$$

In what follows, we will consider positive specifications only, that is, systems of consistent (strictly) positive distributions.

Contrary to the finite case, specification does not always define a random field for which it is a version of conditional distributions. Moreover, if such a random field exists, it may not be unique (see Theorems 1 and 2 in [3]).

Let $\mathbf{Q} = \{q_{\Lambda}^{\bar{x}}, \bar{x} \in X^{\Lambda^c}, \Lambda \Subset \mathbb{Z}^d\}$ be a specification. Since for any $\Lambda \Subset \mathbb{Z}^d$ and $\bar{x} \in X^{\Lambda^c}$, the element $q_{\Lambda}^{\bar{x}}$ of $\mathbf{Q}$ is a probability distribution on X^{Λ} , according to Theorem 1, to it correspond the transition energy $\Delta_{\Lambda}^{\bar{x}}$ such that

$$q_{\Lambda}^{\bar{x}}(x) = \frac{\exp\{\Delta_{\Lambda}^{\bar{x}}(x, u)\}}{\sum_{z \in X^{\Lambda}} \exp\{\Delta_{\Lambda}^{\bar{x}}(z, u)\}}, \quad \Delta_{\Lambda}^{\bar{x}}(x, u) = \ln \frac{q_{\Lambda}^{\bar{x}}(x)}{q_{\Lambda}^{\bar{x}}(u)}, \quad x, u \in X^{\Lambda}.$$

Varying $\Lambda \Subset \mathbb{Z}^d$ and $\bar{x} \in X^{\Lambda^c}$, we obtain a system $\mathbf{\Delta} = \{\Delta_{\Lambda}^{\bar{x}}, \bar{x} \in X^{\Lambda^c}, \Lambda \Subset \mathbb{Z}^d\}$ of transition energies $\Delta_{\Lambda}^{\bar{x}}$ in Λ parameterized by infinite boundary conditions $\bar{x} \in X^{\Lambda^c}$, $\Lambda \Subset \mathbb{Z}^d$. Since the elements of $\mathbf{Q}$ are consistent in the sense (19), the elements of $\mathbf{\Delta}$ must satisfy the appropriate consistency conditions. Indeed, for any $I \subset \Lambda \Subset \mathbb{Z}^d$ and $\bar{x} \in X^{\Lambda^c}$, we have

$$\Delta_{\Lambda}^{\bar{x}}(xy, uy) = \ln \frac{q_{\Lambda}^{\bar{x}}(xy)}{q_{\Lambda}^{\bar{x}}(uy)} = \ln \frac{q_I^{\bar{x}y}(x)}{q_I^{\bar{x}y}(u)} = \Delta_I^{\bar{x}y}(x, u),$$

$x, u \in X^I, y \in X^{\Lambda \setminus I}$.

The system $\mathbf{\Delta} = \{\Delta_{\Lambda}^{\bar{x}}, \bar{x} \in X^{\Lambda^c}, \Lambda \Subset \mathbb{Z}^d\}$ of transition energies $\Delta_{\Lambda}^{\bar{x}}$ in Λ parameterized by infinite boundary conditions $\bar{x} \in X^{\Lambda^c}$, $\Lambda \Subset \mathbb{Z}^d$, is called a

transition energy field (see [2]) if its elements are consistent in the following sense: for any $I \subset \Lambda \Subset \mathbb{Z}^d$, $\bar{x} \in X^{\Lambda^c}$ and $x, u \in X^I$, $y \in X^{\Lambda \setminus I}$, it holds

$$\Delta_{\Lambda}^{\bar{x}}(xy, uy) = \Delta_I^{\bar{x}y}(x, u).$$

Thus, in the case of infinite volume, the duality between energy and probability takes place in the form of one-to-one correspondence between transition energy field and specification.

Theorem 7 [2, Theorem 2.1] *A family $\mathbf{Q} = \{q_{\Lambda}^{\bar{x}}, \bar{x} \in X^{\Lambda^c}, \Lambda \Subset \mathbb{Z}^d\}$ of functions in finite volumes with infinite boundary conditions will be a (strictly) positive specification if and only if its elements have the Gibbs form*

$$q_{\Lambda}^{\bar{x}}(x) = \frac{\exp\{\Delta_{\Lambda}^{\bar{x}}(x, u)\}}{\sum_{z \in X^{\Lambda}} \exp\{\Delta_{\Lambda}^{\bar{x}}(z, u)\}}, \quad x \in X^{\Lambda},$$

where $\Delta = \{\Delta_{\Lambda}^{\bar{x}}, \bar{x} \in X^{\Lambda^c}, \Lambda \Subset \mathbb{Z}^d\}$ is a transition energy field uniquely determined by $\mathbf{Q}$ and $u \in X^{\Lambda}$.

6 Concluding remarks

As we have already mentioned, duality is the principle according to which any true statement of one theory corresponds to a true statement in the dual theory. In particular, this means that knowing properties of one of the dual objects allows establishing properties of the second one.

In the case of energy–probability duality, the additivity of transition energies makes it possible, for infinite-volume systems, to solve Dobrushin’s problem of describing a specification by means of 1–specification (see [2]), and, in the case of finite-volume systems, to find the conditions under which a given system of probability distributions parameterized by boundary conditions is a system of conditional probabilities for some joint probability distribution (see [8]).

On the other hand, knowing the properties of probability distributions, one can obtain properties of the corresponding transition energies. In this paper, we have shown how this principle can be applied to derive the notion of a transition energy field. However, this question requires further study. One possible direction is to consider, instead of the probabilities of individual configurations, the probabilities of events. This would lead to the notion of transition energy between sets of configurations. It is possible that such transition energies may also find applications in the study of quantum systems.

References

- [1] S. Dachian and B.S. Nahapetian, Description of specifications by means of probability distributions in small volumes under condition of very weak positivity. *J. Stat. Phys.*, **117** (2004), pp. 281–300. <https://doi.org/10.1023/b:joss.0000044069.91072.0b>
- [2] S. Dachian and B.S. Nahapetian, On the relationship of energy and probability in models of classical statistical physics. *Markov Processes Relat. Fields*, **25** (2019), no. 4, pp. 649–681.
- [3] R.L. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity. *Theory Probab. Appl.*, **13** (1968), no. 2, pp. 197–224. <https://doi.org/10.1137/1113026>
- [4] H.-O. Georgii, *Gibbs measures and phase transitions*, De Gruyter, Berlin, 1988.
- [5] R. Feynman, *Statistical Mechanics: A Set of Lectures*. W.A. Benjamin, Inc., 1972.
- [6] L.A. Khachatryan and B.S. Nahapetian, On the characterization of a finite random field by conditional distribution and its Gibbs form. *J. Theor. Probab.*, **36** (2023), pp. 1743–1761. <https://doi.org/10.1007/s10959-022-01209-6>
- [7] L.A. Khachatryan and B.S. Nahapetian, Duality of energy and probability in finite-volume models of statistical physics. *Reports of NAS RA*, **123** (2023), no. 3-4, pp. 7–14. <http://math.sci.am/10.54503/0321-1339-2023.123.3-4-7>
- [8] L.A. Khachatryan and B.S. Nahapetian, Additivity of transition energy and its applications. *Armen. J. Math.*, **18** (2026), no. 7, pp. 1–19. <https://doi.org/10.52737/18291163-2025.18.07-1-19>
- [9] C. Preston, *Random Fields*. Springer-Verlag, Berlin, 1976. <https://doi.org/10.1007/BFb0080563>
- [10] Ya.G. Sinai, *Theory of phase transitions — Rigorous results*. Pergamon, Oxford, 1982.

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