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Generalized Viscosity Approximation Methods of Ishikawa Type for Nonexpansive Mappings in Hilbert Spaces

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Abstract. In this paper, by using generalized viscosity mappings, we prove two strong convergence theorems for finding fixed points of a nonexpansive mapping which is also a unique solution of the variational inequality. Our results extend and improve the recent ones announced by some authors.

Key Words: Fixed point; nonexpansive mapping; contractive mapping; convergence theorem
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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|.\|$. Let C be a nonempty closed convex subset of H and $S : C \longrightarrow C$ be a self-mapping on C.
We denote by $F(S)$ the set of fixed points of S and P_C the metric projection of H onto C and a mapping S is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.

A mapping $T : C \longrightarrow C$ with $F(T) \neq \varnothing$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

It is well known that the set of fixed point of a quasi-noexpansive mapping T is closed and convex, see [13].

A mapping $G : C \longrightarrow C$ is a contraction if there exists a constant $\alpha \in (0, 1)$ such that $\|Gx - Gy\| \leq \alpha\|x - y\|$ for all $x \in F(T)$ and $y \in C$.

Some iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first iteration process is Halpern iteration [2].

The following strong convergence theorem of Halpern's type was proved by Wittmann [14]: for any $x_1 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

for all $n \in \mathbb{N} \cup \{0\}$, where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in [0,1], $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ and T is a nonexpansive mapping. Then $\{x_n\}$ converges strongly to a fixed point of T.

The second iteration process is introduced by Mann [12]. We also know the following weak convergence theorem of Mann's type: for any $x_1 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

for all $n \in \mathbb{N} \cup \{0\}$, where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in [0,1] and $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.

The third iteration process is presented by Ishikawa [10], that is

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases}$$

where T is Lipshitzian pseudo-contractive mapping from C into itself. The initial guess x_0 is taken in C arbitrarily and the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are in $[0, 1)$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

In 2000, Moudafi [1] introduced the viscosity approximation method for nonexpansive mappings as following iteration algorithm :

$$x_{n+1} = \alpha_n Gx_n + (1 - \alpha_n)Sx_n,$$

where G is a contraction of C into itself, S is a nonexpansive mapping and $\alpha_n \in (0, 1)$ is a slowly vanishing sequence.

The variational inequality problem is to find a point $x_0 \in C$ such that

$$\langle Gx_0, x - x_0\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

In recent years the theory of variational inequality has been extended to the study of a larg variety of problems arising in structural analysis, economics, engineering sciences, and so on. For more details see [3], [4] and [8].

Peichao Duan and Songnian He [9], introduced a generalized iterative method like viscosity approximation. They prove two following theorems:

Theorem 1 *Let C be a nonempty closed and convex subset of a real Hilbert space H and f_n be a sequence of ρ_n- contractive selfmaps of C with $0 \leq \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n = \rho_u < 1$. Let $S : C \longrightarrow C$ be a nonexpansive mapping. Assume that $F(S) \neq \varnothing$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is a bounded subset of C. Given $x_1 \in C$, let $\{x_n\}$ be generated by the following algorithm:*

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n)Sx_n,$$

where, $\{\alpha_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $x_0 \in F(S)$, which is also the unique solution of the variational inequality.

Theorem 2 *Let C be a nonempty closed and convex subset of a real Hilbert space H and f_n be a sequence of ρ_n- contractive selfmaps of C with $0 \leq \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n = \rho_u < 1$. Let for each $1 \leq i \leq N$; $N \in \mathbb{N}$, $S_i : C \longrightarrow C$ be a nonexpansive mapping. Assume that $F = \bigcap_{i=1}^{N} F(S_i) \neq \varnothing$ and $f_n(w)$ is convergent for any $w \in F$. Given $x_1 \in C$, let $\{x_n\}$ be generated by the following algorithm:*

$$\begin{cases} x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n)S_N^n S_{N-1}^n \dots S_1^n x_n, \\ S_i^n = (1 - \lambda_i^n)I + \lambda_i^n S_i, \quad i = 1, 2, \dots, N. \end{cases}$$

If the parameters $\{\alpha_n\}$ and $\{\lambda_i^n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\lambda_i^n \in (\lambda_l, \lambda_u)$ *for some* $\lambda_l, \lambda_u \in (0, 1)$ *and* $\lim_{n\to\infty} |\lambda_i^n - \lambda_i^{n+1}| = 0, \quad i = 1, 2, \dots, N$,

then the sequence $\{x_n\}$ converges strongly to a point $x_0 \in F(S)$ which is also the unique solution of the variational inequality:

$$\langle f(x_0) - x_0, p - x_0\rangle \leq 0, \qquad \forall p \in F,$$

where

$$f(x_0) := \lim_{n\to\infty} f_n(x_0). \tag{2}$$

In this paper, we prove Theorem 1 and Theorem 2 by Ishikawa iteration scheme that extend and improve those theorems and all of the results that have been obtained in [9].

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. In a Hilbert space H it is known that

(i) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$,

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle \quad \forall x, y \in H$,

for all $x, y \in H$ and for all $t \in [0,1]$. see, [13].

Let $\{x_n\}$ be a sequence in H and $x \in H$. Weak convergence of $\{x_n\}$ is denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \to x$. Let H be a nonempty closed convex subset of a real Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, for all $x \in H$ and $y \in C$

$$\|x - P_C x\| \leq \|x - y\|.$$

A mapping $T : C \longrightarrow C$ is said to be $\lambda-$averaged by K if

$$T = (1-\lambda)I + \lambda K.$$

Lemma 1 ([5]) *Let $\{S_i\}_{i=1}^2$ be γ_i-averaged on C and such that $F(S_1) \bigcap F(S_2) \neq \varnothing$. Then we have*

(i) S_1S_2 and S_2S_1 are γ-averaged, where $\gamma = \gamma_1 + \gamma_2 - \gamma_1\gamma_2$,

(ii) $F(S_1) \bigcap F(S_2) = F(S_1S_2) = F(S_2S_1)$.

Lemma 2 ([6]) *Let C be a nonempty closed and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in c$, we have $y = P_C x$ if and only if*

$$\langle x - y, y - z \rangle \geq 0 \quad \textit{for all } z \in C.$$

Lemma 3 ([15]) *Let $\{b_n\}$ is a sequence of nonnegative real numbers such that*

$$b_{n+1} \leq (1-\alpha_n)b_n + \theta_n,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\theta_n\}$ is a sequence such that,

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\theta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\theta_n| < \infty$,

Then $\lim_{n\to\infty} b_n = 0$.

Lemma 4 ([11]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in Banach space and let $\{\beta_n\}$ be a sequence of $[0,1]$ such that*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 5 ([7]) *Demiclosedness principle. Let C be a nonempty closed convex subset of real Hilbert space H and S be a nonexpansive mapping of C into itself, and $F(S) \neq \varnothing$. Then $I - S$ is demiclosed. That is, if sequence $\{x_n\} \subset C$ converges weakly to some $x \in C$ and the sequence $\{(I-S)x_n\}$ strongly converges to y, then $(I-S)x = y$, where I is the identity operator of H.*

3 Main Result

In this section motivated by Peichao Duan and Songnian He, we prove the following strong convergence theorems of Ishikawa type that extend and improve all of the results of [9].

Theorem 3 *Let C be a nonempty closed and convex subset of a real Hilbert space H, and $\{G_n\}$ be a sequence of selfmaps of C such that $\{G_n\}$ is μ_n-contractive with $0 \leq \mu_l = \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n = \mu_u < 1$. Let $S, T : C \longrightarrow C$ be nonexpansive mappings. Assume that $F = F(S) \bigcap F(T) \neq \varnothing$ and $\{G_n\}$ is uniformly convergent on a bounded subset B of C. Let $x_1 \in C$ and define the sequence $\{x_n\} \subset C$ as follows:*

$$\begin{cases} x_{n+1} = \alpha_n G_n x_n + \beta_n x_n + \gamma_n S y_n, \\ y_n = \delta_n T x_n + (1-\delta_n) x_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1)$. If we have the following conditions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$,

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(c) $\lim_{n\to\infty} \gamma_n \neq 0,\ \lim_{n\to\infty} \delta_n \neq 0\ \lim_{n\to\infty} \mid \delta_n - \delta_{n+1} \mid = 0$,

(d) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to a point $w \in F$, which is also the unique solution of the variational inequality 2.

Proof. We divide the proof into 5 steps.

Step 1: We show that $\{x_n\}$ is bounded.
For any $w \in F$, we have

$$\begin{aligned}\|y_n - w\| &= \|\delta_n T x_n + (1-\delta_n)x_n - w\| \\ &\leq \delta_n \|Tx_n - w\| + (1-\delta_n)\|x_n - w\| \\ &\leq \delta_n \|x_n - w\| + (1-\delta_n)\|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{3}$$

So, from (3) we have

$$\begin{aligned}&\|x_{n+1} - w\| = \|\alpha_n G_n x_n + \beta_n x_n + \gamma_n S y_n - w\| \\ &\leq \alpha_n \|G_n x_n - w\| + \beta_n \|x_n - w\| + \gamma_n \|S y_n - w\| \\ &\leq \alpha_n \|G_n x_n - G_n w\| + \alpha_n \|G_n w - w\| + \beta_n \|x_n - w\| + \gamma_n \|S y_n - w\| \\ &\leq \alpha_n \|G_n x_n - G_n w\| + \alpha_n \|G_n w - w\| + \beta_n \|x_n - w\| + \gamma_n \|y_n - w\| \\ &\leq \alpha_n \mu_n \|x_n - w\| + \beta_n \|x_n - w\| + \gamma_n \|x_n - w\| + \alpha_n \|G_n w - w\| \\ &\leq \alpha_n \mu_n \|x_n - w\| + (1-\alpha_n)\|x_n - w\| + \alpha_n \|G_n w - w\| \\ &\leq (1 - \alpha_n(1-\mu_n))\|x_n - w\| + \alpha_n(1-\mu_n)\frac{\|G_n w - w\|}{(1-\mu_n)}. \end{aligned} \tag{4}$$

Since $\{G_n\}$ is uniformly convergent on B, $\{G_n w\}$ is bounded. Thus, there exist a positive constant M_1, such that $\|G_n w - w\| \leq M_1$. Put $L = \max\{\|x_1 - w\|, \frac{M_1}{1-\mu_u}\}$. Now by induction we show that $\|x_n - w\| \leq L$. Suppose that $\|x_k - w\| \leq L$ for some $k \in \mathbb{N}$. From (4), we have

$$\begin{aligned}\|x_{k+1} - w\| &\leq (1 - \alpha_k(1-\mu_k))\|x_k - w\| + \alpha_k(1-\mu_k)\frac{M_1}{1-\mu_u} \\ &\leq (1 - \alpha_k(1-\mu_k))L + \alpha_k(1-\mu_k)L = L.\end{aligned}$$

So, $\|x_n - w\|$ is bounded. Thus $\{x_n\}$, $\{y_n\}$, $\{Sy_n\}$, $\{Sx_n\}$ and $\{Tx_n\}$ are also bounded.

Step 2: We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Indeed, we define $V_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$. So we have,

$$x_{n+1} = (1-\beta_n)V_n + \beta_n x_n, \tag{5}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}V_{n+1}-V_n &= \frac{\alpha_{n+1}G_nx_{n+1}+\gamma_{n+1}Sx_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nG_nx_n+\gamma_nSy_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}G_{n+1}x_{n+1}+(1-\alpha_{n+1}-\beta_{n+1})Sy_{n+1}}{1-\beta_{n+1}}\\ &\quad -\frac{\alpha_nG_nx_n+(1-\alpha_n-\beta_n)Sy_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}[G_{n+1}x_{n+1}-Sy_{n+1}]-\frac{\alpha_n}{1-\beta_n}[G_nx_n-Sy_n]\\ &\quad +Sy_{n+1}-Sy_n.\end{aligned}$$

Therefore, we have for $x\in\{x_n\}$

$$\begin{aligned}\|V_{n+1}-V_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|G_{n+1}x_{n+1}-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|G_nx_n-Sy_n\|\\ &\quad +\|Sy_{n+1}-Sy_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|G_{n+1}x_{n+1}-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|G_nx_n-Sy_n\|+\|y_{n+1}-y_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|G_{n+1}x_{n+1}-G_{n+1}x\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|G_{n+1}x-Sy_{n+1}\|\\ &\quad +\frac{\alpha_n}{1-\beta_n}\|G_nx_n-G_nx\|+\frac{\alpha_n}{1-\beta_n}\|G_nx-Sy_n\|+\|y_{n+1}-y_n\|\\ &\le \mu_{n+1}\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|G_{n+1}x-Sy_{n+1}\|\\ &\quad +\mu_n\frac{\alpha_n}{1-\beta_n}\|x_n-x\|+\frac{\alpha_n}{1-\beta_n}\|G_nx-Sy_n\|+\|y_{n+1}-y_n\| \end{aligned}\tag{6}$$

On the other hand, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|\delta_{n+1}Tx_{n+1}+(1-\delta_{n+1})x_{n+1}-\delta_nTx_n-(1-\delta_n)x_n\|\\ &= \|\delta_{n+1}Tx_{n+1}-\delta_{n+1}Tx_n+\delta_{n+1}Tx_n-\delta_nTx_n+(1-\delta_{n+1})x_{n+1}\\ &\quad -(1-\delta_{n+1})x_n+(1-\delta_{n+1})x_n-(1-\delta_n)x_n\|\\ &\le \delta_{n+1}\|Tx_{n+1}-Tx_n\|+|\delta_{n+1}-\delta_n|\|Tx_n\|+(1-\delta_{n+1})\|x_{n+1}-x_n\|\\ &\quad +|\delta_{n+1}-\delta_n|\|x_n\|\\ &\le \delta_{n+1}\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\|Tx_n\|+(1-\delta_{n+1})\|x_{n+1}-x_n\|+|\delta_{n+1}\\ &\quad -\delta_n|\|x_n\|\\ &\le \|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\|Tx_n\|+|\delta_{n+1}-\delta_n|\|x_n\|. \end{aligned}\tag{7}$$

from (6) and (7) we have

$$\begin{aligned}\|V_{n+1}-V_n\| &\le \mu_{n+1}\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|G_{n+1}x-Sy_{n+1}\|\\ &\quad +\mu_n\frac{\alpha_n}{1-\beta_n}\|x_n-x\|+\frac{\alpha_n}{1-\beta_n}\|G_nx-Sy_n\|+\|x_{n+1}-x_n\|\\ &\quad +|\delta_{n+1}-\delta_n|\|Tx_n\|+|\delta_{n+1}-\delta_n|\|x_n\|. \end{aligned}\tag{8}$$

Since $\{G_n\}$ is uniformly convergent on B, $\{G_n x\}$ is bounded. Now since $\{Sy_n\}$ is also bounded, there exists a positive constant M_2, such that

$$\|G_n x - Sy_n\| \leq M_2. \tag{9}$$

Therefore from (8) and (9) we have

$$\begin{aligned}\|V_{n+1} - V_n\| \leq \mu_n \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \frac{\alpha_{n+1}M_2}{1-\beta_{n+1}} + \mu_n \frac{\alpha_n}{1-\beta_n}\|x_n - x\| \\ + \frac{\alpha_n M_2}{1-\beta_n} + \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|Tx_n\| + |\delta_{n+1} - \delta_n|\|x_n\|. \end{aligned}\tag{10}$$

Since $\{x_n\}$ is bounded, it follows from conditions (b), (c) and (10)

$$\limsup_{n\longrightarrow\infty}(\|V_{n+1} - V_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 4, that $\lim_{n\longrightarrow\infty}\|V_n - x_n\| = 0$. From (5), we obtain that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| &= \lim_{n\longrightarrow\infty}\|(1-\beta_n)V_n + \beta_n x_n - x_n\| \\ &= \lim_{n\longrightarrow\infty}(1-\beta_n)\|V_n - x_n\| = 0. \end{aligned}\tag{11}$$

Thus $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3: We claim that $\lim_{n\longrightarrow\infty}\|Tx_n - x_n\| = 0$. and $\lim_{n\longrightarrow\infty}\|Sx_n - x_n\| = 0$.
For $w \in F$, we have

$$\begin{aligned}\|x_{n+1} - w\|^2 &= \|\alpha_n G_n x_n + \beta_n x_n + \gamma_n S y_n - w\|^2 \\ &\leq \alpha_n\|G_n x_n - w\|^2 + \beta_n\|x_n - w\|^2 + \gamma_n\|Sy_n - w\|^2 \\ &\leq \alpha_n\|G_n x_n - G_n w + G_n w - w\|^2 + \beta_n\|x_n - w\|^2 + \gamma_n\|Sy_n - w\|^2 \\ &\leq \alpha_n\|G_n x_n - G_n w\|^2 + \alpha_n\|G_n w - w\|^2 + 2\alpha_n\langle G_n x_n - G_n w, G_n w - w\rangle \\ &\quad + \beta_n\|x_n - w\|^2 + \gamma_n\|\delta_n T x_n + (1-\delta_n)x_n - w\|^2 \\ &\leq \alpha_n\|G_n x_n - G_n w\|^2 + \alpha_n\|G_n w - w\|^2 + 2\alpha_n\langle G_n x_n - G_n w, G_n w - w\rangle \\ &\quad + \beta_n\|x_n - w\|^2 + \delta_n\gamma_n\|Tx_n - w\|^2 + \gamma_n(1-\delta_n)\|x_n - w\|^2 \\ &\quad - \gamma_n\delta_n(1-\delta_n)\|Tx_n - x_n\|^2 \\ &\leq \alpha_n\mu_n\|x_n - w\|^2 + \alpha_n\|G_n w - w\|^2 + 2\alpha_n\|G_n x_n - G_n w\|\|G_n w - w\| \\ &\quad + \beta_n\|x_n - w\|^2 + \delta_n\gamma_n\|x_n - w\|^2 + \gamma_n(1-\delta_n)\|x_n - w\|^2 \\ &\quad - \gamma_n\delta_n(1-\delta_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - w\|^2 + \alpha_n\|G_n w - w\|^2 + 2\alpha_n\|G_n x_n - G_n w\|\|G_n w - w\| \\ &\quad - \gamma_n\delta_n(1-\delta_n)\|Tx_n - x_n\|^2. \end{aligned}$$

So, we have

$$\begin{aligned}\gamma_n\delta_n(1-\delta_n)\|Tx_n-x_n\|^2 \le \|x_n-w\|^2 - \|x_{n+1}-w\|^2 + \alpha_n\|G_nw-w\|^2 \\ + 2\alpha_n\|G_nx_n-G_nw\|\|G_nw-w\|.\end{aligned}$$

From condition (a) and (c) we have

$$\lim_{n\longrightarrow\infty}\|Tx_n-x_n\|=0. \tag{12}$$

Since $\{G_n\}$ is μ_n-contractive, by (9) we note that, for $x\in\{x_n\}$

$$\begin{aligned}\|x_n-Sy_n\| &\le \|x_{n+1}-x_n\| + \|x_{n+1}-Sy_n\| \\ &= \|x_{n+1}-x_n\| + \|\alpha_nG_nx_n+\beta_nx_n+\gamma_nSy_n-Sy_n\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\|G_nx_n-Sy_n\| + \beta_n\|x_n-Sy_n\| + \gamma_n\|Sy_n-Sy_n\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\|G_nx_n-Sy_n\| + \beta_n\|x_n-Sy_n\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\|G_nx_n-G_nx\| + \alpha_n\|G_nx-Sy_n\| + \beta_n\|x_n-Sy_n\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\mu_u\|x_n-x\| + \alpha_n\|G_nx-Sy_n\| + \beta_n\|x_n-Sy_n\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\mu_u\|x_n-x\| + \alpha_nM_2 + \beta_n\|x_n-Sy_n\|.\end{aligned}$$

So,

$$(1-\beta_n)\|x_n-Sy_n\| \le \|x_{n+1}-x_n\| + \alpha_n\mu_n\|x_n-x\| + \alpha_nM_2.$$

Therefore from condition (b), (d) and (11)

$$\lim_{n\to\infty}\|x_n-Sy_n\|=0. \tag{13}$$

On the other hand,

$$\begin{aligned}\|y_n-x_n\| &= \|\delta_nTx_n+(1-\delta_n)x_n-x_n\| \\ &\le \delta_n\|Tx_n-x_n\| + (1-\delta_n)\|x_n-x_n\| \\ &= \delta_n\|Tx_n-x_n\|.\end{aligned}$$

So from condition (12)

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{14}$$

But, we have

$$\begin{aligned}\|Sx_n-x_n\| &\le \|Sx_n-Sy_n\| + \|Sy_n-x_n\| \\ &\le \|x_n-y_n\| + \|Sy_n-x_n\|.\end{aligned}$$

So, from (13) and (14), we have

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0. \tag{15}$$

Step 4: We show that $\limsup_{n\longrightarrow\infty}\langle Gw-w, x_n-w\rangle \leq 0$,, where $w = P_{F(S)\cap F(T)}Gw$ is a unique solution of the variational inequality (2). Since $\{G_nx\}$ is uniformly convergent on B, we have $\lim_{n\longrightarrow\infty}(G_nw-w) = Gw-w$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle Gw-w, x_n-w\rangle = \lim_{i\longrightarrow\infty}\langle Gw-w, x_{n_i}-w\rangle. \tag{16}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_k}} \rightharpoonup x_0$. We can assume that $x_{n_i} \rightharpoonup x_0$. Therefore from (15), $Sx_{n_i} \rightharpoonup x_0$ and by Lemma 5, we have $x_0 \in F(S)\bigcap F(T)$ and since $w = P_{F(S)\cap F(T)}Gw$, we have

$$\lim_{i\longrightarrow\infty}\langle Gw-w, x_{n_i}-w\rangle = \langle Gw-w, x_0-w\rangle \leq 0.$$

So, from (16), we have $\limsup_{n\longrightarrow\infty}\langle Gw-w, x_n-w\rangle \leq 0$.

Step 5: We prove that $\lim_{n\to\infty}\|x_n-w\| = 0$.
Notice that, from (3)

$$\begin{aligned}
&\|x_{n+1}-w\|^2 = \langle \alpha_nG_nx_n+\beta_nx_n+\gamma_nSy_n-w, x_{n+1}-w\rangle\\
&= \alpha_n\langle G_nx_n-w, x_{n+1}-w\rangle + \beta_n\langle x_n-w, x_{n+1}-w\rangle\\
&\quad + \gamma_n\langle Sy_n-w, x_{n+1}-w\rangle\\
&\leq \alpha_n\langle G_nx_n-G_nw, x_{n+1}-w\rangle + \alpha_n\langle G_nw-w, x_{n+1}-w\rangle\\
&\quad + \beta_n\|x_n-w\|\|x_{n+1}-w\| + \gamma_n\|Sy_n-w\|\|x_{n+1}-w\|\\
&\leq \alpha_n\|G_nx_n-G_nw\|\|x_{n+1}-w\| + \alpha_n\langle G_nw-w, x_{n+1}-w\rangle\\
&\quad + \beta_n\|x_n-w\|\|x_{n+1}-w\| + \gamma_n\|Sy_n-w\|\|x_{n+1}-w\|\\
&\leq \alpha_n\mu_n\|x_n-w\|\|x_{n+1}-w\| + \alpha_n\langle G_nw-w, x_{n+1}-w\rangle\\
&\quad + \beta_n\|x_n-w\|\|x_{n+1}-w\| + \gamma_n\|Sy_n-w\|\|x_{n+1}-w\|\\
&\leq \alpha_n\mu_n\|x_n-w\|\|x_{n+1}-w\| + \alpha_n\langle G_nw-w, x_{n+1}-w\rangle\\
&\quad + (\beta_n+\gamma_n)\|x_n-w\|\|x_{n+1}-w\|\\
&= [1-\alpha_n(1-\mu_n)]\|x_n-w\|\|x_{n+1}-w\| + \alpha_n\langle G_nw-w, x_{n+1}-w\rangle\\
&\leq \frac{1-\alpha_n(1-\mu_n)}{2}(\|x_n-w\|^2+\|x_{n+1}-w\|^2) + \alpha_n\langle G_nw-w, x_{n+1}-w\rangle\\
&\leq \frac{1-\alpha_n(1-\mu_n)}{2}\|x_n-w\|^2 + \frac{1}{2}\|x_{n+1}-w\|^2 + \alpha_n\langle G_nw-w, x_{n+1}-w\rangle
\end{aligned}$$

So, we have

$$\|x_{n+1}-w\|^2 = [1-\alpha_n(1-\mu_n)]\|x_n-w\|^2 + 2\alpha_n\langle G_nw-w, x_{n+1}-w\rangle.$$

Therefore from step 4 and lemma 3

$$\lim_{n\to\infty}\|x_n-w\| = 0.$$

And from (14)

$$\lim_{n\to\infty} \|y_n - w\| = 0.$$

□

Remark 1 *Theorem 3 improves Theorem 1 by Peichao Duan and Songnian He. It is sufficient to put $T = I$ and $\beta_n = 0$ in Theorem 3.*

Theorem 4 *Let C be a nonempty closed and convex subset of a real Hilbert space H and $\{G_n\}$ be a sequence of self maps of C such that $\{G_n\}$ is μ_n-contractive with $0 \le \mu_l = \liminf_{n\to\infty} \mu_n \le \limsup_{n\to\infty} \mu_n = \mu_u < 1$. Let $S_i, T : C \longrightarrow C$ be nonexpansive mappings where $1 \le i \le N$, $N \in \mathbb{N}$. Assume that $F = (\bigcap_{i=1}^{n} F(S_i)) \bigcap F(T) \neq \varnothing$ and $\{G_n w\}$ is convergent for any $w \in F$. Let $x_1 \in C$ and we define $\{x_n\}$ as follows:*

$$\begin{cases} x_{n+1} = \alpha_n G_n x_n + \beta_n x_n + \gamma_n S_N^n S_{N-1}^n \dots S_1^n y_n, \\ y_n = \delta_n T x_n + (1-\delta_n) x_n, \\ S_i^n = (1-\lambda_i^n) I + \lambda_i^n S_i \qquad i = 1, 2, \dots, N, \end{cases}$$

where

(a) $\{\alpha_n\} \subset [0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lambda_i^n \in (0,1) \quad i = 1, 2, \dots, N$,

(c) $\lim_{n\to\infty} \gamma_n \neq 0$, $\lim_{n\to\infty} \delta_n \neq 0$ $\lim_{n\to\infty} \mid \delta_n - \delta_{n+1} \mid = 0$,

(d) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to a point $w \in F$, which is also the unique solution of the variational inequality (2).

Proof. It is sufficient to prove the theorem in case $N = 2$. The proof is divided into 5 steps.
Step 1: We show that $\{x_n\}$ is bounded.
First we show that S_i^n for $i = 1, 2, \dots, N$ is nonexpansive because

$$\begin{aligned} \|S_i^n x - S_i^n y\| &= \|(1-\lambda_i^n)x + \lambda_i^n S_i x - (1-\lambda_i^n) y - \lambda_i^n S_i y\| \\ &\le (1-\lambda_i^n)\|x-y\| + \lambda_i^n \|S_i x - S_i y\| \\ &\le (1-\lambda_i^n)\|x-y\| + \lambda_i^n \|x-y\| \\ &= \|x-y\| \end{aligned} \tag{17}$$

So, for any $w \in F$, from (3) and (17) we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \|\alpha_n G_n x_n + \beta_n x_n + \gamma_n S_N^n S_{N-1}^n \dots S_1^n y_n - w\| \\
&\leq \alpha_n \|G_n x_n - w\| + \beta_n \|x_n - w\| + \gamma_n \|S_N^n S_{N-1}^n \dots S_1^n y_n - w\| \\
&\leq \alpha_n \|G_n x_n - G_n w\| + \alpha_n \|G_n w - w\| + \beta_n \|x_n - w\| \\
&\quad + \gamma_n \|S_1^n y_n - w\| \\
&\leq \alpha_n \mu_n \|x_n - w\| + \alpha_n \|G_n w - w\| + \beta_n \|x_n - w\| + \gamma_n \|y_n - w\| \\
&\leq \alpha_n \mu_n \|x_n - w\| + \alpha_n \|G_n w - w\| + \beta_n \|x_n - w\| + \gamma_n \|x_n - w\| \\
&\leq \alpha_n \mu_n \|x_n - w\| + (1 - \alpha_n) \|x_n - w\| + \alpha_n \|G_n w - w\| \\
&\leq (1 - \alpha_n (1 - \mu_n)) \|x_n - w\| + \alpha_n (1 - \mu_n) \frac{\|G_n w - w\|}{1 - \mu_n}.
\end{aligned}$$

Similarly to step 1 of proof Theorem 3, $\|x_n - w\|$ is bounded, thus $\{x_n\}$, $\{y_n\}$, and $\{S_2^n S_1^n y_n\}$ are bounded.

Step 2: We show that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

It is sufficient to put $S = S_2^n S_1^n$ in Step 2 from Theorem 3.

Step 3: We claim that $\lim_{n \longrightarrow \infty} \|S_2^n S_1^n x_n - x_n\| = 0$.

It is sufficient to put $S = S_2^n S_1^n$ in step 3. from Theorem 3.

Step 4: We show $\limsup_{n \longrightarrow \infty} \langle Gw - w, x_n - w \rangle \leq 0$.

It is sufficient to put $S = S_2^n S_1^n$ in step 4. from Theorem 3.

Step 5: We prove $\lim_{n \to \infty} \|x_n - w\| = 0$.

It is sufficient to put $S = S_2^n S_1^n$ in step 5 from Theorem 3. $\square$

Remark 2 *Theorem 4 improves Theorem 2 by Peichao Duan and Songnian He. It is sufficient to put $T = I$ and $\beta_n = 0$ in Theorem 4.*

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