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Frequency detection using Padé approximation

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Abstract. The current paper describes an application of Padé approximation to obtain a frequency detection method similar to (and, in a sense, more general than) linear prediction. While generally not applicable for spectrum estimation, the described method is in many cases more precise in finding the frequencies of a harmonic signal. A great number of numerical tests were performed for comparing frequency detection performance of the method under consideration and of linear prediction. Their methodology and results are summarized in the article.

Key words: frequency detection, Padé approximation, linear prediction
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1 Introduction

Frequency detection is one of the best known and the most studied topics in one-dimensional signal processing. One of the possible approaches to that problem is using linear prediction to estimate the spectrum of the signal and then finding the poles of the estimated filter to find frequencies. Some authors have studied variations of this approach, see [1].

The current paper applies a somewhat different technique to arrive at what can be viewed as a generalization of the above mentioned method of linear prediction. Tests performed by the authors show this method favorably when compared to the basic linear-prediction based one.

2 Theoretical motivation for the method

Let us first recall that the spectrum of a discrete signal $x = (x_n)$ is the 2π-periodic function

$$S(e^{i\omega}) = \sum_{n=-\infty}^{\infty} r_n e^{-i\omega n}, \tag{1}$$

where

$$r_n = r_n(x) = \sum_{k=-\infty}^{\infty} x_k \bar{x}_{k-n} \tag{2}$$

is the n-th autocorrelation of (x_n). Setting $z = e^{i\omega}$ we can rewrite (1) as

$$S(z) = \sum_{n=-\infty}^{\infty} r_n z^{-n}. \tag{3}$$

The spectrum is thus a Laurent series in variable $e^{i\omega}$ and one can try to approximate it by a rational function using the Padé approximation technique. If we approximate the series by a rational function

$$S_{[L,M]}(z) = \frac{\sum_{n=0}^{L} a_n z^{-n}}{\sum_{n=0}^{M} b_n z^{-n}}$$

then, under suitable conditions, the poles of $S_{[L,M]}$ tend to the poles of S. As we are approximating a Laurent series, we will instead try to approximate (3) by the function $z^{-a} S_{[L,M]}$.

Thus, we are looking for a solution of the following approximate equation:

$$\sum_{n=-\infty}^{\infty} r_n z^{-n} \approx z^{-a} \frac{\sum_{n=0}^{L} a_n z^{-n}}{\sum_{n=0}^{M} b_n z^{-n}}. \tag{4}$$

For specificity, we assume $b_0 = 1$. After suitable modifications, we obtain from (4) the following approximate equation:

$$\sum_{n=-\infty}^{\infty} \left(\sum_{m=0}^{M} b_m r_{a+n-m} \right) z^{-n} \approx \sum_{n=0}^{L} a_n z^{-n}. \tag{5}$$

Following Padé [3], we will solve (5) by equating the first $L+M+1$ coefficients of powers of z, thus obtaining the following system of linear equations:

$$\sum_{m=1}^{M} b_m r_{a+n-m} = -r_{a+n}, \qquad n = L+1, L+2, \ldots, L+M. \tag{6}$$

$$a_n = \sum_{m=0}^{M} b_m r_{a+n-m}, \qquad n = 0, 1, \ldots, L. \tag{7}$$

In matrix form, these equations are written as

$$\begin{pmatrix} r_{a+L} & r_{a+L-1} & \cdots & r_{a+L-M+1} \\ r_{a+L+1} & r_{a+L} & \cdots & r_{a+L-M+2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{a+L+M-1} & r_{a+L+M-2} & \cdots & r_{a+L} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix} = - \begin{pmatrix} r_{a+L+1} \\ r_{a+L+2} \\ \vdots \\ r_{a+L+M} \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} r_a & r_{a-1} & \cdots & r_{a-M} \\ r_{a+1} & r_a & \cdots & r_{a-M+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{a+L} & r_{a+L-1} & \cdots & r_{a-M+L} \end{pmatrix} \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_M \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_L \end{pmatrix}. \tag{9}$$

We are only interested in the poles of the right side of (4), thus the coefficients a_n are not interesting. (Numerical tests show that the suggested method indeed is not usable for general spectrum estimation.) The equation (8) depends only on $a + L$ and, after denoting $p = a + L$, it can be rewritten as

$$\begin{pmatrix} r_p & r_{p-1} & \dots & r_{p-M+1} \\ r_{p+1} & r_p & \dots & r_{p-M+2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p+M-1} & r_{p+M-2} & \dots & r_p \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix} = - \begin{pmatrix} r_{p+1} \\ r_{p+2} \\ \vdots \\ r_{p+M} \end{pmatrix}. \tag{10}$$

For $a = -L$ (i. e. $p = 0$) the system of equations (6) is exactly the Yule-Walker system for solving the autocorrelation method of the linear prediction model. Thus, the method that we propose can be considered as a generalization of the linear prediction method.

By choosing different values of p in the equation (10), we get different estimates of the location of frequencies in (x_n). We have experimentally determined that the value giving best frequency estimates is $p = M$.

We suppose that the measured signal (x_n) is a sum of an "ideal" signal $(\tilde{x}_n)$ and an error signal (e_n):

$$x_n = \tilde{x}_n + e_n.$$

Then the autocorrelations $r_n(x)$ are expressed as follows:

$$r_n(x) = r_n(\tilde{x}) + r_n(\tilde{x}, e) + r_n(e, \tilde{x}) + r_n(e),$$

where $r_n(x, y) = \sum_{k=-\infty}^{\infty} x_k \bar{y}_{k-n}$. If (e_n) is white noise, then its autocorrelations are all 0 except for $r_0(e)$. One can also assume that noise has little correlation with the signal $(\tilde{x}_n)$, thus the autocorrelation $r_n(x)$ is a good approximation for $r_n(\tilde{x})$ except for $n = 0$. Thus, it seems reasonable to avoid $r_0(x)$ for frequency detection — and the above empirically optimal value $p = M$ is just large enough to avoid $r_0(x)$ in (10).

We will call the described method with $p = M$ the *Padé method* for frequency detection. In the following section, the Padé method is compared to the linear prediction method in a large number of numerical tests.

3 Numerical tests and results

The authors have conducted a large number of automated tests using simulated signals which are compositions of harmonic oscillations of different frequencies, amplitudes and phases, as well as Gaussian white noise. More precisely, for given values of the sample count N, number of significant frequencies M_f and signal-to-noise ratio SNR, a large number of signals with randomly chosen frequencies, amplitudes, phases and white noise were generated and tested using the Padé method, as well as using the linear prediction model (equivalently, using the same algorithm with $p = 0$ instead of $p = M$).

The poles of the approximating functions were then computed. When the order M of the denominator of (4) is significantly greater than the number M_f of frequencies in the signal, we will find more poles than it is necessary. Here, the Padé method has a drawback when it is compared to linear prediction — under the linear prediction model, the excessive poles are further from the unit circle. The Padé method can give wrong poles which are closer to the unit circle than the "real" ones. To discriminate between the poles, we then use Görtzel's algorithm (see [2]) to find and compare the values

$$\left|\sum x_n e^{-i\omega_k n}\right|^2,$$

where ω_k is the argument of the k-th pole of (4). The ω_k's where the values are bigger are declared to be the frequencies found.

In our tests, a frequency is proclaimed to be successfully detected if it is found with an error of less than 0.01. Table 1 shows root mean square errors of the detected frequencies and percentages of frequencies successfully found using linear prediction and the described method. 1000 tests were performed for each (N, SNR) combination. In these tests, $M = 4$ and each signal consists of 2 harmonic oscillations and a specified amount of white noise.

N	SNR	RMS (LP)	RMS (Padé)	% found (LP)	% found (Padé)
100	40	9.4883e-006	1.5201e-006	90.5	97.25
500	40	2.6949e-006	5.0914e-007	97.25	98.65
900	40	2.1684e-006	3.6917e-007	98.9	99.15
100	20	1.1522e-005	4.9392e-006	85.75	89.8
500	20	5.2589e-006	2.552e-006	90.85	95.95
900	20	5.5216e-006	2.2685e-006	88.75	94.2
100	0	2.8288e-005	2.9601e-005	33.6	38.75
500	0	2.5441e-005	1.9471e-005	32.9	49.1
900	0	2.4932e-005	1.6779e-005	32.15	53.65

Table 1: Performance of LP and Padé frequency detection methods of order 4 on signals composed of 2 exponents

The figures below show the ratio of the root mean square detection error of the Padé method to that of the LP method. Again, 1000 tests were performed for each test point. As it can be seen from Table 1 and Figures 1, 2, 3, in all of the presented cases the Padé method gives slightly to significantly better results than linear prediction.

The tests were programed and performed in MATLAB.

4 Conclusion

As it can be seen from the table and graphs above, the proposed Padé method of frequency detection is in many real-life cases more precise than the linear prediction method. Computationally the Padé method is slightly more demanding than linear prediction. It requires computation of $2M$ autocorrelation coefficients, whereas the Toeplitz matrix of the Yule-Walker equation is Hermitian and hence computation of $M+1$ coefficients is sufficient. The equation (10) can be solved efficiently in both cases.

It should also be mentioned that the proposed method is not suitable for spectrum estimation in general. If a signal is generated by passing white noise through a zero-pole filter, then, if the filter has poles far from the unit circle, the Padé method gives quite unpredictable and unusable results.

References

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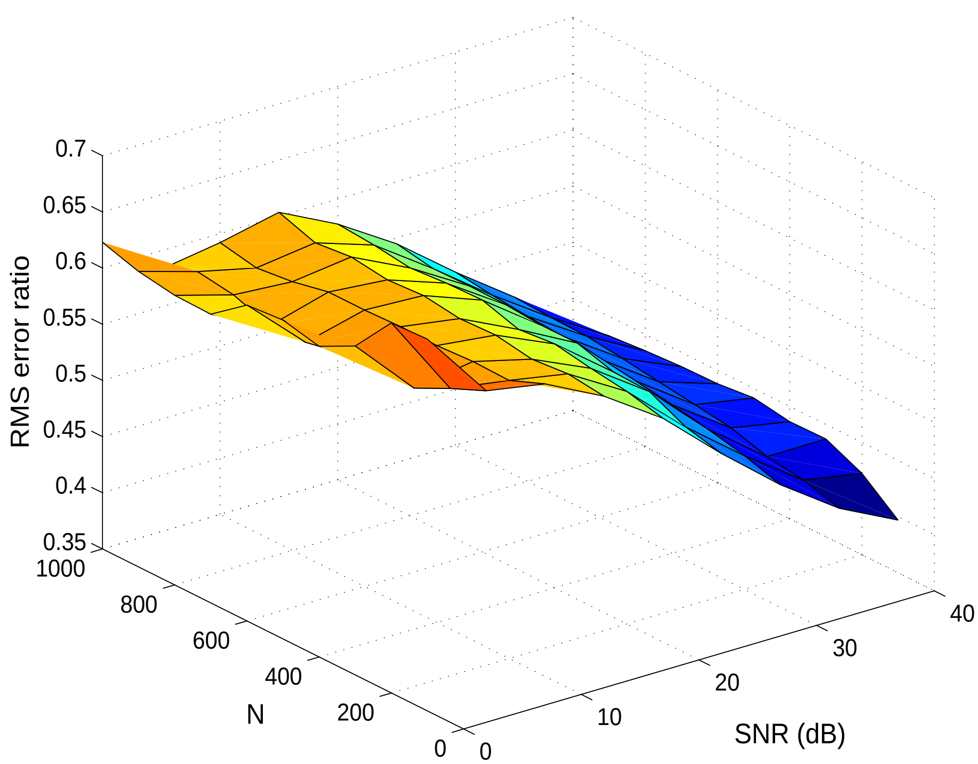

Figure 1: RMS error ratios of Padé and LP frequency estimation methods, 2 frequencies, $M=4$

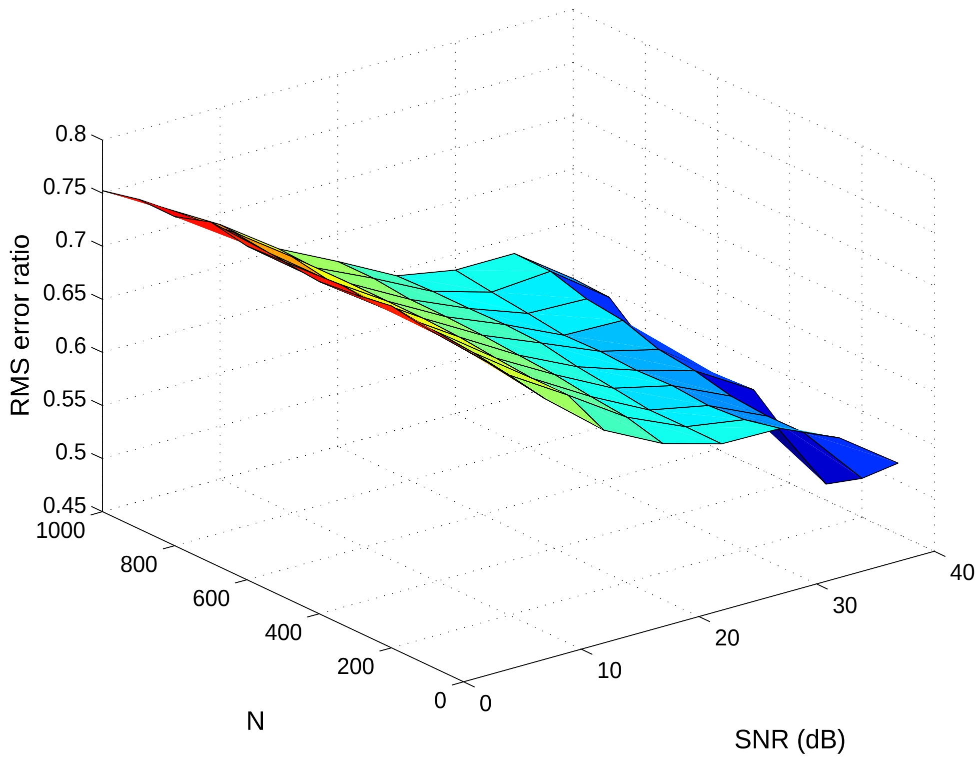

Figure 2: RMS error ratios of Padé and LP frequency estimation methods, 2 frequencies, $M = 10$

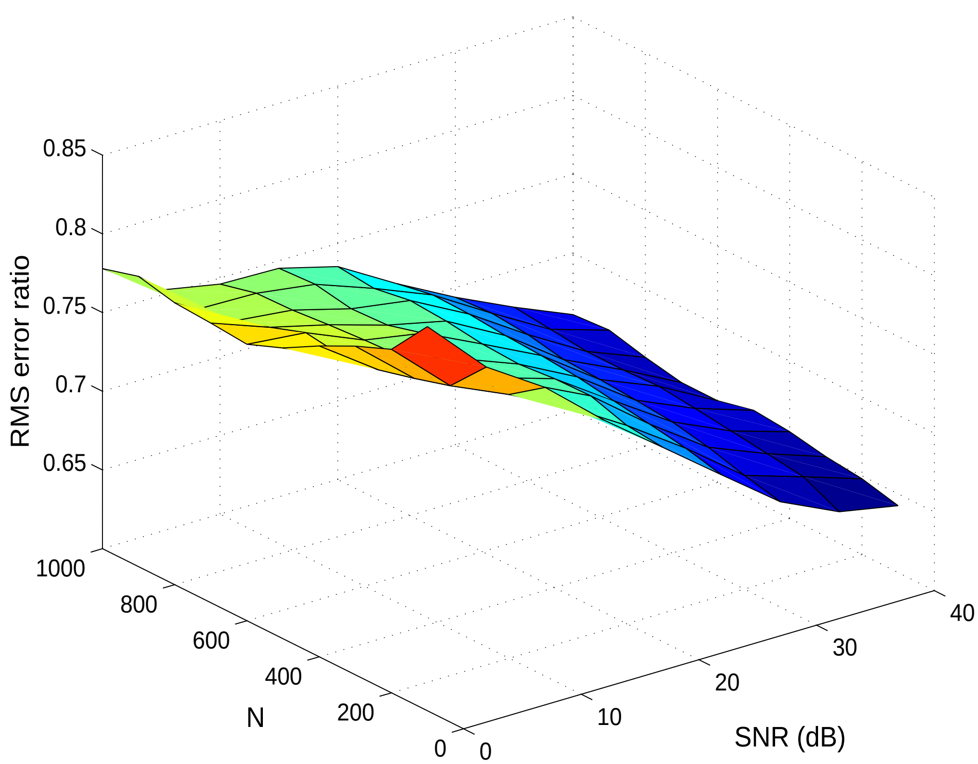

Figure 3: RMS error ratios of Padé and LP frequency estimation methods, 5 frequencies, $M = 10$