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Asymptotic Behavior of Eckhoff's Method for Convergence Acceleration of Dirac Eigenfunction Expansions

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Abstract. The current paper considers the problem of recovering a vector-function on $[-1,1]$ from a limited number of coefficients of its expansion into a series of eigenfunctions of a one-dimensional Dirac system. The Krylov–Lanczos–Eckhoff–Gottlieb acceleration method is examined in the situation when the boundary values it requires have to be computed from the generalized Fourier coefficients themselves. This leads to a $2q\times 2q$ linear system whose matrix is a block Vandermonde matrix; its determinant and inverse are computed explicitly, and the asymptotic L_2-error constant of the method is found, paralleling the classical trigonometric case.

Key Words: Dirac System, Eigenfunction Expansion, Convergence Acceleration, Block Vandermonde Matrix
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Introduction

Let $\{\lambda_n\}_{n\in\mathbb{Z}}$, $\{v_n\}_{n\in\mathbb{Z}}$ be the eigenvalues and L_2-orthonormal eigenfunctions of a self-adjoint boundary value problem on a finite interval, and let f be a square-integrable function on that interval. It is well known that the expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n v_n(x), \qquad c_n = \langle f, v_n\rangle,$$

converges to f in L_2. For practical purposes one approximates f by the truncated series $S_N(f) = \sum_{|n|\le N} c_n v_n$, and the involved error is strongly dependent on the smoothness of f. If f belongs to the domain of the underlying differential operator, this approximation converges uniformly. If, however, f or its derivatives do not satisfy the boundary conditions, the convergence becomes slow, its rate reflects only the smoothness of f, and the Gibbs phenomenon appears at the endpoints.

For the classical Fourier series ($v_n = e^{i\pi nx}/\sqrt{2}$ on $[-1,1]$, the eigenfunctions of $i\,d/dx$ with periodic boundary conditions), increasing the convergence rate by subtracting a polynomial that carries the discontinuities of the function and of its first few derivatives was suggested by A. Krylov as early as 1906 [13] and later, in 1964, by C. Lanczos [14, 15]. More detailed investigation of this approach, together with practical algorithms for computing the "jumps" of f and its derivatives from the Fourier coefficients alone, was performed by K. S. Eckhoff and D. Gottlieb [6, 7, 8, 9, 11, 12]. The method is now usually called the Krylov–Lanczos–Eckhoff–Gottlieb, or simply Eckhoff, method. Its asymptotic L_2 error constants were computed by A. Barkhudaryan, R. Barkhudaryan and A. Poghosyan in [1]. An alternative acceleration technique for nonperiodic functions, based on quasi-periodic approximations and interpolations, was developed in [22, 23, 21].

For more general self-adjoint boundary value problems, in particular for Sturm–Liouville and Dirac systems, the analogue of Krylov's correction polynomial is no longer a Bernoulli polynomial but a combination of derivatives of the Green's function of the underlying operator. The basic algebraic identity behind the method, an integration-by-parts decomposition of c_n into a boundary part P_n and a smoother interior part F_n, was extended by J. Shaw, L. Johnson and R. Riess [24] and, in the form most relevant here, by A. Nersessian and A. Poghosyan [18, 20, 19]. The Gibbs phenomenon in eigenfunction expansions of the one-dimensional Dirac system was studied in [4]. For the same system, R. Barkhudaryan obtained in [2, 3] a generalization of the Eckhoff method under the assumption that the boundary values of f and of its successive Dirac iterates are known *exactly*, and computed the asymptotic L_2 constant of the resulting approximation.

In the current paper, we consider the case when these boundary values are not known in advance and have to be *computed* from the Fourier coefficients $\{c_n\}_{|n|\le N}$ themselves, exactly as Eckhoff proposed in the trigonometric case. The boundary conditions on the eigenfunctions reduce the $4q$ scalar boundary values of f to $2q$ *reduced boundary values* $g_k^{\pm}$, and Eckhoff's recipe, applied at $2q$ selected indices n_s, leads to a linear system for their approximations $\widetilde{g}_k^{\pm}$ — the KEG-D system — whose coefficient matrix is a $2q \times 2q$ *block Vandermonde matrix*, the natural generalization of the $q \times q$ Vandermonde matrix of the original Eckhoff method. Accordingly, in Section 1, we recall the Dirac boundary value problem and the exact-jumps form of the convergence acceleration of [2]; in Section 2, we prove a closed-form expression for the determinant of the block Vandermonde matrix and describe its inverse; in Section 3, we set up the KEG-D system and estimate the errors $\widetilde{g}_k^{\pm} - g_k^{\pm}$, showing, in particular that strict parity balance is necessary and sufficient for the leading-order coefficient matrix to be nonsingular; and in Section 4, we prove the main result of the paper (Theorem 5), an explicit formula for the limit of $N^{2q+1}\|\widetilde{R}_{q,N}(f)\|^2$ in terms of the Dirac jump constants.

1 The Dirac system and exact-jumps method

We work with the canonical Dirac system on $[-1,1]$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} - \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} y = \lambda y, \tag{1}$$

with separated self-adjoint boundary conditions

$$y_2(-1)\cos\alpha + y_1(-1)\sin\alpha = 0, \qquad y_2(1)\cos\beta + y_1(1)\sin\beta = 0, \tag{2}$$

where $p, r \in C^1[-1,1]$ are real-valued and $\alpha, \beta \in [0,\pi]$. We assume that $\lambda = 0$ is not an eigenvalue. This entails no loss of generality, since the spectrum is discrete and a constant spectral shift may always be made so that zero does not belong to the spectrum.

Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be the eigenvalues, indexed in increasing order, with the origin of the indexing chosen consistently with the asymptotic formula below, so that $\lambda_n \to \pm\infty$ as $n \to \pm\infty$. Let $\{v_n\}$ be the corresponding real orthonormal eigenfunctions in $L_2([-1,1];\mathbb{R}^2)$. The asymptotic results of Levitan–Sargsyan [16] give

$$v_{n,1}(x) = K_0 \cos(\xi_n(x) - \alpha) + O(1/n), \tag{3}$$
$$v_{n,2}(x) = K_0 \sin(\xi_n(x) - \alpha) + O(1/n), \tag{4}$$

as $n \to \pm\infty$, uniformly for $x \in [-1,1]$, where

$$\xi_n(x) := \lambda_n(x+1) + \frac{1}{2}\int_{-1}^{x} (p(\tau) + r(\tau))\, d\tau. \tag{5}$$

Choosing the sign of each eigenfunction so that the leading coefficient is positive, the L_2-normalization determines

$$K_0 = \frac{1}{\sqrt{2}},$$

since (3) yields

$$\|v_n\|_{L_2}^2 = \int_{-1}^{1} \left(v_{n,1}^2(x) + v_{n,2}^2(x)\right) dx = 2K_0^2 + O\left(\frac{1}{n}\right) = 1.$$

Moreover, the eigenvalues satisfy (see [16])

$$\lambda_n = \frac{\pi n}{2} - \frac{\widetilde{\theta}}{2} + O\left(\frac{1}{n}\right), \qquad n \to \pm\infty, \tag{6}$$

where

$$\widetilde{\theta} := \beta - \alpha + \frac{1}{2}\int_{-1}^{1} (p(\tau) + r(\tau))\, d\tau.$$

1.1 Reduced boundary values

The eigenfunctions satisfy the boundary conditions (2) exactly. Hence, their endpoint values belong to the corresponding one-dimensional boundary subspaces:

$$v_n(1) = \kappa_n^{(+)} \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix}, \qquad v_n(-1) = \kappa_n^{(-)} \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} \tag{7}$$

for some real scalars $\kappa_n^{(\pm)}$.

We set

$$\mathcal{A}f := Jf' - Qf, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Q(x) = \operatorname{diag}(p(x), r(x)),$$

and define

$$\mathcal{B}f := -Jf = \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}, \qquad f_k := \mathcal{B}\mathcal{A}^k f = -J\mathcal{A}^k f, \qquad k \geq 0. \tag{8}$$

Thus,

$$\mathcal{A}^k f = \begin{pmatrix} (\mathcal{A}^k f)_1 \\ (\mathcal{A}^k f)_2 \end{pmatrix}, \quad f_k = \begin{pmatrix} -(\mathcal{A}^k f)_2 \\ (\mathcal{A}^k f)_1 \end{pmatrix}.$$

For the remainder of this subsection, let $q \geq 1$ and assume in addition that

$$f \in C^q([-1,1];\mathbb{R}^2), \qquad p, r \in C^q[-1,1],$$

so that $\mathcal{A}^k f$ is continuous for $k = 0, \dots, q$. For $k = 0, \dots, q$, we define the *reduced boundary values*

$$g_k^{\pm} := f_{k,1}(\pm 1)\cos\gamma_{\pm} - f_{k,2}(\pm 1)\sin\gamma_{\pm}, \qquad \gamma_+ = \beta, \qquad \gamma_- = \alpha. \tag{9}$$

Equivalently,

$$g_k^{\pm} = -(\mathcal{A}^k f)_2(\pm 1)\cos\gamma_{\pm} - (\mathcal{A}^k f)_1(\pm 1)\sin\gamma_{\pm}.$$

By (7), the boundary terms involving the eigenfunctions reduce to

$$\begin{aligned} v_n^T(\pm 1) f_k(\pm 1) &= \kappa_n^{(\pm)} \begin{pmatrix} \cos\gamma_{\pm} & -\sin\gamma_{\pm} \end{pmatrix} \begin{pmatrix} f_{k,1}(\pm 1) \\ f_{k,2}(\pm 1) \end{pmatrix} \\ &= \kappa_n^{(\pm)} g_k^{\pm}. \end{aligned}$$

It remains to determine the asymptotic behavior of $\kappa_n^{(\pm)}$. By (5) and (6),

$$\xi_n(1) - \alpha = n\pi - \beta + O\left(\frac{1}{n}\right), \qquad \xi_n(-1) - \alpha = -\alpha.$$

Substituting these relations into (3) gives

$$v_n(1) = K_0(-1)^n \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} + O\left(\frac{1}{n}\right)$$

and

$$v_n(-1) = K_0 \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} + O\left(\frac{1}{n}\right).$$

Since the boundary-direction vectors in (7) have unit norm, taking the scalar product with the corresponding direction in the above asymptotic formulas and using (7) yields

$$\kappa_n^{(+)} = K_0(-1)^n + O\left(\frac{1}{n}\right), \qquad \kappa_n^{(-)} = K_0 + O\left(\frac{1}{n}\right). \tag{10}$$

1.2 Approximation with exact boundary corrections

For $q \geq 1$, let

$$f \in C^q([-1,1];\mathbb{R}^2), \qquad p, r \in C^q[-1,1].$$

Define the generalized Fourier coefficients of f with respect to the orthonormal Dirac eigenfunctions $\{v_n\}_{n\in\mathbb{Z}}$ by

$$c_n := \langle f, v_n\rangle_{L_2} = \int_{-1}^{1} v_n^T(x) f(x)\, dx.$$

We first derive a one-step integration-by-parts identity. Since $\mathcal{A}v_n = \lambda_n v_n$, for any sufficiently regular vector-valued function h,

$$\lambda_n \int_{-1}^{1} v_n^T h\, dx = \int_{-1}^{1} (\mathcal{A}v_n)^T h\, dx.$$

Using $\mathcal{A}v_n = Jv_n' - Qv_n$, $J^T = -J$, $Q^T = Q$, we obtain

$$\begin{aligned}\int_{-1}^{1} (\mathcal{A}v_n)^T h\, dx &= \int_{-1}^{1} \left(-v_n'^T Jh - v_n^T Qh\right)\, dx \\ &= -\left[v_n^T Jh\right]_{-1}^{1} + \int_{-1}^{1} v_n^T (Jh' - Qh)\, dx \\ &= v_n^T(-1)Jh(-1) - v_n^T(1)Jh(1) + \int_{-1}^{1} v_n^T \mathcal{A}h\, dx.\end{aligned}$$

Therefore,

$$\int_{-1}^{1} v_n^T h\, dx = \lambda_n^{-1}\left[v_n^T(-1)Jh(-1) - v_n^T(1)Jh(1)\right] + \lambda_n^{-1}\int_{-1}^{1} v_n^T \mathcal{A}h\, dx. \tag{11}$$

We now apply (11) with $h = \mathcal{A}^k f$. Using (8), (7) and the definition (9),

$$\begin{aligned} v_n^T(-1)J\mathcal{A}^k f(-1) - v_n^T(1)J\mathcal{A}^k f(1) \\ &= -v_n^T(-1)f_k(-1) + v_n^T(1)f_k(1) \\ &= -\kappa_n^{(-)}g_k^- + \kappa_n^{(+)}g_k^+. \end{aligned}$$

Define

$$x_n := \frac{1}{\lambda_n}.$$

Then

$$\int_{-1}^{1} v_n^T \mathcal{A}^k f\,dx = x_n\big(\kappa_n^{(+)}g_k^+ - \kappa_n^{(-)}g_k^-\big) + x_n \int_{-1}^{1} v_n^T \mathcal{A}^{k+1} f\,dx. \tag{12}$$

Starting with $k = 0$ and iterating (12) q times yields

$$c_n = P_n + F_n, \tag{13}$$

where

$$P_n = x_n \sum_{k=0}^{q-1} x_n^k\big(\kappa_n^{(+)}g_k^+ - \kappa_n^{(-)}g_k^-\big), \qquad F_n = \lambda_n^{-q}\int_{-1}^{1} v_n^T(x)\mathcal{A}^q f(x)\,dx. \tag{14}$$

If, in addition, $f^{(q)}$ is absolutely continuous, one more integration by parts, combined with the asymptotics (3) and the Riemann–Lebesgue theorem (cf. [2]), gives the sharper expansion

$$F_n = x_n^{q+1}\big[\kappa_n^{(+)}g_q^+ - \kappa_n^{(-)}g_q^-\big] + o(|n|^{-q-1}), \qquad |n| \to \infty. \tag{15}$$

Following [2], we define the exact-jumps approximation

$$S_{q,N}(f) := P(x) + \sum_{|n|\le N} F_n v_n(x), \tag{16}$$

$$P(x) := \sum_{k=0}^{q-1}\big[G_k(x,1,0)f_k(1) - G_k(x,-1,0)f_k(-1)\big],$$

where

$$G_k(x,\xi,0) = \frac{1}{k!}\partial_\lambda^k G(x,\xi,\lambda)\big|_{\lambda=0},$$

with G denoting the Green's matrix of (1)–(2). By the spectral representation of the Green's matrix and its derivatives, the generalized Fourier coefficient of P corresponding to v_n is precisely P_n, that is,

$$\int_{-1}^{1} v_n^T(x)P(x)\,dx = P_n.$$

Substituting (10) into (15) and using the reduced boundary values $g_q^{\pm}$ defined in (9),

$$F_n = K_0\, x_n^{q+1}\big[(-1)^n A + B\big] + o(|n|^{-q-1}), \tag{17}$$

with the *Dirac jump constants*

$$\begin{aligned} A &:= g_q^{+} = f_{q,1}(1)\cos\beta - f_{q,2}(1)\sin\beta, \\ B &:= -g_q^{-} = f_{q,2}(-1)\sin\alpha - f_{q,1}(-1)\cos\alpha. \end{aligned} \tag{18}$$

Hence, by (13), the generalized Fourier coefficients of $f - P$ are precisely F_n. Consequently,

$$R_{q,N}(f) = f - S_{q,N}(f) = \sum_{|n|>N} F_n v_n.$$

By Parseval and (17),

$$\lim_{N\to\infty} N^{2q+1}\|R_{q,N}(f)\|^2 = \frac{2^{2q+2}}{\pi^{2q+2}(2q+1)}(A^2 + B^2), \tag{19}$$

where the constant is obtained by a Riemann-sum argument applied to

$$\sum_{|n|>N} K_0^2 x_n^{2q+2}[(-1)^n A + B]^2,$$

split by parity (see the proof of Theorem 5), using $K_0^2 = 1/2$. Formula (19) replaces Theorem 2 of [2]: the constant $2^{2q+3}/\pi^{2q+3}$ stated there must be replaced by $2^{2q+2}/\pi^{2q+2}$, in accordance with $K_0^2 = 1/2$ rather than $1/\pi$, and the boundary directions entering the constant A there involve an extraneous angle $2\theta - \beta$; the corrected constants are (18).

The approximation (16) has a serious drawback: it assumes the boundary values $f_k(\pm 1)$ to be known in advance, while in many applications they are not. In the next two sections, we develop the tools needed to compute them from the generalized Fourier coefficients themselves.

2 Block Vandermonde matrices

In this section, we study the algebraic structure of the coefficient matrix arising in the linear system used to approximate the reduced boundary values.

Definition 1 *Let $q \in \mathbb{N}$, and let $\alpha_1, \dots, \alpha_{2q}$, $\beta_1, \dots, \beta_{2q}$, and $x_1, \dots, x_{2q}$ be complex numbers. We define the* block Vandermonde matrix of order $2q$ *to be the following $2q \times 2q$ matrix:*

$$\Lambda = \begin{pmatrix} \alpha_1 & \alpha_1 x_1 & \cdots & \alpha_1 x_1^{q-1} & \beta_1 & \beta_1 x_1 & \cdots & \beta_1 x_1^{q-1} \\ \alpha_2 & \alpha_2 x_2 & \cdots & \alpha_2 x_2^{q-1} & \beta_2 & \beta_2 x_2 & \cdots & \beta_2 x_2^{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{2q} & \alpha_{2q} x_{2q} & \cdots & \alpha_{2q} x_{2q}^{q-1} & \beta_{2q} & \beta_{2q} x_{2q} & \cdots & \beta_{2q} x_{2q}^{q-1} \end{pmatrix}. \tag{20}$$

Let

$$V_q = \big(x_i^{j-1}\big)_{1\le i\le 2q,\ 1\le j\le q}$$

denote the rectangular $2q \times q$ Vandermonde matrix, and let

$$D_\alpha = \operatorname{diag}(\alpha_1,\dots,\alpha_{2q}), \qquad D_\beta = \operatorname{diag}(\beta_1,\dots,\beta_{2q}).$$

Then

$$\Lambda = \big(D_\alpha V_q \mid D_\beta V_q\big) = BC, \qquad B = \big(D_\alpha \mid D_\beta\big), \qquad C = \begin{pmatrix} V_q & 0 \\ 0 & V_q \end{pmatrix}. \quad (21)$$

Here,

$$B \in \mathbb{C}^{2q\times 4q}, \qquad C \in \mathbb{C}^{4q\times 2q}.$$

For a subset $T \subseteq [2q] := \{1,\dots,2q\}$, write

$$T = \{t_1 < \cdots < t_m\}, \qquad m = |T|,$$

and define the square Vandermonde matrix associated with T by

$$V_T = \big(x_{t_i}^{j-1}\big)_{1\le i,j\le m}.$$

Its determinant is

$$\det(V_T) = \prod_{1\le i<j\le m} \big(x_{t_j} - x_{t_i}\big).$$

The factorization (21) allows us to compute the determinant of Λ systematically by means of the Cauchy–Binet formula.

2.1 Determinant

Theorem 1 *The determinant of the block Vandermonde matrix* (20) *is*

$$\det\Lambda = \sum_{\substack{T\subseteq[2q]\\ |T|=q}} \varepsilon(T) \left(\prod_{i\in T}\alpha_i\right)\left(\prod_{j\in T^c}\beta_j\right) \det(V_T)\det(V_{T^c}), \qquad (22)$$

where, for each

$$T = \{t_1 < \cdots < t_q\}, \qquad T^c = [2q]\setminus T = \{u_1 < \cdots < u_q\},$$

we set

$$\varepsilon(T) := (-1)^{\sum_{k=1}^q (t_k - k)}.$$

Equivalently,

$$\det\Lambda = \sum_{\substack{T\subseteq[2q]\\ |T|=q}} \varepsilon(T) \left(\prod_{i\in T}\alpha_i\right)\left(\prod_{j\in T^c}\beta_j\right) \prod_{1\le i<j\le q} \big(x_{t_j} - x_{t_i}\big) \prod_{1\le i<j\le q} \big(x_{u_j} - x_{u_i}\big).$$

Proof. Applying the Cauchy–Binet formula (see, for example, [10]) to the factorization $\Lambda = BC$ in (21), we have

$$\det \Lambda = \sum_{\substack{S \subseteq [4q] \\ |S| = 2q}} \det(B_{[:,S]}) \det(C_{[S,:]}),$$

where $B_{[:,S]}$ denotes the submatrix of B formed by all rows and the columns indexed by S, while $C_{[S,:]}$ denotes the submatrix of C formed by the rows indexed by S and all columns.

A subset $S \subseteq [4q]$ with $|S| = 2q$ can contribute only if $B_{[:,S]}$ is nonsingular. Each column of B is a scalar multiple of a standard basis vector $\mathbf{e}_i$: column i is $\alpha_i \mathbf{e}_i$, while column $2q+i$ is $\beta_i \mathbf{e}_i$. Hence $B_{[:,S]}$ is nonsingular only if exactly one column is selected from each pair $\{i, 2q+i\}$, $i = 1, \dots, 2q$.

Define

$$T := S \cap [2q].$$

Since exactly one element is selected from each pair $\{i, 2q+i\}$, $i \in [2q]$, we have

$$T^c := [2q] \setminus T = \{i \in [2q] : 2q + i \in S\}.$$

Since the indices in the first block of C precede those in the second block, the submatrix $C_{[S,:]}$ has the block-diagonal form

$$C_{[S,:]} = \begin{pmatrix} \left(x_i^{j-1}\right)_{\substack{i \in T \\ 1 \le j \le q}} & 0 \\ 0 & \left(x_i^{j-1}\right)_{\substack{i \in T^c \\ 1 \le j \le q}} \end{pmatrix}.$$

If $C_{[S,:]}$ is nonsingular, then its first q columns must be linearly independent. Since these columns are supported only in the first $|T|$ rows, this requires $|T| \ge q$. Likewise, the last q columns are supported only in the remaining $2q - |T|$ rows, so their linear independence requires

$$2q - |T| \ge q.$$

Hence, $|T| \le q$, and therefore, $|T| = q$. Thus, both diagonal blocks are the square Vandermonde matrices V_T and V_{T^c}.

For each q-element subset $T \subseteq [2q]$, the corresponding set S is uniquely determined. If

$$T = \{t_1 < \cdots < t_q\}, \qquad T^c = \{u_1 < \cdots < u_q\},$$

then the selected columns of B occur in the order

$$\alpha_{t_1} \mathbf{e}_{t_1}, \dots, \alpha_{t_q} \mathbf{e}_{t_q}, \beta_{u_1} \mathbf{e}_{u_1}, \dots, \beta_{u_q} \mathbf{e}_{u_q}.$$

The sign of the corresponding permutation is

$$\varepsilon(T) = (-1)^{\sum_{k=1}^{q} (t_k - k)}.$$

Hence,

$$\det(B_{[:,S]}) = \varepsilon(T)\left(\prod_{i\in T}\alpha_i\right)\left(\prod_{j\in T^c}\beta_j\right),$$

while

$$\det(C_{[S,:]}) = \det(V_T)\det(V_{T^c}).$$

Summing over all subsets $T \subseteq [2q]$ with $|T| = q$ gives (22). □

Corollary 1 *The matrix Λ is invertible if and only if the right-hand side of (22) is non-zero. In particular, Λ is singular whenever $\alpha_i = \beta_i$ for all i.*

Proof. The invertibility criterion follows immediately from Theorem 1. If $\alpha_i = \beta_i$ for all i, then the j-th columns of the two blocks in (20) coincide, so Λ is singular. (Note that this is not visible term by term in (22): the terms of T and T^c cancel in pairs only for odd q, since $\varepsilon(T^c) = (-1)^q\varepsilon(T)$.) □

2.2 Inverse via polynomial interpolation

Theorem 2 *Suppose that* $\det(\Lambda) \neq 0$*, and write*

$$\Lambda^{-1} = \begin{pmatrix} Y \\ Z \end{pmatrix}, \qquad Y, Z \in \mathbb{C}^{q\times 2q}.$$

For each $j = 1, \dots, 2q$, define the polynomials

$$Y_j(x) := \sum_{k=0}^{q-1} Y_{k+1,j}x^k, \qquad Z_j(x) := \sum_{k=0}^{q-1} Z_{k+1,j}x^k.$$

Then (Y_j, Z_j) is the unique pair of polynomials of degree at most $q-1$ satisfying

$$\alpha_i Y_j(x_i) + \beta_i Z_j(x_i) = \delta_{ij}, \qquad i = 1, \dots, 2q. \tag{23}$$

Proof. Let $\mathbf{y}_j$ and $\mathbf{z}_j$ denote the j-th columns of Y and Z, respectively. The j-th column of the identity $\Lambda\Lambda^{-1} = I_{2q}$ gives

$$D_\alpha V_q \mathbf{y}_j + D_\beta V_q \mathbf{z}_j = \mathbf{e}_j.$$

By the definitions of Y_j and Z_j, the i-th component of this identity is

$$\alpha_i Y_j(x_i) + \beta_i Z_j(x_i) = \delta_{ij},$$

which proves (23).

Conversely, any pair of polynomials of degree at most $q-1$ satisfying (23) determines coefficient vectors $\mathbf{y}, \mathbf{z} \in \mathbb{C}^q$ such that

$$\Lambda\begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \mathbf{e}_j.$$

Since Λ is invertible, this system has a unique solution. Hence, the pair (Y_j, Z_j) is unique. □

2.3 An explicit cofactor formula

For $j \in [2q]$, let

$$R_j := [2q] \setminus \{j\},$$

equipped with the natural order inherited from $[2q]$. Define the order isomorphism

$$\rho_j : R_j \to [2q-1]$$

by

$$\rho_j(i) := \begin{cases} i, & i < j, \\ i-1, & i > j. \end{cases}$$

For a subset

$$T = \{t_1 < \cdots < t_s\} \subseteq R_j, \qquad s = |T|,$$

let

$$T' := R_j \setminus T.$$

We define the shuffle sign $\varepsilon_j(T)$ by

$$\varepsilon_j(T) := (-1)^{\sum_{k=1}^{s} (\rho_j(t_k)-k)} = (-1)^{\#\{(u,t)\in T' \times T:\ u<t\}}.$$

Equivalently, $\varepsilon_j(T)$ is the sign of the shuffle permutation of the ordered set R_j that places the elements of T before the elements of T', while preserving the natural order within each set.

For any index set

$$S = \{s_1 < \cdots < s_d\} \subseteq [2q]$$

and exponent set

$$E = \{e_1 < \cdots < e_d\} \subset \mathbb{N}_0,$$

we denote the generalized Vandermonde matrix by

$$V_S^E := \left(x_{s_i}^{e_k}\right)_{1 \le i,k \le d}.$$

For variables $y_1, \ldots, y_d$, let

$$e_r(y_1, \ldots, y_d) = \sum_{1 \le i_1 < \cdots < i_r \le d} y_{i_1} \cdots y_{i_r}$$

denote the elementary symmetric polynomial of degree r, with the conventions

$$e_0 = 1, \qquad e_r = 0 \quad \text{for } r < 0 \text{ or } r > d.$$

For $T \subseteq [2q]$, we write

$$e_r((x_t)_{t \in T})$$

for the elementary symmetric polynomial of degree r in the variables x_t, $t \in T$.

We also adopt the standard convention that empty products and determinants of 0×0 matrices equal 1.

For $r = 1, \ldots, q$, set

$$E_r := \{0, 1, \ldots, q-1\} \setminus \{r-1\}.$$

For $1 \le j, m \le 2q$, let $M_{j,m}$ denote the minor obtained from Λ by deleting row j and column m. If $\det(\Lambda) \neq 0$, then the standard cofactor formula gives

$$(\Lambda^{-1})_{m,j} = \frac{(-1)^{j+m}}{\det(\Lambda)} M_{j,m}.$$

Theorem 3 *Fix $j \in [2q]$.*

If $1 \le m \le q$, then

$$M_{j,m} = \sum_{\substack{T \subseteq R_j \\ |T| = q-1}} \varepsilon_j(T) \left(\prod_{i \in T} \alpha_i \right) \left(\prod_{i \in T'} \beta_i \right) \det\bigl(V_T^{E_m}\bigr) \det(V_{T'}), \tag{24}$$

where

$$\det\bigl(V_T^{E_m}\bigr) = \prod_{\substack{r<s \\ r,s \in T}} (x_s - x_r)\, e_{q-m}((x_t)_{t \in T}). \tag{25}$$

If $m = q + \ell$, $1 \le \ell \le q$, then

$$M_{j,q+\ell} = \sum_{\substack{T \subseteq R_j \\ |T| = q}} \varepsilon_j(T) \left(\prod_{i \in T} \alpha_i \right) \left(\prod_{i \in T'} \beta_i \right) \det(V_T) \det\Bigl(V_{T'}^{E_\ell}\Bigr), \tag{26}$$

where

$$\det\Bigl(V_{T'}^{E_\ell}\Bigr) = \prod_{\substack{r<s \\ r,s \in T'}} (x_s - x_r)\, e_{q-\ell}((x_t)_{t \in T'}). \tag{27}$$

Proof. We compute the minors $M_{j,m}$. First suppose that $1 \le m \le q$, so that the deleted column belongs to the α-block. After deleting row j and column m, the remaining rows are indexed by R_j. The remaining α-block has exponent set E_m, while the β-block still has exponent set

$$\{0, 1, \ldots, q-1\}.$$

Applying the same Cauchy–Binet argument as in the proof of Theorem 1 to the matrix obtained after deleting row j and column m, a nonzero term is obtained by assigning $q-1$ indices to the α-block and q indices to the β-block. Hence, the contributing subsets are precisely

$$T \subseteq R_j, \qquad |T| = q-1.$$

Then $|T'| = q$. The indices in T contribute to the α-block, while the indices in T' contribute to the β-block.

For a fixed T, the diagonal weights contribute

$$\left(\prod_{i\in T} \alpha_i\right)\left(\prod_{i\in T'} \beta_i\right).$$

The associated shuffle places the elements of T before the elements of T', while preserving the natural order within each set. By definition, the sign of this shuffle is

$$\varepsilon_j(T) = (-1)^{\sum_{k=1}^{q-1}(\rho_j(t_k)-k)}, \qquad T = \{t_1 < \cdots < t_{q-1}\}.$$

The polynomial part contributes

$$\det\bigl(V_T^{E_m}\bigr) \qquad \text{and} \qquad \det(V_{T'}).$$

Therefore,

$$M_{j,m} = \sum_{\substack{T\subseteq R_j\\ |T|=q-1}} \varepsilon_j(T)\left(\prod_{i\in T}\alpha_i\right)\left(\prod_{i\in T'}\beta_i\right)\det\bigl(V_T^{E_m}\bigr)\det(V_{T'}).$$

This proves (24).

It remains to evaluate $\det(V_T^{E_m})$. By the classical alternant formula for Schur polynomials (see, for example, [17]), the generalized Vandermonde determinant factors as the ordinary Vandermonde determinant times the Schur polynomial associated with the exponent set. For E_m, this Schur polynomial is the elementary symmetric polynomial e_{q-m}. Hence,

$$\det\bigl(V_T^{E_m}\bigr) = \prod_{\substack{r<s\\ r,s\in T}} (x_s - x_r)\, e_{q-m}((x_t)_{t\in T}).$$

This proves (25).

Now suppose that

$$m = q+\ell, \qquad 1 \le \ell \le q.$$

Then the deleted column belongs to the β-block. The α-block keeps the full exponent set

$$\{0,1,\ldots,q-1\},$$

whereas the β-block has exponent set E_ℓ. Thus, a nonzero Cauchy–Binet term must assign q indices to the α-block and $q-1$ indices to the β-block. Hence, the contributing subsets satisfy

$$T \subseteq R_j, \qquad |T| = q.$$

Then $|T'| = q-1$. The same shuffle-sign argument gives

$$M_{j,q+\ell} = \sum_{\substack{T\subseteq R_j\\ |T|=q}} \varepsilon_j(T) \left(\prod_{i\in T} \alpha_i\right) \left(\prod_{i\in T'} \beta_i\right) \det(V_T) \det\left(V_{T'}^{E_\ell}\right).$$

This proves (26).

Similarly, the same alternant formula gives

$$\det\left(V_{T'}^{E_\ell}\right) = \prod_{\substack{r<s\\ r,s\in T'}} (x_s - x_r)\, e_{q-\ell}((x_t)_{t\in T'}).$$

This proves (27) and completes the proof. □

Remark 1 *For large q, the $\binom{2q}{q}$ terms in (22) are too numerous for the determinant formula to be useful for numerical computation. In practice, one may solve the system*

$$\Lambda \mathbf{u} = \mathbf{e}_j$$

by Gaussian elimination, which requires $O(q^3)$ operations. For ordinary Vandermonde systems, specialized algorithms such as the Björck–Pereyra algorithm reduce the computational complexity; see, for example, [5]. Whether the particular block Vandermonde structure of (20) can be exploited to obtain a comparable fast algorithm is an interesting numerical question, which we do not pursue here.

3 The KEG-D system with approximate boundary values

3.1 Setting up the system

Let $q \geq 1$, let

$$f \in C^q([-1,1];\mathbb{R}^2), \qquad p, r \in C^q[-1,1],$$

and assume that $f^{(q)}$ is absolutely continuous. For sufficiently large N, choose $2q$ distinct indices $n_1, \ldots, n_{2q} \in \mathbb{Z} \setminus \{0\}$ satisfying

$$\alpha_0 N \leq |n_s| \leq N, \qquad s = 1, \ldots, 2q, \tag{28}$$

for some fixed constant $0 < \alpha_0 < 1$, and form the following linear system, obtained by discarding the remainder term F_{n_s} in (13)–(14):

$$c_{n_s} = x_{n_s} \sum_{k=0}^{q-1} x_{n_s}^k \big[\kappa_{n_s}^{(+)} \widetilde{g}_k^+ - \kappa_{n_s}^{(-)} \widetilde{g}_k^-\big], \qquad s = 1, \dots, 2q. \tag{29}$$

Introducing the polynomial pair

$$\widetilde{G}^+(x) := \sum_{k=0}^{q-1} \widetilde{g}_k^+ x^k, \qquad \widetilde{G}^-(x) := \sum_{k=0}^{q-1} \widetilde{g}_k^- x^k, \qquad \deg \widetilde{G}^\pm \le q-1,$$

we can rewrite the system (29) as the interpolation problem

$$\kappa_{n_s}^{(+)} \widetilde{G}^+(x_{n_s}) - \kappa_{n_s}^{(-)} \widetilde{G}^-(x_{n_s}) = \lambda_{n_s} c_{n_s}, \qquad s = 1, \dots, 2q. \tag{30}$$

The coefficient matrix of (30) is

$$\Lambda_N :=$$
$$\begin{pmatrix} \kappa_{n_1}^{(+)} & \kappa_{n_1}^{(+)} x_{n_1} & \cdots & \kappa_{n_1}^{(+)} x_{n_1}^{q-1} & -\kappa_{n_1}^{(-)} & -\kappa_{n_1}^{(-)} x_{n_1} & \cdots & -\kappa_{n_1}^{(-)} x_{n_1}^{q-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \kappa_{n_{2q}}^{(+)} & \kappa_{n_{2q}}^{(+)} x_{n_{2q}} & \cdots & \kappa_{n_{2q}}^{(+)} x_{n_{2q}}^{q-1} & -\kappa_{n_{2q}}^{(-)} & -\kappa_{n_{2q}}^{(-)} x_{n_{2q}} & \cdots & -\kappa_{n_{2q}}^{(-)} x_{n_{2q}}^{q-1} \end{pmatrix}.$$

Thus, Λ_N is the block Vandermonde matrix (20) with the identification

$$\alpha_s := \kappa_{n_s}^{(+)}, \qquad \beta_s := -\kappa_{n_s}^{(-)}, \qquad x_s := x_{n_s}.$$

Since $\lambda_{n_s} \neq 0$, multiplying the s-th equation of (29) by λ_{n_s} does not change its solvability. Thus, (29) and (30) are equivalent linear systems, and Λ_N is the coefficient matrix of the latter. Theorem 1 therefore gives an explicit criterion for the unique solvability of (29). By Theorem 2, when Λ_N is invertible, the solution is the unique pair of polynomials $\widetilde{G}^\pm$ of degree at most $q-1$ satisfying (30).

3.2 Error polynomials

Comparing (29) with (13)–(14), we see that the error polynomials

$$\Delta^\pm(x) := \sum_{k=0}^{q-1} \big(\widetilde{g}_k^\pm - g_k^\pm\big) x^k \quad (\deg \le q-1)$$

satisfy

$$\kappa_{n_s}^{(+)} \Delta^+(x_{n_s}) - \kappa_{n_s}^{(-)} \Delta^-(x_{n_s}) = \lambda_{n_s} F_{n_s}, \qquad s = 1, \dots, 2q. \tag{31}$$

From (31) and (15), setting

$$\Phi^{\pm}(x) := \Delta^{\pm}(x) - g_q^{\pm} x^q \qquad (\deg \le q), \tag{32}$$

the system becomes the almost homogeneous system

$$\kappa_{n_s}^{(+)} \Phi^+(x_{n_s}) - \kappa_{n_s}^{(-)} \Phi^-(x_{n_s}) = \varepsilon_s, \qquad s = 1, \dots, 2q, \tag{33}$$

where, by (15) and (28),

$$\varepsilon_s = \lambda_{n_s} \big(F_{n_s} - x_{n_s}^{q+1} [\kappa_{n_s}^{(+)} g_q^+ - \kappa_{n_s}^{(-)} g_q^-] \big) = o(N^{-q}) \qquad \text{uniformly in } s. \tag{34}$$

The leading-order behavior of $\Delta^{\pm}$ is thus governed by how small the left-hand side of (33) can be, and this depends on the parity of the indices n_s, which we study in the next subsection.

3.3 Parity classification and kernel basis

By (10),

$$\kappa_n^{(+)} = (-1)^n K_0 + O\bigg(\frac{1}{n}\bigg), \qquad -\kappa_n^{(-)} = -K_0 + O\bigg(\frac{1}{n}\bigg), \tag{35}$$

where $K_0 = 1/\sqrt{2}$. We partition the selected indices according to parity:

$$E := \{ s \in [2q] : n_s \text{ is even} \}, \qquad O := \{ s \in [2q] : n_s \text{ is odd} \},$$

and set

$$q_e := |E|, \qquad q_o := |O|, \qquad q_e + q_o = 2q.$$

We now examine the leading-order structure of the KEG-D coefficient matrix Λ_N. In view of (35), define

$$\Lambda_0 := \\ K_0 \begin{pmatrix} (-1)^{n_1} & (-1)^{n_1} x_{n_1} & \cdots & (-1)^{n_1} x_{n_1}^{q-1} & -1 & -x_{n_1} & \cdots & -x_{n_1}^{q-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n_{2q}} & (-1)^{n_{2q}} x_{n_{2q}} & \cdots & (-1)^{n_{2q}} x_{n_{2q}}^{q-1} & -1 & -x_{n_{2q}} & \cdots & -x_{n_{2q}}^{q-1} \end{pmatrix}.$$

Thus, Λ_0 is obtained from Λ_N by replacing $\kappa_{n_s}^{(+)}$ and $-\kappa_{n_s}^{(-)}$ by their leading terms

$$(-1)^{n_s} K_0 \qquad \text{and} \qquad -K_0,$$

respectively. The notation Λ_0 refers to this leading-order replacement; the matrix still depends on N through the nodes x_{n_s}.

We determine which parity distribution makes Λ_0 non-singular. For each exponent $j = 0, \dots, q-1$, consider the corresponding pair of columns

$$K_0\big((-1)^{n_s} x_{n_s}^j\big)_{s=1}^{2q}, \qquad -K_0\big(x_{n_s}^j\big)_{s=1}^{2q},$$

and replace them by their sum and difference. The determinant of each such two-column transformation is -2, so the rank is unchanged.

For the sum, the s-th component is

$$K_0\big((-1)^{n_s} - 1\big)x_{n_s}^j = \begin{cases} 0, & s \in E, \\ -2K_0 x_{n_s}^j, & s \in O, \end{cases}$$

whereas for the difference it is

$$K_0\big((-1)^{n_s} + 1\big)x_{n_s}^j = \begin{cases} 2K_0 x_{n_s}^j, & s \in E, \\ 0, & s \in O. \end{cases}$$

Hence, after a permutation of rows and columns, the transformed matrix has the block-diagonal form

$$\begin{pmatrix} -2K_0 V_O & 0 \\ 0 & 2K_0 V_E \end{pmatrix},$$

where V_O is the $q_o \times q$ Vandermonde matrix on the nodes x_{n_s}, $s \in O$, and V_E is the $q_e \times q$ Vandermonde matrix on the nodes x_{n_s}, $s \in E$.

For Λ_0 to have full rank $2q$, both V_O and V_E must have rank q. Hence, necessarily

$$q_o \geq q, \qquad q_e \geq q.$$

Since

$$q_e + q_o = 2q,$$

these two inequalities hold simultaneously if and only if

$$q_e = q_o = q. \tag{36}$$

Conversely, under (36), both V_E and V_O are ordinary $q \times q$ Vandermonde matrices. Since the indices $n_1, \dots, n_{2q}$ are distinct, the eigenvalues $\lambda_{n_1}, \dots, \lambda_{n_{2q}}$, and hence the nodes $x_{n_1}, \dots, x_{n_{2q}}$, are pairwise distinct. Therefore, both V_E and V_O are non-singular. Since the row and column transformations used above are invertible, it follows that

$$\det \Lambda_0 \neq 0.$$

Thus, the strict parity balance (36) is necessary and sufficient for the leading-order matrix Λ_0 to be non-singular. This conclusion concerns Λ_0 only; the invertibility of the actual KEG-D coefficient matrix Λ_N requires a separate comparison with Λ_0. In particular, (36) explains the choice of exactly q even and q odd indices.

We now study the leading-order structure of the error system (33). Replacing $\kappa_{n_s}^{(+)}$ and $\kappa_{n_s}^{(-)}$ by their leading terms from (10) and dividing by K_0, we are led to the linear evaluation map

$$\mathcal{E} : \mathcal{P}_q \times \mathcal{P}_q \longrightarrow \mathbb{C}^{2q},$$

defined by

$$\mathcal{E}(\Phi^+, \Phi^-) := \big((-1)^{n_s}\Phi^+(x_{n_s}) - \Phi^-(x_{n_s})\big)_{s=1}^{2q},$$

where $\mathcal{P}_q$ denotes the space of polynomials of degree at most q.

A pair (Φ^+, Φ^-) belongs to $\ker \mathcal{E}$ if and only if

$$\begin{aligned} W_E(x_{n_s}) &= 0, \quad s \in E, \\ W_O(x_{n_s}) &= 0, \quad s \in O, \end{aligned}$$

where

$$W_E(x) := \Phi^+(x) - \Phi^-(x), \qquad W_O(x) := -\Phi^+(x) - \Phi^-(x). \tag{37}$$

Both W_E and W_O have degree at most q. We also define

$$\Pi_E(x) := \prod_{s \in E}(x - x_{n_s}), \qquad \Pi_O(x) := \prod_{s \in O}(x - x_{n_s}). \tag{38}$$

Lemma 1 (Kernel dimension) *The kernel of $\mathcal{E}$ has dimension 2 if and only if*

$$q_e \le q + 1, \qquad q_o \le q + 1.$$

Proof. The correspondence

$$(\Phi^+, \Phi^-) \longleftrightarrow (W_E, W_O)$$

defined by (37) is a linear bijection of $\mathcal{P}_q \times \mathcal{P}_q$ onto itself.

For a pair $(\Phi^+, \Phi^-) \in \ker \mathcal{E}$, the corresponding polynomial $W_E \in \mathcal{P}_q$ must vanish at the q_e distinct nodes x_{n_s}, $s \in E$. Therefore, the space of possible W_E has dimension

$$\max\{0, q + 1 - q_e\}.$$

Similarly, the space of possible W_O has dimension

$$\max\{0, q + 1 - q_o\}.$$

Hence,

$$\dim \ker \mathcal{E} = \max\{0, q + 1 - q_e\} + \max\{0, q + 1 - q_o\}.$$

If

$$q_e \le q+1, \qquad q_o \le q+1,$$

then

$$\dim\ker\mathcal{E} = (q+1-q_e) + (q+1-q_o) = 2,$$

because $q_e + q_o = 2q$.

If, for example, $q_e \ge q+2$, then the first term vanishes and

$$\dim\ker\mathcal{E} = q+1-q_o.$$

Since $q_o = 2q - q_e$,

$$q+1-q_o = q_e - q + 1 \ge 3.$$

The case $q_o \ge q+2$ is analogous. This proves the claim. □

Under the strict-balance condition (36),

$$|E| = |O| = q.$$

Hence, for every $(\Phi^+, \Phi^-) \in \ker\mathcal{E}$,

$$W_E = c_E \Pi_E, \qquad W_O = c_O \Pi_O$$

for some constants c_E, c_O. Since Π_E and Π_O are monic of degree q, if

$$\Phi^{\pm}(x) = \sum_{k=0}^{q} \Phi_k^{\pm} x^k,$$

then by (37),

$$c_E = \Phi_q^+ - \Phi_q^-, \qquad c_O = -\Phi_q^+ - \Phi_q^-.$$

Thus, if

$$(\Phi_q^+, \Phi_q^-) = (g^+, g^-),$$

then

$$W_E = (g^+ - g^-)\Pi_E, \qquad W_O = (-g^+ - g^-)\Pi_O.$$

Conversely, these formulas determine $\Phi^{\pm}$ uniquely through

$$\Phi^+ = \frac{W_E - W_O}{2}, \qquad \Phi^- = -\frac{W_E + W_O}{2}.$$

Therefore, the map

$$(\Phi^+, \Phi^-) \longmapsto (\Phi_q^+, \Phi_q^-)$$

is a linear bijection from $\ker\mathcal{E}$ onto $\mathbb{C}^2$.

We may therefore define (U^+, U^-) to be the unique element of $\ker\mathcal{E}$ satisfying

$$U_q^+ = 1, \qquad U_q^- = 0,$$

and (V^+, V^-) to be the unique element satisfying

$$V_q^+ = 0, \qquad V_q^- = 1,$$

where $U_q^\pm$ and $V_q^\pm$ denote the coefficients of x^q in $U^\pm$ and $V^\pm$, respectively.

By (32),

$$\Phi^\pm(x) = \Delta^\pm(x) - g_q^\pm x^q,$$

and since $\deg \Delta^\pm \leq q-1$, the coefficients of x^q in $\Phi^\pm$ are

$$\Phi_q^+ = -g_q^+, \qquad \Phi_q^- = -g_q^-.$$

Hence, the kernel element having the same leading coefficients as the error pair (Φ^+, Φ^-) is

$$-g_q^+(U^+, U^-) - g_q^-(V^+, V^-).$$

This suggests the approximation

$$\Phi^\pm(x) \approx -g_q^+ U^\pm(x) - g_q^- V^\pm(x),$$

which is made precise in Lemma 3 and Theorem 4 below.

The marginal distributions

$$(q_e, q_o) = (q+1, q-1) \qquad \text{or} \qquad (q_e, q_o) = (q-1, q+1)$$

also give

$$\dim \ker \mathcal{E} = 2,$$

but they do not satisfy the leading-order non-singularity condition (36). They will therefore be considered separately in Case B below.

Case A: strict balance $q_e = q_o = q$

Lemma 2 *Under the strict-balance condition* $q_e = q_o = q$,

$$\begin{aligned} U^+(x) &= \frac{1}{2}\big[\Pi_E(x) + \Pi_O(x)\big], & U^-(x) &= -\frac{1}{2}\big[\Pi_E(x) - \Pi_O(x)\big], \\ V^+(x) &= -\frac{1}{2}\big[\Pi_E(x) - \Pi_O(x)\big], & V^-(x) &= \frac{1}{2}\big[\Pi_E(x) + \Pi_O(x)\big]. \end{aligned} \tag{39}$$

Proof. For a kernel element with coefficients of x^q given by (g^+, g^-), the preceding characterization of $\ker \mathcal{E}$ gives

$$W_E = (g^+ - g^-)\Pi_E, \qquad W_O = (-g^+ - g^-)\Pi_O.$$

For (U^+, U^-), we have $(g^+, g^-) = (1, 0)$, and hence,

$$W_E = \Pi_E, \qquad W_O = -\Pi_O.$$

For (V^+, V^-), we have $(g^+, g^-) = (0, 1)$, and hence,

$$W_E = -\Pi_E, \qquad W_O = -\Pi_O.$$

Using (37), equivalently,

$$\Phi^+ = \frac{W_E - W_O}{2}, \qquad \Phi^- = -\frac{W_E + W_O}{2},$$

gives (39). $\square$

Case B: marginal imbalance $q_e = q + 1$ or $q_o = q + 1$

Suppose first that

$$q_e = q + 1, \qquad q_o = q - 1.$$

Since $W_E \in \mathcal{P}_q$ vanishes at $q + 1$ distinct nodes, $W_E \equiv 0$. Hence, by (37), $\Phi^+ = \Phi^-$, and therefore,

$$\Phi_q^+ = \Phi_q^-.$$

Using (32), this gives the compatibility condition

$$g_q^+ = g_q^-,$$

which need not hold for a general f. Thus, the leading-order system is degenerate; equivalently, Λ_0 is singular.

Any invertibility of the full KEG-D matrix must therefore come from the $O(1/n)$-corrections in (35). For $p = r = 0$, these corrections vanish, so the coefficient matrix remains singular for every N.

The case $q_o = q + 1$, $q_e = q - 1$ is analogous. Hence, the subsequent analysis is restricted to the strict-balance condition $q_e = q_o = q$.

Finally, for odd q, the symmetric choice

$$\pm N,\ \pm(N-1), \ldots$$

falls into the marginal case: among the $2q$ selected indices, $q + 1$ have the same parity as N, while $q - 1$ have the opposite parity.

3.4 Asymptotic estimates for $\widetilde{g}_k^{\pm} - g_k^{\pm}$

We now restrict to the strictly balanced case and assume, in addition, that the limiting nodes are distinct within each parity class. Under these assumptions, the following lemma shows that a solution of (33) remains quantitatively close to the unique element of $\ker \mathcal{E}$ whose coefficients of x^q are $-g_q^+$ and $-g_q^-$.

Lemma 3 (Stability) *Assume $q_e = q_o = q$, and let the indices satisfy* (28) *with*

$$\lim_{N\to\infty} \frac{n_s}{N} = c_s \neq 0, \qquad s = 1, \dots, 2q.$$

Assume that the numbers $\{c_s : s \in E\}$ are pairwise distinct, as are $\{c_s : s \in O\}$. Let

$$\Phi^{\pm}(x) = \sum_{k=0}^{q} \Phi_k^{\pm} x^k$$

satisfy (33) *with*

$$\Phi_q^{\pm} = -g_q^{\pm},$$

and put

$$\varepsilon := \max_s |\varepsilon_s|.$$

Then, for all sufficiently large N,

$$\Phi_k^{\pm} + g_q^{+} U_k^{\pm} + g_q^{-} V_k^{\pm} = O\big(N^k \varepsilon + N^{k-q-1}\big), \qquad k = 0, \dots, q-1, \qquad (40)$$

where $U_k^{\pm}$ and $V_k^{\pm}$ denote the coefficients of x^k in the polynomials $U^{\pm}$ and $V^{\pm}$ given by (39)*. Moreover, the matrix Λ_N is invertible for all sufficiently large N.*

Proof. Define

$$R_E := \Phi^+ - \Phi^- + (g_q^+ - g_q^-)\Pi_E, \qquad R_O := -\Phi^+ - \Phi^- - (g_q^+ + g_q^-)\Pi_O, \quad (41)$$

where Π_E and Π_O are given by (38). Since

$$\Phi_q^+ = -g_q^+, \qquad \Phi_q^- = -g_q^-,$$

and Π_E, Π_O are monic polynomials of degree q, we have

$$\deg R_E, \deg R_O \le q - 1.$$

Using (39), we obtain

$$\begin{aligned} \Phi^+ + g_q^+ U^+ + g_q^- V^+ &= \frac{1}{2}(R_E - R_O), \\ \Phi^- + g_q^+ U^- + g_q^- V^- &= -\frac{1}{2}(R_E + R_O). \end{aligned} \qquad (42)$$

Thus, it suffices to estimate the coefficients of R_E and R_O.

By (35) and (28), we may write

$$\kappa_{n_s}^{(+)} = (-1)^{n_s} K_0 + \delta_s^+, \qquad -\kappa_{n_s}^{(-)} = -K_0 + \delta_s^-,$$

where

$$\delta_s^{\pm} = O(N^{-1})$$

uniformly in s. Substituting these expressions into (33), and using $\Pi_E(x_{n_s}) = 0$ for $s \in E$ and $\Pi_O(x_{n_s}) = 0$ for $s \in O$, gives

$$\begin{aligned} K_0 R_E(x_{n_s}) &= \varepsilon_s - \delta_s^+ \Phi^+(x_{n_s}) - \delta_s^- \Phi^-(x_{n_s}), \quad s \in E, \\ K_0 R_O(x_{n_s}) &= \varepsilon_s - \delta_s^+ \Phi^+(x_{n_s}) - \delta_s^- \Phi^-(x_{n_s}), \quad s \in O. \end{aligned} \tag{43}$$

Set

$$m := \max \left\{ \max_{s\in E} |R_E(x_{n_s})|, \max_{s\in O} |R_O(x_{n_s})| \right\}. \tag{44}$$

By (6),

$$N x_{n_s} = \frac{N}{\lambda_{n_s}} \longrightarrow \frac{2}{\pi c_s}.$$

Hence, there exist constants $C, d > 0$ such that, for all sufficiently large N,

$$|x_{n_s}| \le \frac{C}{N}, \qquad |x_{n_s} - x_{n_t}| \ge \frac{d}{N} \tag{45}$$

whenever $s \neq t$ belong to the same parity class.

Let ℓ_t^E, $t \in E$, be the Lagrange basis polynomials associated with the E-nodes. Writing

$$\ell_t^E(x) = \sum_{k=0}^{q-1} \ell_{t,k}^E x^k,$$

the node estimates (45) imply

$$|\ell_t^E(y)| \le C_1, \qquad |y| \le \frac{C}{N}, \qquad \ell_{t,k}^E = O(N^k), \quad k = 0, \dots, q-1. \tag{46}$$

The same estimates hold for the Lagrange basis polynomials ℓ_t^O, $t \in O$.

Since $\deg R_E, \deg R_O \le q - 1$, Lagrange interpolation gives

$$R_E(x) = \sum_{t\in E} R_E(x_{n_t}) \ell_t^E(x), \qquad R_O(x) = \sum_{t\in O} R_O(x_{n_t}) \ell_t^O(x).$$

By (44) and (46),

$$|R_E(y)| + |R_O(y)| \le C_2 m, \qquad |y| \le \frac{C}{N}.$$

If

$$R_E(x) = \sum_{k=0}^{q-1} R_{E,k} x^k, \qquad R_O(x) = \sum_{k=0}^{q-1} R_{O,k} x^k,$$

then (46) also gives

$$R_{E,k}, R_{O,k} = O(mN^k), \qquad k = 0, \dots, q-1. \tag{47}$$

For every selected node $y = x_{n_s}$, the definitions of R_E and R_O give

$$\Phi^+(y) = \frac{1}{2}\Big[R_E(y) - R_O(y) - (g_q^+ - g_q^-)\Pi_E(y) - (g_q^+ + g_q^-)\Pi_O(y)\Big],$$
$$\Phi^-(y) = -\frac{1}{2}\Big[R_E(y) + R_O(y) - (g_q^+ - g_q^-)\Pi_E(y) + (g_q^+ + g_q^-)\Pi_O(y)\Big].$$

By (45) and (38),

$$|\Pi_E(y)|,\ |\Pi_O(y)| = O(N^{-q}),$$

and therefore,

$$|\Phi^\pm(y)| \le C_3\big(N^{-q} + m\big), \qquad y = x_{n_s}. \tag{48}$$

Combining (43) with (48) and using $\delta_s^\pm = O(N^{-1})$, we obtain

$$K_0 m \le \varepsilon + C_4 N^{-1}\big(N^{-q} + m\big).$$

Hence, for all sufficiently large N,

$$m \le C_5\big(\varepsilon + N^{-q-1}\big). \tag{49}$$

Combining (49) with (47), we obtain

$$R_{E,k}, R_{O,k} = O\big(N^k\varepsilon + N^{k-q-1}\big), \qquad k = 0, \dots, q-1. \tag{50}$$

Combining (50) with (42) yields (40).

It remains to prove that Λ_N is invertible for all sufficiently large N. Suppose that the homogeneous version of (31) has a solution

$$\Delta^\pm \in \mathcal{P}_{q-1}.$$

Then (33) holds with

$$\varepsilon_s = 0, \qquad g_q^\pm = 0.$$

Hence, by (41),

$$R_E = \Delta^+ - \Delta^-, \qquad R_O = -\Delta^+ - \Delta^-.$$

Since $\varepsilon = 0$ and $g_q^\pm = 0$, the estimate (48) reduces to

$$|\Phi^\pm(y)| \le C_3 m.$$

Substituting this into (43) gives

$$m \le \frac{C_4}{K_0 N}\, m.$$

For sufficiently large N, this forces $m = 0$. Thus, R_E and R_O vanish at the q nodes of their respective parity classes. Since

$$\deg R_E, \deg R_O \le q - 1,$$

we have
$$R_E \equiv R_O \equiv 0.$$
Hence,
$$\Delta^+ = \Delta^- = 0.$$
Thus, the homogeneous system has only the trivial solution, and Λ_N is invertible for all sufficiently large N. □

Combining Lemma 3 with (34), we obtain the decomposition

$$\Phi^{\pm}(x) = -g_q^+ U^{\pm}(x) - g_q^- V^{\pm}(x) + \rho^{\pm}(x), \tag{51}$$

where

$$\rho^{\pm}(x) = \sum_{k=0}^{q-1} \rho_k^{\pm} x^k, \qquad \rho_k^{\pm} = o(N^{k-q}), \qquad k = 0, \ldots, q-1.$$

In particular, the coefficient of x^q in $\rho^{\pm}$ is exactly zero.

Theorem 4 *Suppose*

$$f \in C^q([-1,1]; \mathbb{R}^2), \qquad p, r \in C^q[-1,1],$$

with $f^{(q)}$ absolutely continuous, where $q \geq 1$. Assume that the indices $n_s = n_s(N)$ satisfy (28) *with*

$$\lim_{N\to\infty} \frac{n_s}{N} = c_s \neq 0, \qquad s = 1, \ldots, 2q,$$

that the strict-balance condition (36) *holds, and that the limits $\{c_s : s \in E\}$ are pairwise distinct, as are $\{c_s : s \in O\}$. Then*

$$\widetilde{g}_k^{\pm} = g_k^{\pm} - g_q^+ U_k^{\pm} - g_q^- V_k^{\pm} + o(N^{k-q}), \qquad k = 0, \ldots, q-1, \tag{52}$$

where $U_k^{\pm}$ and $V_k^{\pm}$ denote the coefficients of x^k in the polynomials $U^{\pm}$ and $V^{\pm}$ given by (39)*. In particular,*

$$\widetilde{g}_k^{\pm} - g_k^{\pm} = O(N^{k-q}), \qquad k = 0, \ldots, q-1.$$

Proof. By Lemma 3, the matrix Λ_N is invertible for all sufficiently large N, so the KEG-D system has a unique solution. Moreover, by (34),

$$\varepsilon = \max_s |\varepsilon_s| = o(N^{-q}).$$

Hence, (51) gives

$$\Phi^{\pm}(x) = -g_q^+ U^{\pm}(x) - g_q^- V^{\pm}(x) + \rho^{\pm}(x),$$

where

$$\rho^{\pm}(x) = \sum_{k=0}^{q-1} \rho_k^{\pm} x^k, \qquad \rho_k^{\pm} = o(N^{k-q}).$$

By (32), for $k = 0, \ldots, q-1$,

$$\Phi_k^{\pm} = \widetilde{g}_k^{\pm} - g_k^{\pm}.$$

Comparing the coefficients of x^k therefore gives (52).

Finally, by (39) and the node estimates (45),

$$U_k^{\pm}, V_k^{\pm} = O(N^{k-q}), \qquad k = 0, \ldots, q-1.$$

The final assertion follows. □

Remark 2 *The distinctness assumption on the limits within each parity class excludes configurations in which several of the ratios n_s/N converge to the same limit within E or within O. This is analogous to the trigonometric case considered in [1], where a multiplicity α among the limiting values requires*

$$f \in C^{q+\alpha-1}$$

with $f^{(q+\alpha-1)}$ absolutely continuous.

One expects that (52) and the corresponding L_2-error asymptotics remain valid for clustered configurations when the maximal multiplicity α is counted separately within each parity class, provided

$$f \in C^{q+\alpha-1}([-1, 1]; \mathbb{R}^2),$$

with $f^{(q+\alpha-1)}$ absolutely continuous, and p, r are sufficiently smooth for the asymptotic expansions in (10) to be continued to the required order in $1/n$.

We do not prove this extension here. As in [1, Theorem 3.3], further integrations by parts extending (15) would introduce the additional reduced boundary values

$$g_{q+1}^{\pm}, \ldots, g_{q+\alpha-1}^{\pm},$$

which would enter the interpolation equations through higher powers of x_{n_s}.

4 Asymptotic L_2 error estimates

As in (16), define

$$\widetilde{S}_{q,N}(f) := \widetilde{P}(x) + \sum_{|n| \le N} \widetilde{F}_n v_n(x),$$

where

$$\widetilde{P}(x) := \sum_{k=0}^{q-1} \big[G_k(x,1,0)\widetilde{f}_k(1) - G_k(x,-1,0)\widetilde{f}_k(-1)\big].$$

For the reduced boundary values $\widetilde{g}_k^{\pm}$ obtained from (29), we choose

$$\widetilde{f}_k(\pm 1) := \widetilde{g}_k^{\pm} \begin{pmatrix} \cos\gamma_\pm \\ -\sin\gamma_\pm \end{pmatrix}, \qquad \gamma_+ = \beta, \qquad \gamma_- = \alpha,$$

so that, consistently with (9),

$$\big(\cos\gamma_\pm \quad -\sin\gamma_\pm\big)\,\widetilde{f}_k(\pm 1) = \widetilde{g}_k^{\pm}.$$

We then set

$$\widetilde{P}_n := \langle \widetilde{P}, v_n\rangle_{L_2}, \qquad \widetilde{F}_n := c_n - \widetilde{P}_n.$$

The *KEG-D error* is defined by

$$\widetilde{R}_{q,N}(f) := f - \widetilde{S}_{q,N}(f).$$

4.1 Main theorem

Theorem 5 *Under the hypotheses of Theorem 4 (in particular, the strict parity balance $q_e = q_o = q$),*

$$\lim_{N\to\infty} N^{2q+1}\|\widetilde{R}_{q,N}(f)\|^2 = \frac{2^{2q}}{\pi^{2q+2}}\big[(A+B)^2\, J_E + (A-B)^2\, J_O\big], \tag{53}$$

where A, B are the Dirac jump constants (18) *and*

$$J_E := \int_{-1}^{1} \prod_{s\in E}\Big(x - \frac{1}{c_s}\Big)^2 dx, \qquad J_O := \int_{-1}^{1} \prod_{s\in O}\Big(x - \frac{1}{c_s}\Big)^2 dx.$$

Proof. Since $\widetilde{F}_n$ is the generalized Fourier coefficient of $f - \widetilde{P}$, we have

$$\widetilde{R}_{q,N}(f) = \sum_{|n|>N} \widetilde{F}_n v_n.$$

Hence, by Parseval,

$$\|\widetilde{R}_{q,N}(f)\|^2 = \sum_{|n|>N} |\widetilde{F}_n|^2.$$

Since

$$\widetilde{F}_n = F_n + (P_n - \widetilde{P}_n),$$

we first compute the second term. By the definitions of P_n and $\widetilde{P}_n$,

$$\begin{aligned} P_n - \widetilde{P}_n &= -x_n\big[\kappa_n^{(+)}\Delta^+(x_n) - \kappa_n^{(-)}\Delta^-(x_n)\big] \\ &= -x_n\big[\kappa_n^{(+)}\Phi^+(x_n) - \kappa_n^{(-)}\Phi^-(x_n)\big] \\ &\quad - x_n^{q+1}\big[\kappa_n^{(+)}g_q^+ - \kappa_n^{(-)}g_q^-\big], \end{aligned}$$

where we used (32). Define

$$\mathcal{Q}_n := \kappa_n^{(+)}\Phi^+(x_n) - \kappa_n^{(-)}\Phi^-(x_n).$$

Then (15) gives

$$P_n - \widetilde{P}_n = -x_n\mathcal{Q}_n - F_n + r_n, \qquad |n| > N,$$

where

$$r_n = o(|n|^{-q-1})$$

uniformly for $|n| > N$. Therefore,

$$\widetilde{F}_n = -x_n\mathcal{Q}_n + r_n. \tag{54}$$

By (51),

$$\begin{aligned} \mathcal{Q}_n = &-g_q^+\big[\kappa_n^{(+)}U^+(x_n) - \kappa_n^{(-)}U^-(x_n)\big] \\ &- g_q^-\big[\kappa_n^{(+)}V^+(x_n) - \kappa_n^{(-)}V^-(x_n)\big] \\ &+ \kappa_n^{(+)}\rho^+(x_n) - \kappa_n^{(-)}\rho^-(x_n). \end{aligned}$$

Since

$$\rho^\pm(x) = \sum_{k=0}^{q-1}\rho_k^\pm x^k, \qquad \rho_k^\pm = o(N^{k-q}),$$

and $|x_n| = O(N^{-1})$ uniformly for $|n| > N$, it follows that

$$\rho^\pm(x_n) = o(N^{-q})$$

uniformly for $|n| > N$ as $N \to \infty$.

Using (35) and (39), we now distinguish the parity of the tail index n. For $|n| > N$, we have

$$U^\pm(x_n), V^\pm(x_n) = O(N^{-q}).$$

Hence, if n is even,

$$\kappa_n^{(+)}U^+(x_n) - \kappa_n^{(-)}U^-(x_n) = K_0\Pi_E(x_n) + o\big(N^{-q}\big)\,,$$

and

$$\kappa_n^{(+)}V^+(x_n) - \kappa_n^{(-)}V^-(x_n) = -K_0\Pi_E(x_n) + o\big(N^{-q}\big)\,.$$

If n is odd,

$$\kappa_n^{(+)} U^+(x_n) - \kappa_n^{(-)} U^-(x_n) = -K_0 \Pi_O(x_n) + o\big(N^{-q}\big)\,,$$

and

$$\kappa_n^{(+)} V^+(x_n) - \kappa_n^{(-)} V^-(x_n) = -K_0 \Pi_O(x_n) + o\big(N^{-q}\big)\,.$$

Hence,

$$\mathcal{Q}_n = \begin{cases} -K_0(g_q^+ - g_q^-)\Pi_E(x_n) + o(N^{-q})\,, & n \text{ even,} \\ \;\; K_0(g_q^+ + g_q^-)\Pi_O(x_n) + o(N^{-q})\,, & n \text{ odd.} \end{cases} \tag{55}$$

Combining (54) and (55), and using (38), we obtain

$$\widetilde{F}_n = \begin{cases} K_0(g_q^+ - g_q^-)\, x_n^{q+1} \displaystyle\prod_{s\in E}\left(1 - \frac{n}{n_s}\right) + x_n \sigma_{n,N}, & n \text{ even,} \\ -K_0(g_q^+ + g_q^-)\, x_n^{q+1} \displaystyle\prod_{s\in O}\left(1 - \frac{n}{n_s}\right) + x_n \sigma_{n,N}, & n \text{ odd,} \end{cases} \tag{56}$$

where

$$\sup_{|n|>N} |\sigma_{n,N}| = o(N^{-q}).$$

Here we used (38) and (6). Indeed,

$$n x_n = \frac{2}{\pi} + O\left(\frac{1}{n}\right),$$

and hence, uniformly for $|n| > N$,

$$(x_n - x_{n_s}) - x_n\left(1 - \frac{n}{n_s}\right) = \frac{n x_n - n_s x_{n_s}}{n_s} = O(N^{-2}).$$

It follows that

$$x_n \Pi_E(x_n) = x_n^{q+1} \prod_{s\in E}\left(1 - \frac{n}{n_s}\right) + x_n O(N^{-q-1}),$$

and analogously for Π_O.

The remainders in (56) are of the form $x_n \sigma_{n,N}$, where

$$\sup_{|n|>N} |\sigma_{n,N}| = o(N^{-q}).$$

Since

$$\sum_{|n|>N} |x_n|^2 = O(N^{-1}),$$

we have

$$\sum_{|n|>N} |x_n \sigma_{n,N}|^2 = o(N^{-2q-1}).$$

The squared sum of the leading terms in (56) is $O(N^{-2q-1})$, so the corresponding cross terms are also $o(N^{-2q-1})$ by the Cauchy–Schwarz inequality. Thus, only the leading terms in (56) contribute to the limit.

For the even indices, the Riemann-sum argument gives

$$N^{2q+1} \sum_{\substack{|n|>N \\ n \text{ even}}} |\widetilde{F}_n|^2 \longrightarrow K_0^2 (g_q^+ - g_q^-)^2 \frac{(2/\pi)^{2q+2}}{2} \int_{|u|>1} u^{-2q-2} \prod_{s\in E} \left(1 - \frac{u}{c_s}\right)^2 du.$$

The factor 1/2 accounts for the density of the even integers. With the change of variables $u = 1/x$,

$$\int_{|u|>1} u^{-2q-2} \prod_{s\in E} \left(1 - \frac{u}{c_s}\right)^2 du = \int_{-1}^{1} \prod_{s\in E} \left(x - \frac{1}{c_s}\right)^2 dx = J_E.$$

Similarly,

$$N^{2q+1} \sum_{\substack{|n|>N \\ n \text{ odd}}} |\widetilde{F}_n|^2 \longrightarrow K_0^2 (g_q^+ + g_q^-)^2 \frac{(2/\pi)^{2q+2}}{2} J_O.$$

Using $K_0^2 = 1/2$, we conclude that

$$N^{2q+1} \|\widetilde{R}_{q,N}(f)\|^2 \longrightarrow \frac{2^{2q}}{\pi^{2q+2}} \left[(g_q^+ - g_q^-)^2 J_E + (g_q^+ + g_q^-)^2 J_O \right].$$

Finally, by (18),

$$g_q^+ - g_q^- = A + B, \qquad g_q^+ + g_q^- = A - B,$$

and this is precisely (53). □

Remark 3 *For comparison, applying the same Riemann-sum argument to the exact-jumps coefficients F_n gives* (19)*:*

$$\lim_{N\to\infty} N^{2q+1} \|R_{q,N}(f)\|^2 = \frac{2^{2q+2}}{\pi^{2q+2}(2q+1)} (A^2 + B^2).$$

Thus, so long as the parity balance holds, computing the boundary values from the generalized Fourier coefficients does not degrade the rate of convergence, but changes the asymptotic constant.

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