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On Mixing Properties of the Limiting Gibbs Process

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Abstract. We consider locally compact second countable Abelian topological group E with Haar measure and pair potentials in E satisfying some natural stability and regularity conditions. We show that the limiting Gibbs process G with empty boundary conditions is strongly Brillinger-mixing in the spirit of Lothar Heinrich and thereby mixing with respect to the group of translations and ergodic.

Key Words: Mixing, Strong Brillinger Mixing, Limiting Gibbs Process
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Introduction

In statistical mechanics mixing conditions for weakly dependent Gibbs random fields are an essential tool for establishing limit theorems. A large part of the papers on the subject were using various mixing conditions to derive central limit theorems for discrete systems. A valuable systematic presentation of different type mixing conditions for weakly dependent random fields and their applications to central limit theorems for lattice gases was given by Nahapetian [13].

The number of works on mixing properties of classical continuous systems is rather limited. The starting point is two papers of Minlos from 1967 [11,12]. In the first paper he gave a construction of the limiting Gibbs process G in $\mathbb{R}^d$ with empty boundary conditions. In the second paper, under quite general assumptions on the pair potential and for sufficiently small activity, he established the regularity of G. This mixing property means that events occurring in well-separated bounded domains become asymptotically independent. To prove this the author considers the conditional limiting Gibbs distribution, given that the particle positions of a finite configuration are fixed. He then estimates the pointwise difference between the conditional and unconditional limiting correlation functions. This estimation is done using the Kirkwood–Salsburg equations for the respective limiting correlation functions. The overall procedure is technically complex and cumbersome.

Next we mention the paper of Spohn from 1986 (cf. [22]) where he proved exponential decay of correlations for positive potentials of finite range. This result was then extended by Minlos in 1999 in collaboration with Kondratiev, Rockner and Shchepan'uk [7].

We note also the paper by Rebenko and Tertychnyi in 2017, where the authors establish the decay of correlations at low activity for strong superstable, finite-range pair potentials. They prove this by approximating the classical system with a continuous cell gas model and using the Kirkwood–Salsburg equations.

In the present paper, we consider stationary point processes generated by interacting identical particles in locally compact second countable Abelian topological group E with the Haar measure. For such processes, Brillinger mixing conditions offer an ideal framework to quantify weak dependence using cumulants (also called truncated correlation functions or semi-invariants) of all orders. Our primary objective is to establish strong Brillinger mixing for the limiting Gibbs process G with empty boundary conditions by imposing appropriate constraints on the interaction potential. To achieve this, we employ direct point process methods, thereby avoiding artificial reliance on lattice gas techniques.

Traditional α or β mixing conditions measure dependence between events from two σ algebras. The Brillinger mixing is reflecting dependence between points thus requiring certain decay of k-order, $k > 1$, joint cumulant functions in terms of their integrability properties. Strong Brillinger mixing is more restrictive since it requires, roughly speaking, absolute integrability of joint cumulant densities. Our formulation of this condition is intimately related to the work of Heinrich [3].

The main result of this paper is Theorem 1, in which we show that under quite general conditions on the potential Φ, which we call strong normal, G is strongly Brillinger-mixing. On the other hand, the property of strong Brillinger-mixing implies in Theorem 2 that G is mixing and thereby ergodic.

1 Synopsis of notions and methods

Notions not explained here can be found in [18].

1.1 Point processes

The underlying phase space $(X, \mathcal{B}(X), \mathcal{B}_0(X))$ is a lcscH, i.e., locally compact, second countable Hausdorff topological space. $\mathcal{B}(X)$ denotes its Borel sets and $\mathcal{B}_0(X)$ the bounded Borel sets. $\mathcal{M}(X)$ is the space of Radon measures (measures that are bounded on bounded measurable subsets) on X. $\mathcal{M}(X)$ is Polish space with respect to vague convergence, and $\mathcal{M}^{\cdot}(X)$ the Borel subspace of simple point measures, i.e., locally finite subsets of X. $\mathfrak{X} = \mathcal{M}^{\cdot}_f(X)$ denotes the measurable subset of finite elements ξ in $\mathcal{M}^{\cdot}(X)$ and $\mathfrak{X}'$ all $\xi \in \mathfrak{X}$ with $\xi \neq o$, where o denotes the zero measure or vacuum.

For notational convenience, we often do not indicate the underlying space in the sequel. Δ_o is the Dirac measure on $\mathcal{M}^{\cdot}$ in o. Dirac measures on X are denoted by ε_x.

Laws P on $(\mathcal{M}^{\cdot}, \mathcal{B}(\mathcal{M}^{\cdot}))$ are called (simple point) processes in X (in the sequel we call them processes). We use different function spaces: U is the collection of all bounded, non-negative and measurable functions f on X with bounded support, and F denotes the set of all non-negative measurable functions on the underlying space which is not always indicated. For $f \in U$, the functional $\zeta_f : \mu \mapsto \mu(f) = \int_X f(x)\mu(\mathrm{d}\, x)$ is a well-defined measurable function on $\mathcal{M}^{\cdot}$, which is vaguely continuous for continuous $f \in U$.

In general two types of bounded subsets in $\mathfrak{X}$ are considered, the collection of all finite configurations in K, denoted by $\mathfrak{X}(K)$, and the collection of finite configurations that hit K, denoted by $\mathfrak{X}_K$, $\mathfrak{X}_K = (\zeta_K \geq 1)$. Note that $\mathfrak{X}(K) \subset \mathfrak{X}_K$.

Functionals of point processes. $\mathcal{L}_P(f) = P(\mathrm{e}^{-\zeta_f})$, $f \in U$, denotes the Laplace transform of process P. Another important functional is the Campbell measure of P, defined by

$$\mathcal{C}_P(h) = \int_{\mathcal{M}^{\cdot}} \int_X h(x,\mu)\; \mu(\mathrm{d}\, x) P(\mathrm{d}\, \mu), \qquad h \in F(X \times \mathcal{M}^{\cdot}).$$

P is uniquely determined by both functionals. We also need the Campbell measure of P of order $n > 1$. It is defined by

$$\mathcal{C}_P^n(h) = \int_{\mathcal{M}^{\cdot}(X)} \int_{X^n} h(x,\mu)\; \mu^{\otimes n}(\mathrm{d}\, x) P(\mathrm{d}\, \mu), \qquad h \in F(X^n \times \mathcal{M}^{\cdot}),$$

where $\mu^{\otimes n}$ denotes the n-times product of μ with itself. The Campbell measure of a process P determines all moments measures of P. Indeed, the k-th moment measure of P is defined by

$$\nu_P^k(f_1 \otimes \cdots \otimes f_k) = \int_{\mathcal{M}^{\cdot}(X)} \int_{X^k} f_1 \otimes \cdots \otimes f_k(x) \mu^{\otimes n}(\mathrm{d}\, x) P(\mathrm{d}\, \mu),$$

then evidently

$$\nu_P^k(f_1 \otimes \cdots \otimes f_k) = \mathcal{C}_P(f_1 \otimes (\zeta_{f_2} \cdots \zeta_{f_k})), \quad f_1, \ldots, f_k \in F.$$

If ν_P^k is a Radon measure, in the sense that it is bounded on bounded events, then P is called of order k. If P is of order k for all $k \geq 1$, then P is called of infinite order. The factorial measure of P of order k is defined as

$$\tilde{\nu}_P^k(f) = \\ = \int_{\mathcal{M}^\cdot} P(\mathrm{d}\,\mu) \int_{X^k} \mu(\mathrm{d}\,x_1)(\mu - \varepsilon_{x_1})(\mathrm{d}\,x_2)\dots(\mu - \sum_{\ell=1}^{k-1} \varepsilon_{x_\ell})(\mathrm{d}\,x_k) f(x_1, \dots, x_k)$$

for $f \in F(X^k)$. If P is of order k, then, obviously, also $\tilde{\nu}_P^k$ is a Radon measure.

In case where P is simple, the factorial measure of P of order k can be defined as the restriction of ν_P^k to $\tilde{X}^k = \{(x_1, \dots, x_k) : (i \neq j \Rightarrow x_i \neq x_j)\}$.

1.2 KMM-processes

The following construction of processes, which we call KMM-processes after Kerstan, Mecke and Matthes, is based on the concept of a generator, similarly to the one in the theory of stochastic processes (cf. [8, 17]).

Denote by $\mathcal{E}_{sm}$ the collection of all finite signed measures L on $\mathfrak{X}$. With respect to convolution $*$ and the total variation norm, this space is a real Banach algebra with unit Δ_o.

For all L, the series

$$\exp \mathrm{L} = \sum_{n=0}^{\infty} \frac{1}{n!} \, \mathrm{L}^{*n}$$

with ($\mathrm{L}^{*o} = \Delta_o$) converges absolutely in $\mathcal{E}_{sm}$. Furthermore,

$$\exp(\mathrm{L}_1 + \mathrm{L}_2) = \exp(\mathrm{L}_1) * \exp(\mathrm{L}_2).$$

The finite case. We start the construction with a finite signed measure L on $\mathfrak{X}$ satisfying $\mathrm{L}\{o\} = 0$. L generates within $\mathcal{E}_{sm}$ the signed measure

$$\Im_{\mathrm{L}} = \exp(\mathrm{L} - \mathrm{L}(\mathbf{1})\Delta_o) = \frac{1}{\Xi} \cdot \exp \mathrm{L},$$

where $\Xi = \exp\, \mathrm{L}(\mathbf{1})$. (Without confusing the reader, we use the same symbol both for the convolution and usual exponent.) Its Laplace transform is given by

$$\mathcal{L}_{\Im_{\mathrm{L}}} f = \exp(-\mathcal{K}_{\mathrm{L}}\, f), \qquad f \in U,$$

where

$$\mathcal{K}_{\mathrm{L}}\, f = \mathrm{L}(1 - \mathrm{e}^{-\zeta_f})$$

is the modified Laplace transform of L of Mecke [9]. It determines the measure L completely. Note that $\mathcal{K}_{\mathrm{L}}$ is real-valued, because the variation $|\,\mathrm{L}\,|$ is a finite positive measure on $\mathfrak{X}$, and thereby,

$$\mathcal{K}_{|\,\mathrm{L}\,|}\, f = |\,\mathrm{L}\,|(1 - \mathrm{e}^{-\zeta_f}) < +\infty, \qquad f \in U.$$

If, particularly, $\mathrm{L} = \mathrm{k}\,.\Lambda_\varrho$, where ϱ is a finite measure, k is real-valued with $\mathrm{k}(o) = 0$ and the measure Λ_ϱ is defined on $\mathfrak{X}$ by

$$\Lambda_\varrho(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \varphi(\varepsilon_{x_1} + \cdots + \varepsilon_{x_n})\, \varrho(\mathrm{d}\, x_1) \ldots \varrho(\mathrm{d}\, x_n), \qquad \varphi \in F,$$

then $\Im_{\mathrm{L}}$ has the representation

$$\Im_{\mathrm{L}} = \frac{1}{\Xi} \cdot \Gamma(\mathrm{k}).\Lambda_\varrho,$$

where $\Gamma(\mathrm{k})$ denotes the algebraic exponential of k (the details can be found in [17]). Thus, a sufficient condition for $\Im_{\mathrm{L}}$ to be a process is the condition

$$\Gamma\,(\mathrm{k}) \geq 0.$$

We then say that k is of positive type.

The infinitely extended case. We consider L of the form $\mathrm{k}\,.\Lambda_\varrho$ where ϱ is an infinite Radon measure, k is of positive type, $\mathrm{k}(o) = 0$, and assume the condition

$$\mathcal{K}_{|\,\mathrm{L}\,|}\, f < \infty, \qquad f \in U, \tag{1}$$

which is due to Mecke (cf. [9], Formula (5.1)). A sufficient condition for (1) is that $|\,\mathrm{L}\,|$ is of first order. Indeed, using $1 - \mathrm{e}^{-a} \leq a$ for $a \geq 0$, we have

$$|\,\mathrm{L}\,|(1 - \mathrm{e}^{-\zeta_f}) \leq |\,\mathrm{L}\,|(\zeta_f) = \nu^1_{|\,\mathrm{L}\,|}(f).$$

We will work under the condition that $|\,\mathrm{L}\,|$ is of first order since it implies that L is a Radon measure.

Consider the variation of L, i.e., the positive measure $|\,\mathrm{L}\,| = |\,\mathrm{k}\,|.\Lambda_\varrho$ on $\mathfrak{X}$. It may be infinite. In this case, to construct the associated process $\Im_{\mathrm{L}}$, we localize L and then go to the thermodynamic limit. To be more precise: Let $(K_n)_n$ be an increasing sequence of bounded Borel sets in X exhausting X if $n \to \infty$, i.e., $K_n \uparrow X$. Consider

$$\mathrm{L}_n = \mathbf{1}_{\mathfrak{X}_{K_n}} \cdot \mathrm{L}\,.$$

L_n is a finite signed measure since

$$|\,\mathrm{L}(\mathfrak{X}_{K_n})| \leq |\,\mathrm{L}\,|(\zeta_{K_n} \geq 1) = \nu^1_{|\,\mathrm{L}\,|}(K_n) < \infty.$$

Here we used that $\mathrm{k}(o) = 0$. Therefore, we are in the finite case and we can consider the sequence of processes $\Im_{\mathrm{L}_n}$. By means of Mecke's continuity theorem (cf. [10]), one obtains

Lemma 1 *For functions* k, $\mathrm{k}(o) = 0$, *of positive type and satisfying condition* (1), *there exists a limiting process* $\Im_{\mathrm{L}}$ *in* X *such that* $\Im_{\mathrm{L}_n}$ *converges weakly to* $\Im_{\mathrm{L}}$ *if* $n \to \infty$. *We then write* $\Im_{\mathrm{L}_n} \Rightarrow \Im_{\mathrm{L}}$. *The Laplace transform of* $\Im_{\mathrm{L}}$ *is given by* $\exp(-\mathcal{K}_{\mathrm{L}})$.

Remark 1 *It is worth to note that the condition* (1) *is the key assumption for the construction of KMM process* $P = \Im_{\mathrm{L}}$. *On the other hand,* $|\,\mathrm{L}\,|$ *to be of first order is a sufficient condition for* (1) *and can be proved under normal conditions on* Φ *via cluster expansion method in case where* k *is the Ursell function (see formula* (5) *below).*

1.3 The cluster equation

Let L be some σ-finite signed measure on $\mathfrak{X}$ with $\mathrm{L}\{o\} = 0$ of first order. Then $\Im_{\mathrm{L}}$ is of first order and is a solution P of the cluster equation (cf. Mecke [10])

$$\mathcal{C}_P = \mathcal{C}_{\mathrm{L}} \star P. \tag{2}$$

Here the operation $\star$ is defined by

$$\begin{aligned} \mathcal{C}_{\mathrm{L}} \star P\,(h) &= \int_{\mathcal{M}^{\cdot}} \int_X \int_{\mathfrak{X}} h(x, \xi + \mu)\, \mathcal{C}_{\mathrm{L}}(\mathrm{d}\, x\, \mathrm{d}\, \xi) P(\mathrm{d}\, \mu) \\ &= \int_{\mathcal{M}^{\cdot}} \int_X \int_{\mathfrak{X}} h(x, \xi + \mu)\, \xi(\mathrm{d}\, x)\, \mathrm{L}(\mathrm{d}\, \xi) P(\mathrm{d}\, \mu). \end{aligned}$$

Conversely, if L is of first order and P a process which solves (2), then $P = \Im_{\mathrm{L}}$ (cf. [14]).

Below we consider KMM-processes $P = \Im_{\mathrm{L}}$, where $\mathrm{L} = \mathrm{k}\,.\Lambda_\varrho$ with k of positive type and $\mathrm{k}(o) = 0$. We assume that ϱ is a positive, diffuse measure on X and $|\,\mathrm{L}\,|$ is of first order. In view of ϱ being diffuse, L has this property too. This implies that $\Im_{\mathrm{L}}$ is simple (cf. [14], Proposition 2.6.).

Conditions on the underlying potential. Let Φ be a pair potential on X, i.e., a measurable symmetric function $\Phi : X \times X \mapsto (-\infty, +\infty]$. We consider the following stability and regularity conditions:

> *(stability)* There exists a non-negative, measurable function $\mathrm{c} : X \to [0, +\infty)$ such that
>
> $$E_\Phi(\xi) \geq -\xi(\mathrm{c}), \qquad \xi \in \mathfrak{X}.$$
>
> Recall that $\xi(\mathrm{c}) = \sum_{x \in \xi} \mathrm{c}(x)$.

If c is given, the *regularity conditions* are *(weak* c*-regularity)* There exists a non-negative, measurable function $\mathrm{a} : X \to [0, +\infty)$ such that

$$\int_X (1 - \mathrm{e}^{-|\Phi|(x,y)})\, \mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\ \varrho(\mathrm{d}\, y) \le \mathrm{a}(x), \qquad x \in X.$$

*(*c*-regularity)* There exists a non-negative, measurable function $\mathrm{a} : X \to [0, +\infty)$ such that

$$\int_X |\omega|(x,y)\, \mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\ \varrho(\mathrm{d}\, y) \le \mathrm{a}(x), \qquad x \in X,$$

where $\omega(x,y) = \mathrm{e}^{-\Phi(x,y)} - 1$ is the Mayer function.

We assume also that the measure ϱ with density $\mathrm{e}^{\mathrm{c}+\mathrm{a}}$ is a positive Radon measure on X.

1.4 Stationary processes

Let $X = E$ be a lcscH space, which is also an Abelian group, and λ be some Haar measure on it. Write $\lambda(\mathrm{d}\, x) = \mathrm{d}\, x$. We then use the notation (E, λ) and have mainly in mind the Euclidean space $E = \mathbb{R}^d$ with Lebesgue measure.

The group operation in E is written additively: $T_x\, y = y - x$. It induces on the space of simple configurations $\mathcal{M}^{\cdot}(E) = \mathcal{M}^{\cdot}$ the transformation

$$(T_x\mu)(B) = \mu(B + x), \qquad B \in \mathcal{B}_0.$$

Write also $T_x\mu = \mu - x$. This in turn induces on the level of processes P the transformation $T_x P$, i.e., the image of P under T_x. P is called stationary if

$$T_x P = P, \qquad x \in E.$$

We shall consider below also σ-finite measures L on $\mathcal{M}^{\cdot}_f(E)$ which are stationary. If the first moment measure ν^1_P of P is stationary Radon measure, then it is of the form $\imath_P \cdot \lambda$ for some $\imath_P > 0$ (throughout processes are assumed to be of first order without further mentioning). $\imath_P$ thus is the expected number of particles in a region B with $\lambda(B) = 1$.

To a stationary P there belongs the Palm measure defined by

$$P^0(\varphi) = \int_{\mathcal{M}^{\cdot}} \int_E g(x)\varphi(\mu - x)\ \mu(\mathrm{d}\, x) P(\mathrm{d}\, \mu), \qquad \varphi \in F. \tag{3}$$

Here $g \in F$ is such that $\lambda(g) = 1$; and the right hand side does not depend on the choice of g. $P^0(\mathcal{M}^{\cdot})$ is called the intensity of P and is equal to $\nu^1_P(g) = \imath_P$. Thus, in general, P^0 is not a distribution. We denote by

$$P_0 = \frac{1}{\imath_P} \cdot P^0$$

the associated Palm probability of P.

Stationary classical continuous systems. Let Φ be a translation-invariant pair potential. Thus, $\Phi(x,y) = \phi(x-y)$ for some measurable function ϕ defined on E with $\phi(-x) = \phi(x)$. In this case, the stability function is a constant $\mathrm{c} \geq 0$, and we take $\mathrm{a} \equiv 1$.

We assume that Φ is c-stable and weakly c-regular:

$$C_{|\phi|} \cdot \mathrm{e}^{\mathrm{c}+1} \leq 1, \quad C_{|\phi|} = \int_E (1 - \mathrm{e}^{-|\phi|(y)})\,\mathrm{d}\,y. \tag{4}$$

These two conditions are called normal and in classical case include the important class of the Lennard–Jones type potentials. If in (4) inequality is strict, we speak about strong normal conditions.

Let $\mathrm{L} = \varkappa.\Lambda_\varrho$ where $\varkappa$ is the Ursell function defined by $\varkappa(o) = 0, \varkappa(x) \equiv 1$ and

$$\varkappa(\varepsilon_{x_1} + \cdots + \varepsilon_{x_n}) = \sum_{g \in \mathcal{C}_n} \prod_{\{i,j\} \in g} (\mathrm{e}^{-\phi(x_j - x_i)} - 1),$$

where $\mathcal{C}_n$ denotes the set of all simple unoriented, connected graphs g with n vertices, and the product is taken over all edges in g. It is well known (cf. Ruelle [21]) that $\varkappa$ is the cluster function of the Boltzmann factor, i.e., the Boltzmann factor is the algebraic exponential of $\varkappa$, or explicitly written

$$\mathrm{e}^{-E_\phi(\varepsilon_{x_1} + \cdots + \varepsilon_{x_k})} = \sum_{\mathcal{J} \in \wp[k]} \prod_{J \in \mathcal{J}} \varkappa((x_j)_{j \in J}),$$

where

$$E_\phi(\varepsilon_{x_1} + \cdots + \varepsilon_{x_k}) = \sum_{1 \leq i < j \leq k} \phi(x_i - x_j)$$

is the energy of the configuration $\varepsilon_{x_1} + \cdots + \varepsilon_{x_k}$ and the sum is taken over all partitions of $[k] := \{1, \ldots, k\}$ into disjoint non-empty subsets J. This implies that $\varkappa$ is of positive type, so that $\Im_{\mathrm{L}}$ is locally given by the Gibbs process for ϕ with empty boundary conditions.

Let us see that L satisfies the integrability condition (1). It is enough to show that L is of first order. Being of first order means

$$|\,\mathrm{L}\,|(\zeta_f) = \int_{\mathfrak{X}'} \xi(f) \cdot |\varkappa|(\xi)\ \Lambda_\varrho(\mathrm{d}\,\xi) < +\infty, \qquad f \in U. \tag{5}$$

This follows by partial integration:

$$\begin{aligned} |\,\mathrm{L}\,|(\zeta_f) &= \int_{\mathfrak{X}} \int_{\mathbb{R}^d} f(x) \mathbf{1}_{\mathfrak{X}'}(\xi) |\varkappa|(\xi)\ \xi(\mathrm{d}\,x) \Lambda_\varrho(\mathrm{d}\,\xi) \\ &= \int_{\mathbb{R}^d} f(x) \int_{\mathfrak{X}} |\varkappa|(\xi + \varepsilon_x)\ \Lambda_\varrho(\mathrm{d}\,\xi) \varrho(\mathrm{d}\,x) \\ &\leq \varrho(f) \cdot \mathrm{e}^{\mathrm{c}+1} < +\infty. \end{aligned}$$

In the last step, we used that under normal conditions on ϕ the inner integral can be estimated from above by $\mathrm{e}^{\mathrm{c}+1}$. The proof of this result for the case of c-stable and 2 c-regular potentials is given by Theorem 2.1 in [16] or Lemma 9 in [18]. For the case of c-stable and weak c-regular potentials this can be proved with the help of improved tree-graph estimate of Procacci and Yuhjtman [19] (see also [24]) following the same lines in proof of Theorem 2.1 in [16].

Then Lemma 1 implies the existence of the limiting Gibbs process $\Im_{\mathrm{L}}$ with empty boundary conditions, which will be denoted by G (in the sequel, G is called shortly the Gibbs process).

On the other hand, since Φ is translation-invariant, the σ-finite measure L is stationary too, and thereby, G has the same property. Indeed, the Laplace transform of $T_x\,\mathrm{G}$ does not depend on x, as is easily seen.

If the potential Φ is c-stable and c-regular, then G is even a Gibbs process for (Φ, ϱ). This means that G is a solution P of the equation

$$\mathcal{C}_P\, h = \int_X \int h(x, \mu + \varepsilon_x)\, \mathrm{e}^{-W_\Phi(x,\mu)}\; P(\mathrm{d}\,\mu)\varrho(\mathrm{d}\,x), \qquad h \in F. \tag{6}$$

This is the partial-integration equation of Nguyen and Zessin. It is equivalent to the DLR-equation (cf. [15]). We then write $P \in \mathcal{G}(\Phi, \varrho)$.

If we know that $P \in \mathcal{G}_0(\Phi, \varrho)$, i.e., $P \in \mathcal{G}(\Phi, \varrho)$ and stationary, and $\varrho = z\lambda$, $z > 0$, then the definition (3) implies, in view of its Gibbsian character,

$$P_0\, \varphi = \frac{z}{\imath_P} \cdot \int_{\mathcal{M}^\cdot} \varphi(\mu + \varepsilon_0)\, \mathrm{e}^{-W_\Phi(0,\mu)}\; P(\mathrm{d}\,\mu), \tag{7}$$

so that $\imath_P$ is determined by

$$\imath_P = z \cdot \int_{\mathcal{M}^\cdot} \mathrm{e}^{-W_\Phi(0,\mu)}\; P(\mathrm{d}\,\mu). \tag{8}$$

Equation (7) even characterizes stationary Gibbs processes (cf. Georgii [2] and Takahashi [23]). Indeed, let P be a stationary process in E of first order solving (7). Equation (7) implies condition (6). By Satz 2.3 in [9] one obtains

$$\begin{aligned}\mathcal{C}_P\, h &= \imath_P \cdot \int_E \int_{\mathcal{M}^\cdot} h(x, \mu + x)\; P_0(\mathrm{d}\,\mu)\lambda(\mathrm{d}\,x)\\ &= \int_E \int_{\mathcal{M}^\cdot} h(x, \mu + \varepsilon_x)\, \mathrm{e}^{-W_\Phi(x,\mu)}\; P(\mathrm{d}\,\mu) z\lambda(\mathrm{d}\,x),\end{aligned}$$

which shows that $P \in \mathcal{G}_0(\Phi, \varrho)$.

Mixing properties of stationary processes. We consider the dynamical systems $(\mathcal{M}^{\cdot}(E), P, (T_x)_{x\in E})$, where P is a stationary process P in E. (More generally, we shall consider below also σ-finite measures L which are stationary.) Recall that P is always assumed to be of first order. Then it is stationary and therefore of the form $\imath_P \cdot \lambda$ for some $\imath_P > 0$.

A process P is called ergodic if every invariant measurable set in $\mathcal{M}^{\cdot}$ has probability 0 or 1. P is called mixing if for all events $\mathcal{N}_1, \mathcal{N}_2$ in $\mathcal{M}^{\cdot}$,

$$\lim_{|y|\to\infty} P(\mathcal{N}_1 \cap T_y\,\mathcal{N}_2) = P(\mathcal{N}_1)\cdot P(\mathcal{N}_2).$$

Mixing implies ergodicity. We shall consider another mixing condition which is due to Brillinger [1]: A stationary process P is Brillinger-mixing if all its factorial cumulant measures, to be considered below in the next section, exist and are σ-finite.

2 Moments and cumulants

Let P be a process in X of infinite order. Its cumulant measure of order k is defined by the following cumulant-to-moments formula:

$$\gamma_P^k(\otimes_{j=1}^k f_j) = \sum_{\mathcal{J}\in\wp[k]} (-1)^{|\mathcal{J}|-1}(|\mathcal{J}|-1)! \prod_{J\in\mathcal{J}} \nu_P^{|J|}(\otimes_{j\in J} f_j), \tag{9}$$

$f_1,\dots,f_k \in U$. Conversely, the moment measure of order n can be expressed in terms of its cumulant measures as follows:

$$\nu_P^n(\otimes_{j=1}^n f_j) = \sum_{\mathcal{J}\in\wp[n]} \prod_{J\in\mathcal{J}} \gamma_P^{|J|}(\otimes_{j\in J} f_j). \tag{10}$$

Equations (9) or (10), respectively, express a one-to-one correspondence between moment and cumulant measures of a process P.

If we assume even that $|\,\mathrm{L}\,|$ is of infinite order, this implies that $P = \Im_{\mathrm{L}}$ is of infinite order too. Indeed, let $n \geq 1$ and $f \in U$. It is enough to show that $\nu_P^n\,(f^{\otimes n}) < +\infty$ if $f \in U(X^n)$. Using the cluster equation one obtains by induction

$$\begin{aligned}\nu_P^n(f^{\otimes n}) &= \mathcal{C}_P(f\otimes\zeta_{f^{\otimes(n-1)}}) \leq \mathcal{C}_{|\,\mathrm{L}\,|}\star P\,(f\otimes\zeta_{f^{\otimes(n-1)}})\\ &= \sum_{J\subset\{2,\dots,n\}} \nu_{|\,\mathrm{L}\,|}^{|J|+1}(f^{\otimes(|J|+1)})\nu_P^{|J^c|}(f^{\otimes|J^c|}) < \infty.\end{aligned}$$

The following basic result is a general version of the cluster equation (2).

Lemma 2 (Nehring [14]) *Assume that $|\,\mathrm{L}\,|$ is of infinite order. Let $f_1,\dots,f_n, g \in U$. Then*

$$\mathcal{C}_P^n(\otimes_{j=1}^n f_j\otimes \mathrm{e}^{-\zeta_g}) = \mathcal{L}_P(g)\cdot \sum_{\mathcal{J}\in\wp[n]} \prod_{J\in\mathcal{J}} \mathcal{C}_{\mathrm{L}}^{|J|}\left(\otimes_{j\in J} f_j\otimes \mathrm{e}^{-\zeta_g}\right).$$

Taking $g \equiv 0$ implies directly the following formula, which we shall need below:

$$\nu_P^n(\otimes_{j=1}^n f_j) = \sum_{\mathcal{J} \in \wp[n]} \prod_{J \in \mathcal{J}} \nu_{\mathrm{L}}^{|J|}(\otimes_{j \in J} f_j).$$

Proof. We indicate Nehring's reasoning of his proof of Theorem 4.3.1 in [14]. The proof is given by induction with respect to n. If $n = 1$, the assertion reduces to the cluster equation (2).

The induction hypotheses is introduced by means of the representation

$$\mathcal{C}_P^n(\otimes_{j=1}^n f_j \otimes \mathrm{e}^{-\zeta_g}) = -\frac{\mathrm{d}}{\mathrm{d}\, s}\mathcal{C}_P^{n-1}((\otimes_{j=1}^{n-1} f_j) \otimes \mathrm{e}^{-\zeta_{sf_n+g}})|_{s=0}. \tag{11}$$

Interchanging differentiation and integration, which is admissible since $|\,\mathrm{L}\,|$ and thereby P has moments of all orders, one has to evaluate the derivative with respect to s of the following product for $s = 0$:

$$\mathcal{L}_P(sf_n + g) \cdot \sum_{\mathcal{J} \in \wp[n-1]} \prod_{J \in \mathcal{J}} \mathcal{C}_{\mathrm{L}}^{|J|}((\otimes_J f_j) \otimes \mathrm{e}^{-\zeta_{sf_n+g}}).$$

One obtains the sum of two terms. For the derivative of the first term, one obtains at $s = 0$

$$\mathcal{L}_P(g) \cdot \mathcal{C}_{\mathrm{L}}(f_n \otimes \mathrm{e}^{-\zeta_g}) \cdot \sum_{\mathcal{J} \in \wp[n-1]} \prod_{J \in \mathcal{J}} \mathcal{C}_{\mathrm{L}}^{|J|}((\otimes_J f_j) \otimes \mathrm{e}^{-\zeta_g}).$$

For the second term, one uses Leibniz' rule combined with formula (11) with P replaced by L. $\square$

In view of the 1-1-correspondence between moment and cumulant measures in (9), equation (10) yields

Corollary 1 *If* $|\,\mathrm{L}\,|$ *is of infinite order, then*

$$\gamma_{\Im_{\mathrm{L}}}^{\ell} = \nu_{\mathrm{L}}^{\ell}, \qquad \ell \geq 1.$$

Thus, the moment measure of L of order ℓ coincides with the cumulant measure of $\Im_{\mathrm{L}}$ of the same order. And this implies directly the same for the associated factorial measures:

$$\widetilde{\gamma}_{\Im_{\mathrm{L}}}^{\ell} = \widetilde{\nu}_{\mathrm{L}}^{\ell}, \tag{12}$$

which are given by the restrictions of the measures in question to $\tilde{X}^{\ell}$.

We next evaluate the density of $\widetilde{\nu}^{\ell}_{|\mathrm{L}|}$ with respect to ϱ^{ℓ}, without assuming that $|\mathrm{L}|$ is of order ℓ. We can write

$$
\begin{aligned}
&\widetilde{\nu}^{\ell}_{|\mathrm{L}|}(\otimes_{j=1}^{\ell} f_j)\\
&= \sum_{n\geq\ell} \frac{1}{n!} \int_{X^n} \sum_{i_1,\dots,i_\ell\in[n]}^{\neq} f_1(x_{i_1})\cdots f_\ell(x_{i_\ell}) |\,\mathrm{k}\,|(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n})\varrho(\mathrm{d}\,x_1)\dots\varrho(\mathrm{d}\,x_n)\\
&= \sum_{n\geq\ell} \frac{1}{(n-\ell)!} \int_{X^n} f_1(x_1)\cdots f_\ell(x_\ell)\,|\,\mathrm{k}\,|(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n})\;\varrho(\mathrm{d}\,x_1)\dots\varrho(\mathrm{d}\,x_n)\\
&= \int_{X^\ell} f_1(x_1)\cdots f_\ell(x_\ell)\; r^{\ell}_{|\mathrm{L}|}(x_1,\dots,x_\ell)\;\varrho(\mathrm{d}\,x_1)\dots\varrho(\mathrm{d}\,x_\ell),
\end{aligned}
$$

where the sum under the integral sign denotes the sum over all ℓ-tuples $(i_1,\dots,i_\ell)$ with pairwise different i_j. It follows from (12) that the cumulant densities coincide with the correlation functions of L.

To summarize, we obtained

Lemma 3 *The Radon–Nikodym derivatives of the factorial moment measures of $|\mathrm{L}|$ of order ℓ with respect to the measures ϱ^{ℓ} are given by*

$$
\frac{d\widetilde{\nu}^{\ell}_{|\mathrm{L}|}}{d\varrho^{l}}(x_1,\dots,x_\ell) = \; r^{\ell}_{|\mathrm{L}|}(x_1,\dots,x_\ell) = \int_{\mathfrak{X}} |\,\mathrm{k}\,|(\xi+\varepsilon_{x_1}+\cdots+\varepsilon_{x_\ell})\;\Lambda_{\varrho}(\mathrm{d}\,\xi).
$$

It follows that this formula remains true without the modulus-sign if $|\mathrm{L}|$ is of order ℓ.

3 Strong Brillinger-mixing of G

Below we assume that k is given by the Ursell function $\varkappa$ and the Haar measure λ is a diffuse Radon measure. We write shortly $\mathrm{d}\,x$ for $\lambda(\mathrm{d}\,x)$.

The following result is a generalized and improved version of Ruelle's bound (cf. [21], estimate (4.37) of Theorem 4.4.8). Note that the radius of convergence in Lemma 4 is strictly larger then that of Ruelle's bound.

Lemma 4 *If Φ satisfies strong normal conditions*

$$
\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|} < 1,
$$

then the following estimate holds uniformly in x:

$$
\int_{E^{m-1}} r^{m}_{|\mathrm{L}|}(x,x_2,\dots,x_m)\,\mathrm{d}\,x_2\dots\mathrm{d}\,x_m \leq (m-1)!\,\mathrm{e}^{\mathrm{c}}\,\frac{\left(\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|}\right)^{m-1}}{\left(1-\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|}\right)^{m}}. \quad (13)
$$

Proof. We give a proof which uses tree-graph estimate of the Ursell function (cf. [16] or [17]). The left hand side of (12) satisfies

$$\int_{\mathfrak{X}}\int_{E^{m-1}} |\varkappa|(\xi+\varepsilon_x+\varepsilon_{x_2}+\cdots+\varepsilon_{x_m})\ \mathrm{d}\,x_2\ldots\mathrm{d}\,x_m\,\Lambda_\lambda(\mathrm{d}\,\xi) \\ \le \mathrm{e}^{\mathrm{c}}(\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|})^\ell\cdot\ell!$$

Indeed,

$$\begin{aligned}\int_{E^\ell}&|\varkappa|(\varepsilon_{y_0}+\varepsilon_{y_1}+\cdots+\varepsilon_{y_\ell})\ \mathrm{d}\,y_1\ldots\mathrm{d}\,y_\ell \\ &\le \mathrm{e}^{(\ell+1)\,\mathrm{c}}\cdot\int_{E^\ell}\sum_{\gamma\in\mathcal{T}(0,1,\ldots,\ell)}\prod_{\{i,j\}\in\gamma}\left(1-\mathrm{e}^{-|\phi(y_i-y_j)|}\right)\ \mathrm{d}\,y_1\ldots\mathrm{d}\,y_\ell \qquad (14)\\ &= \mathrm{e}^{\mathrm{c}}(\mathrm{e}^{\mathrm{c}}\,C_{|\phi|})^\ell\cdot(\ell+1)^{\ell-1}\le \mathrm{e}^{\mathrm{c}}(\mathrm{e}^{\mathrm{c}}\,C_{|\phi|})^\ell\cdot\ell!\,\mathrm{e}^\ell = \mathrm{e}^{\mathrm{c}}(\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|})^\ell\cdot\ell!\end{aligned}$$

Here we used that $(\ell+1)^{\ell-1}\le\ell^\ell$ is the number of trees in $\mathcal{T}(0,1,\ldots,\ell)$ combined with Stirling's formula.

By means of (14), the left hand side of (13) can be estimated from above as follows:

$$\sum_{n\ge 0}\frac{1}{n!}\int_{E^{n+m-1}}|\varkappa|(x,x_2\ldots,x_m,y_1,\ldots,y_n)\ \mathrm{d}\,x_2\ldots\mathrm{d}\,x_m\ \mathrm{d}\,y_1\ldots\mathrm{d}\,y_n \\ \le \mathrm{e}^{\mathrm{c}}\left(\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|}\right)^{m-1}\sum_{n\ge 0}\frac{\left(\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|}\right)^n}{n!}\,(n+m-1)!$$

This implies the assertion because for $\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|}<1$, the series on the right hand side converges to $\dfrac{(m-1)!}{(1-\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|})^m}$. □

Remark 2 *Estimate* (13) *contains the mixing condition in the sense of Brillinger [1], which means that the integral on the left hand side of* (13) *is finite.*

To be able to apply the results of Section 3, we need

Corollary 2 *Under strong normal conditions the measure* $|\,\mathrm{L}\,|=|\varkappa|.\Lambda_\lambda$ *is of infinite order.*

Proof. The moment measures of $|\,\mathrm{L}\,|$ can be represented as the sum of their factorial moment measures of lower order (cf. Theorem 4 of Krickeberg [6]). Thus, the assertion is reduced to showing that the factorial moment measures of $|\,\mathrm{L}\,|$ are Radon measures.

Lemma 4 combined with Lemma 3 implies for every $f_1,\ldots,f_m\in U$,

$$\tilde{\nu}^m_{|\,\mathrm{L}\,|}(\otimes_{j=1}^m f_j)\le\varrho(f_1)\|f_2\|\cdots\|f_m\|\cdot(m-1)!\,\mathrm{e}^{\mathrm{c}}\,\frac{(z\,\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|})^{m-1}}{(1-z\,\mathrm{e}^{\mathrm{c}+1}\,C_{|\phi|})^m},$$

and thereby the assertion. Here $\|f\|$, $f\in U$, denotes the sup norm of f. □

Combining Corollary 1 with Corollary 2, we obtain

Theorem 1 *Under strong normal conditions,* G *is strongly Brillinger mixing in the following sense: for all* $f \in U$*,*

$$\sum_{m=1}^{\infty} \frac{1}{m!} \cdot \tilde{\gamma}_{\mathrm{G}}^{m} (1 - \mathrm{e}^{-f})^{\otimes m} < +\infty.$$

This notion of strong Brillinger-mixing is related to the work of Lothar Heinrich (cf. [3], e.g.).

4 Mixing of G

In Theorem 1, we saw that the limiting Gibbs process G is strongly Brillinger-mixing and thereby Brillinger-mixing. We will now show that strong Brillinger-mixing implies that G is mixing and thereby ergodic. It has been shown recently that Brillinger-mixing processes need not be ergodic (cf. [4]).

To verify the mixing property for the Gibbs process $\mathrm{G} = \Im_{\mathrm{L}}$, we use the following criterium due to Westcott [25]: the probability generating functional of G is defined on the function space $V = \{\mathrm{v} = 1 - u | u \in U, u < 1\}$ by

$$\mathcal{W}_{\mathrm{G}}(\mathrm{v}) = \mathcal{L}_{\mathrm{G}}(-\log\ \mathrm{v}), \qquad \mathrm{v} \in V.$$

$\mathcal{W}_{\mathrm{G}}$ can be extended to $\overline{V} = \{\mathrm{v} = 1 - u | u \in U, u \leq 1\}$ (cf. [20]).

Lemma 5 (Westcott [25]) G *is mixing iff*

$$\lim_{|y| \to \infty} \mathcal{W}_{\mathrm{G}}(\mathrm{v}\, T_y\, \mathrm{w}) = \mathcal{W}_{\mathrm{G}}(\mathrm{v}) \cdot \mathcal{W}_{\mathrm{G}}(\mathrm{w}), \qquad \mathrm{v}, \mathrm{w} \in \overline{V}. \tag{15}$$

Here $T_y\, \mathrm{w}(x) = \mathrm{w}(x - y)$*.*

Since G is the KMM-process $\Im_{\mathrm{L}}$, its probability generating functional is of the form

$$\mathcal{W}_{\mathrm{G}}(\mathrm{v}) = \exp(-\mathcal{K}_{\mathrm{L}}(-\log \mathrm{v})), \qquad \mathrm{v} \in \overline{V}. \tag{16}$$

In view of equation (16), an equivalent formulation of equation (15) in terms of the modified Laplace transform of L is

$$\lim_{|y| \to \infty} \mathcal{K}_{\mathrm{L}}(f_1 + T_y f_2) = \mathcal{K}_{\mathrm{L}}(f_1) + \mathcal{K}_{\mathrm{L}}(f_2)$$

for all functions $f_j \in U$ of the form $f_j = -\log \mathrm{v}_j, \mathrm{v}_j \in \overline{V}$.

To evaluate the left hand side of (15), we use a representation of $\mathcal{K}_{\mathrm{L}}$ in terms of the factorial moment measures of L of Mecke (cf. Proposition 22 in [10]) and obtain in view of Corollary 1

Lemma 6 *For every $f \in U$,*

$$\mathcal{K}_{\mathrm{L}}\, f = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!} \widetilde{\gamma}_{\mathrm{L}}^{m} (1 - \mathrm{e}^{-f})^{\otimes m} \tag{17}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!} \widetilde{\nu}_{\mathrm{G}}^{m} (1 - \mathrm{e}^{-f})^{\otimes m}. \tag{18}$$

Remark that the right hand side of (18) or (17) is absolutely convergent by Theorem 1, which implies the identities.

We need to estimate $\widetilde{\nu}_{\mathrm{L}}^{m}(1 - \mathrm{e}^{-f})^{\otimes m}$ for functions $f = -\log(\mathrm{v}_1\, T_y\, \mathrm{v}_2)$ with $\mathrm{v}_j = 1 - u_j$. Observe first that

$$\widetilde{\nu}_{\mathrm{L}}^{m}(1 - \mathrm{e}^{-f})^{\otimes m} = \int_{E^m} \prod_{j=1}^{m} (u_1 + T_y u_2)(x_j)\; r_{\mathrm{L}}^{m}(x_1, \ldots, x_m)\; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m.$$

Here we used the device of Ivanoff [5] that for $|y|$ large enough

$$\mathrm{v}_1(x) T_y\, \mathrm{v}_2(x_j) = 1 - (\mathrm{u}_1 + T_y\, \mathrm{u}_2)(x_j).$$

We expand the product under the integral and obtain a sum of three different terms. One is of the form

$$\int_{E^m} \prod_{j=1}^{m} u_1(x_j)\; r_{\mathrm{L}}^{m}(x_1, \ldots, x_m)\; \mathrm{d}\, x_1 \ldots \varrho\, \mathrm{d}\, x_m.$$

Another is

$$\int_{E^m} \prod_{j=1}^{m} (T_y u_2)(x_j)\; r_{\mathrm{L}}^{m}(x_1, \ldots, x_m)\; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m,$$

and does not depend on y because the Lebesgue measure is translation invariant and r_{L}^{m} is invariant under the diagonal group. And the remaining third contribution is the sum of integrals of products containing at least one term of the form $u_1(x_j)$ and one of the form $T_y u_2(x_k)$. It remains to show that they vanish in the limit $|y| \to \infty$ and that the right hand side of (18) converges for the special functions f. The latter is clear because $u_1 + T_y u_2 \in U$.

For this, we may use Theorem 1, i.e., that G is strongly Brilinger-mixing. It implies that the integral of the third term vanishes in the limit $|y| \to +\infty$ on account of the dominated convergence theorem, because $T_y u_2 \in U$.

To summarize, we obtained another important result of this note, namely

Theorem 2 *Under strong normal conditions on the potential the limiting Gibbs process* G *is mixing.*

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References

[1] D.R. Brillinger, Statistical inference for stationary point processes. In: M.L. Puri (ed.), *Stochastic Processes and Related Topics*, Academic Press, pp. 55–99, 1975.

[2] H.-O. Georgii, Canonical and grand canonical Gibbs states for continuum systems. Commun. Math. Phys., **48** (1976), pp. 31–51. https://doi.org/10.1007/BF01609410

[3] L. Heinrich, On the strong Brillinger-mixing property of α-determinantal point processes and some applications. Appl. Math., **61** (2016), pp. 443–461. https://doi.org/10.1007/s10492-016-0141-y

[4] L. Heinrich, Brillinger-mixing point processes need not to be ergodic. Stat. Probab. Lett., **138** (2018), pp. 31–35. https://doi.org/10.1016/j.spl.2018.02.029

[5] G. Ivanoff, Central limit theorems for point processes. Stoch. Proc. Appl., **12** (1982), no. 2, pp. 171–186. https://doi.org/10.1016/0304-4149(82)90040-0

[6] K. Krickeberg, Moments of point processes. In: E.F. Harding and D.G. Kendall (ed.), *Stochastic Geometry*, Wiley, pp. 243–313, 1974.

[7] Y.G. Kondratiev, R.A. Minlos, M. Rockner and G.V. Shchepan'uk, Exponential mixing for classical continuous systems. Preprint of Diskrete Strukturen in der Mathematik SFB 343, Universitat Bielefeld, pp. 243–254 (1999).

[8] K. Matthes, J. Kerstan and J. Mecke, *Infinitely Divisible Point Processes*. Wiley, 1978.

[9] J. Mecke, Stationäre zufällige Maße auf lokalkompakten Abelschen gruppen. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **9** (1967), pp. 36–58. https://doi.org/10.1007/BF00535466

[10] J. Mecke, *Random Measures*. Classical Lectures, Walter Warmuth Verlag, 2011.

[11] R.A. Minlos, Gibbs limit distribution. Funct. Anal. Appl., **1** (1967), pp. 141–150. https://doi.org/10.1007/BF01076086

[12] R.A. Minlos, Regularity of the Gibbs limit distribution. Funct. Anal. Appl., **1** (1967), no. 3, pp. 206–217. https://doi.org/10.1007/BF01076905

[13] B. Nahapetian, *Limit Theorems and Some Applications in Statistical Physics*, Teubner Texte zur Mathematik Band **123**, 1991.

[14] B. Nehring, *Construction of classical and quantum Gases. The method of cluster expansions.* Mathematical lessons, Walter Warmuth Verlag, 2013.

[15] X.X. Nguyen and H. Zessin, Integral and differential characterizations of the Gibbs process. Math. Nachr., **88** (1979), pp. 105–115. https://doi.org/10.1002/mana.19790880109

[16] S. Poghosyan and D. Ueltschi, Abstract cluster expansion with applications to statistical mechanical systems. J. Math. Phys., **50** (2009), no. 5, article 053509. https://doi.org/10.1063/1.3124770

[17] S. Poghosyan and H. Zessin, Cluster representation of classical and quantum processes. Moscow Math. J., **19** (2019), pp. 1–19. https://doi.org/10.17323/1609-4514-2019-19-1-133-151

[18] S. Poghosyan and H. Zessin, Penrose-stable interactions in classical statistical mechanics. Ann. Henri Poincaré, **23** (2022), pp. 739–771.

[19] A. Procacci and S. Yuhjtman, Convergence of Mayer and Virial expansions and the Penrose tree-graph identity. Lett. Math. Phys., **107** (2017), no. 1, pp. 31–46. https://doi.org/10.1007/s11005-016-0918-7

[20] P. Ressel and W. Schmidtchen,A new characterization of Laplace functionals and probability generating functionals. Probab. Theory Relat. Fields, **88** (1991), pp. 195–213. https://doi.org/10.1007/BF01212559

[21] D. Ruelle, *Statistical Mechanics, Rigorous Results*, 3rd edition, Imperial College Press and World Scientific Publishing, 1999.

[22] H. Spohn, Equilibrium fluctuations for interacting Brownian particles. Commun. Math. Phys., **103** (1986), pp. 1–33. https://doi.org/10.1007/BF01464280

[23] Y. Takahashi, Random point fields revisited: Fock space associated with Poisson measures, fermion (determinantal) processes and Gibbs measures. Research Institute for Mathematical Sciences, Kyoto University, **1681** (2009).

[24] D. Ueltschi, An improved tree-graph bound, preprint arXiv:1705.05353, 2017.

[25] M. Westcott, The probability generating functional. J. Australian Math. Soc., **14** (1972), pp. 448–466. https://doi.org/10.1017/S1446788700011095

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