Armenian Journal of Mathematics
Volume 18, Number 7, 2026, 1–19
https://doi.org/10.52737/18291163-2026.18.07-1-19

Additivity of Transition Energy and its Applications

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Abstract. In this paper, we study the additivity property of the transition energy in a finite volume. In particular, we show how this property can be applied to solve the well-known problem of finding the conditions under which a given system of probability distributions parameterized by boundary conditions is a system of conditional probabilities for some finite random field (joint probability distribution).

Key Words: Transition Energy, Additivity, Finite Random Field, Conditional Probabilities
Mathematics Subject Classification 2020: 82B03, 60G60, 60E05, 62H99

Introduction

The Gibbs formula (probability distribution), which relates the probability of a given state of a physical system to the energy (Hamiltonian) of that state, was the foundation upon which classical statistical physics was built. Much attention has been paid to the justification of the Gibbs formula using physical reasoning, and this process still continues. However, this formula can be justified based on mathematical reasonings as was illustrated in [8]. More precisely, in [8], it was shown that any finite-volume probability distribution necessarily admits the Gibbs form and any function can be considered as a Hamiltonian.

The mentioned results are based on the concept of a transition energy field introduced in [8] as part of the development of a new approach to the mathematical foundations of the statistical physics of infinite-volume systems. This concept proved useful in studying systems in both infinite [8, 12, 15–18] and finite [13, 14] volumes.

At the core of our approach is the following idea: instead of the Hamiltonian, which is a function of one variable, we consider the transition energy, that is, a function of two variables whose value can be interpreted as the amount of energy required to change the state of the system from one state to another.

In this paper, we study the additivity property of the transition energy in a finite volume and show how this property can be applied to solve the following well-known problem from probability theory: to find conditions under which a given system of probability distributions parameterized by boundary conditions is a system of conditional probabilities for some finite random field (joint probability distribution). Such (intrinsic) conditions are usually called *compatibility conditions* or, following Dobrushin [6], consistency conditions, and the corresponding random field is called *compatible with* the given system.[1]

There is a large number of works devoted to the above mentioned problem (see, for example, [1–5,9]). As it was mentioned by Geman and Geman [10], "it is extremely difficult to spot local characteristics, i.e., to determine when a given set of functions ... are conditional probabilities for some (necessarily unique) distribution." The relevance of these studies is connected not only with theoretical interest but also with the huge number of practical problems discussed, for example, in works by Besag [3] and Geman and Geman [10]. We will show that this problem has a simple and natural solution using the existing fundamental relationship between energy and probability.

1 Notations

Let Λ be a finite subset of the integer lattice $\mathbb{Z}^d$, $d \geq 1$. Following physical terminology, Λ is sometimes called the finite volume. We write $V \subsetneq \Lambda$ to denote a proper subset V of Λ. For one-point sets $\{t\}$, $t \in \Lambda$, braces will be omitted.

Let each point $t \in \Lambda$ be associated with a set X^t, which is a copy of some finite set X, $1 < |X| < \infty$. By X^Λ we denote the set of all configurations on Λ, that is, the set of all functions on Λ taking values in X: $X^\Lambda = \{x = (x_t, t \in \Lambda) : x_t \in X^t, t \in \Lambda\}$. We assume that $X^\varnothing = \{\boldsymbol{\varnothing}\}$ where $\boldsymbol{\varnothing}$ denotes an empty configuration. By $x_V = (x_t, t \in V)$ we denote the restriction of configuration $x \in X^\Lambda$ on $V \subsetneq \Lambda$. For any $V, I \subset \Lambda$, $V \cap I = \varnothing$ and any $x \in X^V$, $y \in X^I$, by xy we denote the concatenation of x and y, that is, the configuration on $V \cup I$ equal to x on V and to y on I. In particular, for $y = \boldsymbol{\varnothing}$, we put $x\boldsymbol{\varnothing} = x$.

[1]Note that, in the formulation of this problem, some authors use the term *compatible* to mean that there exists a random field compatible with the system, although the term more naturally refers to a system satisfying the compatibility conditions.

2 Transition energy

We start by the definition of the transition energy (of a physical system) in Λ. This concept was introduced in [8] and was further developed in [13, 14].

A real-valued function Δ_Λ on $X^\Lambda \times X^\Lambda$ is called a *transition energy* in Λ if

$$\Delta_\Lambda(x,u) = \Delta_\Lambda(x,y) + \Delta_\Lambda(y,u), \qquad x,u,y \in X^\Lambda. \tag{1}$$

The following statement provides the representation for the transition energy which allows it to be constructed from any positive function on X^Λ.

Theorem 1 *A function Δ_Λ on $X^\Lambda \times X^\Lambda$ is a transition energy in Λ if and only if it has the following form:*

$$\Delta_\Lambda(x,u) = \ln \frac{\varphi_\Lambda(x)}{\varphi_\Lambda(u)}, \qquad x,u \in X^\Lambda, \tag{2}$$

where φ_Λ is a positive function on X^Λ.

Proof. It is obvious that function Δ_Λ defined by (2) satisfies (1), and thus, is a transition energy.

Now, let Δ_Λ be a transition energy in Λ. Then, due to (1), for any $x,u,y \in X^\Lambda$, we have

$$e^{\Delta_\Lambda(y,u)} = e^{\Delta_\Lambda(y,x)} e^{\Delta_\Lambda(x,u)}.$$

Taking sum over all $y \in X^\Lambda$, we obtain

$$\sum_{y\in X^\Lambda} e^{\Delta_\Lambda(y,u)} = e^{\Delta_\Lambda(x,u)} \sum_{y\in X^\Lambda} e^{\Delta_\Lambda(y,x)},$$

and therefore,

$$e^{\Delta_\Lambda(x,u)} = \frac{\sum\limits_{y\in X^\Lambda} e^{\Delta_\Lambda(y,u)}}{\sum\limits_{y\in X^\Lambda} e^{\Delta_\Lambda(y,x)}} = \frac{\varphi_\Lambda(x)}{\varphi_\Lambda(u)},$$

where $\varphi_\Lambda(x) = \left(\sum\limits_{y\in X^\Lambda} e^{\Delta_\Lambda(y,x)} \right)^{-1}$. $\square$

2.1 Additivity of transition energy

Transition energy in Λ possess an important and useful additivity property. Namely, it can be calculated as the sum of transition energies with boundary conditions in the elements of some partition of Λ.

Let Δ_Λ be a transition energy in Λ. For a proper subset (sub-volume) V of Λ and configuration $\bar{x} \in X^{\Lambda \setminus V}$, consider the following function:

$$\Delta_V^{\bar{x}}(x,u) = \Delta_\Lambda(x\bar{x}, u\bar{x}), \qquad x,u \in X^V. \tag{3}$$

In the case $V = \emptyset$, we assume that $\Delta_\emptyset^{\bar{x}}(\boldsymbol{\emptyset}, \boldsymbol{\emptyset}) = 0$ for any $\bar{x} \in X^\Lambda$. It is not difficult to see that for each $\bar{x} \in X^{\Lambda \setminus V}$, $\Delta_V^{\bar{x}}$ is a transition energy in V, that is, for all $x, u, y \in X^V$, we have

$$\Delta_V^{\bar{x}}(x,u) = \Delta_V^{\bar{x}}(x,y) + \Delta_V^{\bar{x}}(y,u).$$

Indeed, due to (1),

$$\Delta_V^{\bar{x}}(x,u) = \Delta_\Lambda(x\bar{x}, u\bar{x}) = \Delta_\Lambda(x\bar{x}, y\bar{x}) + \Delta_\Lambda(y\bar{x}, u\bar{x}) = \Delta_V^{\bar{x}}(x,y) + \Delta_V^{\bar{x}}(y,u).$$

The value of the function $\Delta_V^{\bar{x}}(x,u)$ corresponds to the amount of energy required to transfer the system from state x to state u in part V of volume Λ, considering the unchangeable part $\bar{x}$ of the configuration of the system as a fixed boundary condition (environment). The function $\Delta_V^{\bar{x}}$ will be called a *transition energy in V with boundary condition $\bar{x} \in X^{\Lambda \setminus V}$*.

Note that if we take $V = \Lambda$ in (3), we will have

$$\Delta_\Lambda^{\boldsymbol{\emptyset}}(x,u) = \Delta_\Lambda(x\boldsymbol{\emptyset}, u\boldsymbol{\emptyset}) = \Delta_\Lambda(x,u), \qquad x,u \in X^\Lambda. \tag{4}$$

The following result shows that a sum of transition energies in two disjoint sub-volumes of Λ forms a transition energy in the union of these sub-volumes.

Proposition 1 *For any disjoint $V_1, V_2 \subsetneq \Lambda$ and any $\bar{x} \in X^{\Lambda \setminus (V_1 \cup V_2)}$, it holds*

$$\Delta_{V_1}^{\bar{x}x_2}(x_1,u_1) + \Delta_{V_2}^{\bar{x}u_1}(x_2,u_2) = \Delta_{V_1 \cup V_2}^{\bar{x}}(x_1x_2, u_1u_2),$$

$x_1, u_1 \in X^{V_1}$, $x_2, u_2 \in X^{V_2}$.

Proof. Let $V_1, V_2 \subsetneq \Lambda$ be such that $V_1 \cap V_2 = \emptyset$ and let $\bar{x} \in X^{\Lambda \setminus (V_1 \cup V_2)}$. Then, using (3) and (1), for any $x_1, u_1 \in X^{V_1}$ and $x_2, u_2 \in X^{V_2}$, we can write

$$\begin{aligned}
\Delta_{V_1}^{\bar{x}x_2}(x_1,u_1) + \Delta_{V_2}^{\bar{x}u_1}(x_2,u_2) &= \Delta_\Lambda(x_1x_2\bar{x}, u_1x_2\bar{x}) + \Delta_\Lambda(u_1x_2\bar{x}, u_1u_2\bar{x}) \\
&= \Delta_\Lambda(x_1x_2\bar{x}, u_1u_2\bar{x}) \\
&= \Delta_{V_1 \cup V_2}^{\bar{x}}(x_1x_2, u_1u_2).
\end{aligned}$$

□

Varying $V \subsetneq \Lambda$ and $\bar{x} \in X^{\Lambda \setminus V}$, we see that each transition energy Δ_Λ in Λ defines the family $\boldsymbol{D}(\Delta_\Lambda) = \{\Delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \subsetneq \Lambda\}$ of transition energies in sub-volumes $V \subsetneq \Lambda$ with boundary conditions $\bar{x} \in X^{\Lambda \setminus V}$.

Proposition 1 allows us to conclude that the elements of $\boldsymbol{D}(\Delta_\Lambda)$ possess the following property.

Proposition 2 *For any transition energy Δ_Λ in Λ, the elements of $\boldsymbol{D}(\Delta_\Lambda)$ are consistent in the following sense: for any disjoint $V_1, V_2 \subsetneq \Lambda$ and configuration $\bar{x} \in X^{\Lambda\setminus(V_1\cup V_2)}$, it holds*

$$\Delta_{V_1}^{\bar{x}x_2}(x_1,u_1) + \Delta_{V_2}^{\bar{x}u_1}(x_2,u_2) = \Delta_{V_2}^{\bar{x}x_1}(x_2,u_2) + \Delta_{V_1}^{\bar{x}u_2}(x_1,u_1), \tag{5}$$

$x_1, u_1 \in X^{V_1}$, $x_2, u_2 \in X^{V_2}$.

Consistency conditions (5) have a simple physical meaning. In order to change the state of the system in $V_1 \cup V_2$ from x to u under condition $\bar{x} \in X^{\Lambda\setminus(V_1\cup V_2)}$, one can proceed in two ways. First, change the state of the part of the system in sub-volume V_1 from x_{V_1} to u_{V_1} with the state $\bar{x}x_{V_2}$ in $\Lambda\setminus V_1$ remaining unchanged, and thus, considering as a boundary condition, and then change the state in V_2 from x_{V_2} to u_{V_2} under the boundary condition $\bar{x}u_{V_1}$ in $\Lambda\setminus V_2$. Or change the state in V_2 from x_{V_2} to u_{V_2} under the boundary condition $\bar{x}x_{V_1}$ in $\Lambda\setminus V_2$, then change the state in V_1 from x_{V_1} to u_{V_1} under the condition that in $\Lambda\setminus V_1$ we have $\bar{x}u_{V_2}$. Naturally, the same amount of energy is required in both cases.

Now let $\Pi = \{V_1, V_2, ..., V_n\}$ be a partition of the set Λ, i.e., $V_j \cap V_k = \varnothing$ for $j \neq k$ and $\bigcup\limits_{j=1}^{n} V_n = \Lambda$. Consider the family

$$\boldsymbol{D}_\Pi(\Delta_\Lambda) = \{\Delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \in \Pi\}$$

of transition energies with boundary conditions in the elements of the partition Π defined by formula (3). Using Proposition 1, it is not difficult to obtain the following result.

Theorem 2 *Let Δ_Λ be a transition energy in Λ. Then for any partition $\Pi = \{V_1, V_2, ..., V_n\}$ of Λ, it holds*

$$\begin{aligned}&\Delta_\Lambda(x,u) = \\ &= \Delta_{V_1}^{x_{\Lambda\setminus V_1}}(x_{V_1}, u_{V_1}) + \Delta_{V_2}^{u_{V_1}x_{\Lambda\setminus(V_1\cup V_2)}}(x_{V_2}, u_{V_2}) + ... + \Delta_{V_n}^{u_{\Lambda\setminus V_n}}(x_{V_n}, u_{V_n})\end{aligned} \tag{6}$$

for all $x, u \in X^\Lambda$.

Proof. Let $\Pi = \{V_1, V_2, ..., V_n\}$ be a partition of Λ. Then for any $x, u \in X^\Lambda$, subsequently applying Proposition 1, we can write

$$\begin{aligned}&\Delta_{V_1}^{x_{\Lambda\setminus V_1}}(x_{V_1}, u_{V_1}) + \Delta_{V_2}^{u_{V_1}x_{\Lambda\setminus(V_1\cup V_2)}}(x_{V_2}, u_{V_2}) + ... + \Delta_{V_n}^{u_{\Lambda\setminus V_n}}(x_{V_n}, u_{V_n}) \\ &\quad = \Delta_{V_1\cup V_2}^{x_{\Lambda\setminus(V_1\cup V_2)}}(x_{V_1\cup V_2}, u_{V_1\cup V_2}) + \Delta_{V_3}^{u_{V_1\cup V_2}x_{\Lambda\setminus(V_1\cup V_2\cup V_3)}}(x_{V_3}, u_{V_3}) \\ &\qquad\quad + ... + \Delta_{V_n}^{u_{\Lambda\setminus V_n}}(x_{V_n}, u_{V_n}) = \ldots = \\ &\quad = \Delta_{V_1\cup...\cup V_{n-1}}^{x_{\Lambda\setminus(V_1\cup...\cup V_{n-1})}}(x_{V_1\cup...\cup V_{n-1}}, u_{V_1\cup...\cup V_{n-1}}) + \Delta_{V_n}^{u_{\Lambda\setminus V_n}}(x_{V_n}, u_{V_n}) \\ &\quad = \Delta_\Lambda^{\varnothing}(x,u) = \Delta_\Lambda(x,u).\end{aligned}$$

In the last equality, we used notation (4). $\square$

From Theorem 2, we obtain the following important result.

Corollary 1 *For any transition energy Δ_Λ in Λ, it holds*

$$\Delta_\Lambda(x,u) = \Delta_{t_1}^{x_{\Lambda\setminus t_1}}(x_{t_1},u_{t_1}) + \Delta_{t_2}^{u_{t_1}x_{\Lambda\setminus\{t_1,t_2\}}}(x_{t_2},u_{t_2}) + ... + \Delta_{t_n}^{u_{\Lambda\setminus t_n}}(x_{t_n},u_{t_n}),$$

$x,u \in X^\Lambda$, where $\Lambda = \{t_1,t_2,...,t_n\}$ is some enumeration of points in Λ, $n = |\Lambda|$.

2.2 Transition energy field

The following natural question arises: is it possible to construct a transition energy in Λ by means of some family of consistent transition energies in the elements of partition of Λ parameterized by boundary conditions? Below, we show that the answer to this question is positive.

Let $\Pi = \{V_1, V_2, \ldots, V_n\}$ be a partition of Λ. Denote by $\mathscr{W}_\Pi$ the family of all sets obtained as unions of the elements of Π.

Consider an *a priory* (autonomously) given family $\mathbf{D}_\mathscr{W} = \{\delta_W^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus W}, W \in \mathscr{W}_\Pi\}$ of transition energies parameterized by boundary conditions, that is, functions $\delta_W^{\bar{x}}$ on $X^W \times X^W$, $W \in \mathscr{W}_\Pi$, such that for any $\bar{x} \in X^{\Lambda\setminus W}$, it holds

$$\delta_W^{\bar{x}}(x,u) = \delta_W^{\bar{x}}(x,y) + \delta_W^{\bar{x}}(y,u), \qquad x,u,y \in X^W. \tag{7}$$

The family $\mathbf{D}_\mathscr{W}$ will be called a *transition energy field in* Λ *under partition* Π if for any $W_1, W_2 \in \mathscr{W}_\Pi$ and $\bar{x} \in X^{\Lambda\setminus(W_1\cup W_2)}$, it holds

$$\delta_{W_1}^{\bar{x}x_2}(x_1,u_1) + \delta_{W_2}^{\bar{x}u_1}(x_2,u_2) = \delta_{W_2}^{\bar{x}x_1}(x_2,u_2) + \delta_{W_1}^{\bar{x}u_2}(x_1,u_1), \tag{8}$$

$x_1, u_1 \in X^{W_1}$, $x_2, u_2 \in X^{W_2}$.

The family $\mathbf{D}_\Pi = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus V}, V \in \Pi\}$ of transition energies in the elements of Π parameterized by boundary conditions and consistent in the sense (8) will be called an *elementary transition energy field in* Λ *under partition* Π.

It is clear, that each transition energy field $\mathbf{D}_\mathscr{W}$ contains the elementary transition energy field $\mathbf{D}_\Pi$. The following result shows that any elementary transition energy field $\mathbf{D}_\Pi$ can be uniquely extended to the corresponding transition energy field $\mathbf{D}_\mathscr{W}$.

Theorem 3 *Let Π be a partition of Λ and $\mathbf{D}_\Pi$ be the corresponding elementary transition energy field. Then there exists the unique transition energy field $\mathbf{D}_\mathscr{W} = \{\delta_W^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus W}, W \in \mathscr{W}_\Pi\}$ in Λ under partition Π such that for any $W = \bigcup_{j=1}^{k} V_{i_j}$, $V_{i_j} \in \Pi$, $\bar{x} \in X^{\Lambda\setminus W}$ and $x,u \in X^W$, it holds*

$$\delta_W^{\bar{x}}(x,u) = \delta_{V_{i_1}}^{\bar{x}x_2\ldots x_k}(x_1,u_1) + \delta_{V_{i_2}}^{\bar{x}u_1x_3\ldots x_k}(x_2,u_2) + ... + \delta_{V_{i_k}}^{\bar{x}u_1\ldots u_{k-1}}(x_k,u_k),$$

where z_j *denotes the restriction of configuration* $z \in X^W$ *on* V_{i_j}, $1 \le j \le k$, *and therefore,* $\mathbf{D}_\Pi \subset \mathbf{D}_{\mathscr{W}}$.

Proof. Let $\mathbf{D}_\Pi = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \in \Pi\}$ be an elementary transition energy field corresponding to partition Π of Λ. To construct $\mathbf{D}_{\mathscr{W}}$, we proceed by induction as follows.

For any $V_1, V_2 \in \Pi$ and any $\bar{x} \in X^{\Lambda \setminus (V_1 \cup V_2)}$, consider the function

$$\delta_{V_1 \cup V_2}^{\bar{x}}(x_1 x_2, u_1 u_2) = \delta_{V_1}^{\bar{x} x_2}(x_1, u_1) + \delta_{V_2}^{\bar{x} u_1}(x_2, u_2),$$

$x_1, u_1 \in X^{V_1}$, $x_2, u_2 \in X^{V_2}$. This function is correctly defined since, according to (8), we have

$$\begin{aligned} \delta_{V_2 \cup V_1}^{\bar{x}}(x_2 x_1, u_2 u_1) &= \delta_{V_2}^{\bar{x} x_1}(x_2, u_2) + \delta_{V_1}^{\bar{x} u_2}(x_1, u_1) \\ &= \delta_{V_1}^{\bar{x} x_2}(x_1, u_1) + \delta_{V_2}^{\bar{x} u_1}(x_2, u_2) \\ &= \delta_{V_1 \cup V_2}^{\bar{x}}(x_1 x_2, u_1 u_2). \end{aligned}$$

For any $y_1 \in X^{V_1}$, $y_2 \in X^{V_2}$, using (8), we can write

$$\begin{aligned} &\delta_{V_1 \cup V_2}^{\bar{x}}(x_1 x_2, y_1 y_2) + \delta_{V_1 \cup V_2}^{\bar{x}}(y_1 y_2, u_1 u_2) = \\ &\quad = \delta_{V_1}^{\bar{x} x_2}(x_1, y_1) + \delta_{V_2}^{\bar{x} y_1}(x_2, y_2) + \delta_{V_2}^{\bar{x} y_1}(y_2, u_2) + \delta_{V_1}^{\bar{x} u_2}(y_1, u_1) \\ &\quad = \delta_{V_1}^{\bar{x} x_2}(x_1, y_1) + \delta_{V_2}^{\bar{x} y_1}(x_2, u_2) + \delta_{V_1}^{\bar{x} u_2}(y_1, u_1) \\ &\quad = \delta_{V_1}^{\bar{x} x_2}(x_1, y_1) + \delta_{V_1}^{\bar{x} x_2}(y_1, u_1) + \delta_{V_2}^{\bar{x} u_1}(x_2, u_2) \\ &\quad = \delta_{V_1}^{\bar{x} x_2}(x_1, u_1) + \delta_{V_2}^{\bar{x} u_1}(x_2, u_2) \\ &\quad = \delta_{V_1 \cup V_2}^{\bar{x}}(x_1 x_2, u_1 u_2). \end{aligned}$$

Hence, for each $\bar{x} \in X^{\Lambda \setminus (V_1 \cup V_2)}$, function $\delta_{V_1 \cup V_2}^{\bar{x}}$ is a transition energy in $V_1 \cup V_2$.

Further, for any $V_3 \in \Pi$, $V_3 \neq V_1, V_2$, $\bar{x} \in X^{\Lambda \setminus (V_1 \cup V_2 \cup V_3)}$ and $x_1, u_1 \in X^{V_1}$, $x_2, u_2 \in X^{V_2}$, $x_3, u_3 \in X^{V_3}$, applying consistency conditions (8), we can write

$$\begin{aligned} &\delta_{V_1 \cup V_2}^{\bar{x} x_3}(x_1 x_2, u_1 u_2) + \delta_{V_3}^{\bar{x} u_1 u_2}(x_3, u_3) = \\ &\quad = \delta_{V_1}^{\bar{x} x_3 x_2}(x_1, u_1) + \delta_{V_2}^{\bar{x} x_3 u_1}(x_2, u_2) + \delta_{V_3}^{\bar{x} u_1 u_2}(x_3, u_3) \\ &\quad = \delta_{V_1}^{\bar{x} x_3 x_2}(x_1, u_1) + \delta_{V_3}^{\bar{x} u_1 x_2}(x_3, u_3) + \delta_{V_2}^{\bar{x} u_1 u_3}(x_2, u_2) \\ &\quad = \delta_{V_3}^{\bar{x} x_2 x_1}(x_3, u_3) + \delta_{V_1}^{\bar{x} x_2 u_3}(x_1, u_1) + \delta_{V_2}^{\bar{x} u_1 u_3}(x_2, u_2) \\ &\quad = \delta_{V_3}^{\bar{x} x_1 x_2}(x_3, u_3) + \delta_{V_1 \cup V_2}^{\bar{x} u_3}(x_1 x_2, u_1 u_2). \end{aligned}$$

Hence, transition energies $\delta_{V_1 \cup V_2}^{\bar{x} x_3}$ and $\delta_{V_3}^{\bar{x} x_1 x_2}$ are consistent in the sense (8).

In general, if we have already defined transition energies parameterized by boundary conditions in the disjoint sets $W_1, W_2 \in \mathscr{W}_\Pi$, then, using the similar arguments, we show that the function

$$\delta_{W_1 \cup W_2}^{\bar{x}}(x_1 x_2, u_1 u_2) = \delta_{W_1}^{\bar{x} x_2}(x_1, u_1) + \delta_{W_2}^{\bar{x} u_1}(x_2, u_2),$$

$x_1, u_1 \in X^{W_1}$, $x_2, u_2 \in X^{W_2}$, $\bar{x} \in X^{\Lambda\setminus(W_1\cup W_2)}$, is correctly defined, forms a transition energy in $W_1 \cup W_2$ and is consistent with any other defined transition energy in $W_3 \in \mathscr{W}_\Pi$, $W_3 \cap W_j = \varnothing$, $j = 1, 2$.

Thus, for any $W = \bigcup_{j=1}^{k} V_{i_j}$, $V_{i_j} \in \Pi$, $\bar{x} \in X^{\Lambda\setminus W}$ and $x, u \in X^W$, we obtain

$$\delta_W^{\bar{x}}(x,u) = \delta_{V_{i_1}}^{\bar{x}x_2\ldots x_k}(x_1,u_1) + \delta_{V_{i_2}}^{\bar{x}u_1x_3\ldots x_k}(x_2,u_2) + \ldots + \delta_{V_{i_k}}^{\bar{x}u_1\ldots u_{k-1}}(x_k,u_k),$$

where z_j denotes the restriction of configuration $z \in X^W$ on V_{i_j}, $1 \le j \le k$. The values of $\delta_W^{\bar{x}}$ do not depend on the enumeration of the sets $V_{i_1}, V_{i_2}, \ldots, V_{i_k}$ from Π. Indeed, any permutation of $V_{i_1}, V_{i_2}, \ldots, V_{i_k}$ can be decomposed into a finite number of transpositions of adjacent elements $V_{i_{j-1}}$ and V_{i_j}, $j = 2, \ldots, k$. Conditions (8) guaranties that for each such transposition, the value of $\delta_W^{\bar{x}}$ does not change. Hence, we have constructed the unique transition energy field $\mathbf{D}_{\mathscr{W}} = \{\delta_W^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus W}, W \in \mathscr{W}_\Pi\}$. $\square$

In particular, since for any partition $\Pi = \{V_1, V_2, \ldots, V_n\}$ of Λ, we have $\Lambda = \bigcup_{j=1}^{n} V_j \in \mathscr{W}_\Pi$, the system $\mathbf{D}_{\mathscr{W}}$ contains the transition energy $\Delta_\Lambda^{\varnothing} = \Delta_\Lambda$ in Λ (see notations (4)). Therefore, the following result holds true.

Corollary 2 *Any elementary transition energy field $\mathbf{D}_\Pi$ corresponding to some partition Π of Λ uniquely determines transition energy Δ_Λ in Λ such that for any $V \in \Pi$ and $\bar{x} \in X^{\Lambda\setminus V}$, it holds*

$$\Delta_\Lambda(x\bar{x}, u\bar{x}) = \delta_V^{\bar{x}}(x,u), \qquad x, u \in X^V.$$

In this case, for all $x, u \in X^\Lambda$,

$$\Delta_\Lambda^{\bar{x}}(x,u) = \delta_{V_1}^{\bar{x}x_{\Lambda\setminus V_1}}(x_{V_1}, u_{V_1}) + \delta_{V_2}^{\bar{x}u_{V_1}x_{\Lambda\setminus(V_1\cup V_2)}}(x_{V_2}, u_{V_2}) + \ldots + \delta_{V_n}^{\bar{x}u_{\Lambda\setminus V_n}}(x_{V_n}, u_{V_n}).$$

The case when the elements of Π are one-point subsets of Λ is of a special interest.

A set $\mathbf{D}_1 = \{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ of real-valued functions $\delta_t^{\bar{x}}(x,u)$, $x, u \in X^t$, parameterized by boundary conditions $\bar{x} \in X^{\Lambda\setminus t}$, $t \in \Lambda$, is called a *one-point transition energy field* in Λ if

1. for each $t \in \Lambda$ and $\bar{x} \in X^{\Lambda\setminus t}$, the function $\delta_t^{\bar{x}}$ is a transition energy, that is,
$$\delta_t^{\bar{x}}(x,u) = \delta_t^{\bar{x}}(x,y) + \delta_t^{\bar{x}}(y,u), \qquad x, u, y \in X^t;$$
2. for all $t, s \in \Lambda$ and $\bar{x} \in X^{\Lambda\setminus\{t,s\}}$, the following consistency conditions hold:
$$\delta_t^{\bar{x}y}(x,u) + \delta_s^{\bar{x}u}(y,v) = \delta_s^{\bar{x}x}(y,v) + \delta_t^{\bar{x}v}(x,u), \qquad x, u \in X^t, y, v \in X^s. \quad (9)$$

It is clear that if the elements of Π are one-point subsets of Λ, the family $\mathscr{W}_\Pi$ contains all subsets of Λ. The direct consequence of Theorem 3 are the following results.

Corollary 3 *Let $\mathbf{D}_1$ be a one-point transition energy field in Λ. Then there exists a unique family $\mathbf{D} = \{\delta_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \subsetneq \Lambda\}$ of transition energies in the subsets of Λ parameterized by boundary conditions and consistent in the sense (8). In this case, for any $V \subsetneq \Lambda$, $\bar{x} \in X^{\Lambda \setminus V}$ and $x, u \in X^V$, it holds*

$$\delta_V^{\bar{x}}(x,u) = \delta_{t_1}^{x_{\Lambda \setminus t_1}}(x_{t_1}, u_{t_1}) + \delta_{t_2}^{u_{t_1} x_{\Lambda \setminus \{t_1,t_2\}}}(x_{t_2}, u_{t_2}) + ... + \delta_{t_k}^{u_{\Lambda \setminus t_n}}(x_{t_k}, u_{t_k}),$$

where $V = \{t_1, t_2, ..., t_k\}$ is some enumeration of points in V, $k = |V|$.

Corollary 4 *Let $\mathbf{D}_1$ be a one-point transition energy field in Λ. Then there exists a unique transition energy Δ_Λ in Λ such that for any $t \in \Lambda$ and $\bar{x} \in X^{\Lambda \setminus t}$, it holds*

$$\Delta_\Lambda(x\bar{x}, u\bar{x}) = \delta_t^{\bar{x}}(x,u), \qquad x, u \in X^t.$$

In this case, for any $x, u \in X^\Lambda$,

$$\Delta_\Lambda(x,u) = \delta_{t_1}^{x_{\Lambda \setminus t_1}}(x_{t_1}, u_{t_1}) + \delta_{t_2}^{u_{t_1} x_{\Lambda \setminus \{t_1,t_2\}}}(x_{t_2}, u_{t_2}) + ... + \delta_{t_n}^{u_{\Lambda \setminus t_n}}(x_{t_n}, u_{t_n}), \quad (10)$$

where $\Lambda = \{t_1, t_2, ..., t_n\}$ is some enumeration of points in Λ, $n = |\Lambda|$.

3 Transition energy and finite random fields

A finite random field in Λ is a probability distribution P_Λ on X^Λ, that is, a non-negative function, sum of whose values is equal to one:

$$P_\Lambda(x) \geq 0,\, x \in X^\Lambda, \qquad \sum_{x \in X^\Lambda} P_\Lambda(x) = 1.$$

In what follows, we will consider only (strictly) positive finite random fields, that is, random fields for which $P_\Lambda(x) > 0$ for all $x \in X^\Lambda$.

There is a deep connection between two fundamental objects, transition energy and probability (see [8] as well as [13, 14]).

Theorem 4 *For a function P_Λ to be a (positive) probability distribution on X^Λ, it is necessary and sufficient that it has the following form:*

$$P_\Lambda(x) = \frac{\exp\{\Delta_\Lambda(x,u)\}}{\sum\limits_{z \in X^\Lambda} \exp\{\Delta_\Lambda(z,u)\}}, \qquad x \in X^\Lambda, \quad (11)$$

where Δ_Λ is the transition energy in Λ and $u \in X^\Lambda$ is an arbitrary configuration.

Proof. Suppose P_Λ is a probability distribution on X^Λ and put

$$\Delta_\Lambda^P(x,u) = \ln \frac{P_\Lambda(x)}{P_\Lambda(u)}, \qquad x, u \in X^\Lambda. \tag{12}$$

By Theorem 1, Δ_Λ^P is a transition energy. From (12) it follows that

$$P_\Lambda(x) = \exp\{\Delta_\Lambda^P(x,u)\} P_\Lambda(u). \tag{13}$$

Hence,

$$P_\Lambda(u) = \left(\sum_{z \in X^\Lambda} \exp\{\Delta_\Lambda^P(z,u)\} \right)^{-1}.$$

Substituting the value of $P_\Lambda(u)$ in (13), we obtain (11).

Now let Δ_Λ be a transition energy. Define function P_Λ by formula (11). This function is well-defined since its values do not depend on $u \in X^\Lambda$. Indeed, for any $y \in X^\Lambda$, using (1) we can write

$$\begin{aligned} P_\Lambda(x) &= \frac{\exp\{\Delta_\Lambda(x,u)\}}{\sum\limits_{z \in X^\Lambda} \exp\{\Delta_\Lambda(z,u)\}} = \frac{\exp\{\Delta_\Lambda(x,y) + \Delta_\Lambda(y,u)\}}{\sum\limits_{z \in X^\Lambda} \exp\{\Delta_\Lambda(z,y) + \Delta_\Lambda(y,u)\}} \\ &= \frac{\exp\{\Delta_\Lambda(x,y)\}}{\sum\limits_{z \in X^\Lambda} \exp\{\Delta_\Lambda(z,y)\}}. \end{aligned}$$

It remains to note that P_Λ is a probability distribution on X^Λ. $\square$

3.1 Markov–Dobrushin scheme

In practical problems, information about a probability distribution is often specified by means of conditional probabilities. This raises the problem of describing distributions using systems of conditional distributions.

The idea of characterization of a multivariate probability distribution by means of conditional probabilities is very old and goes back to the concept of the finite Markov chain. Already here, from the very beginning, we are faced with two fundamentally important characterization problems: the problem of restoring a chain from the system of its transition probabilities and the problem of the existence of a Markov chain corresponding to a given initial distribution and a stochastic matrix.

Both of these problems are relevant in the general setting, when one deals with an arbitrary random field in a finite or infinite volume. Having solved the problem of restoring a random field from the system of its conditional probabilities, we can establish the general properties of the random field in these terms. The existence of a random field with a given system of conditional distributions allows constructing models of random fields with the required properties.

There is a large number of works devoted to these problems in the case of finite volumes (see, for example, [1–5, 9]). The main issue considered here is to find the conditions under which a given system of probability distributions uniquely determines a random field whose conditional probabilities coincide with the elements of the given system. Finding such conditions (consistency conditions of the elements of the system) will allow one to define a conditional distribution autonomously, that is, regardless to any joint probability distribution generating it.

Dobrushin [6] was the first to consider the problem of characterization of a random field on an infinite integer lattice by specification — a system of consistent probability distributions parameterized by infinite boundary conditions. He obtained criteria for both the existence and uniqueness of a random field with a given specification.

The methods used to solve the above mentioned characterization problems in finite and infinite volumes are fundamentally different. Apparently, this difference led to the situation where the research in these two directions (finite and infinite cases) was carried out independently. At the same time, as it was shown in [13], the methods developed in [7] and [8] for infinite random fields can be successfully applied to random fields in finite volumes.

We now proceed to a rigorous formulation of the aforementioned problems. Let P_Λ be a finite random field. By $P_V = (P_\Lambda)_V$ we denote the restriction of P_Λ on X^V, $V \subsetneq \Lambda$, that is, $P_V(x) = \sum\limits_{y \in X^{\Lambda \setminus V}} P_\Lambda(xy)$, $x \in X^V$.
For P_Λ, conditional probability on X^V under condition $\bar{x} \in X^{\Lambda \setminus V}$ is defined as

$$Q_V^{\bar{x}}(x) = \frac{P_\Lambda(x\bar{x})}{P_{\Lambda \setminus V}(\bar{x})} = \frac{P_\Lambda(x\bar{x})}{\sum\limits_{u \in X^V} P_\Lambda(u\bar{x})}, \qquad x \in X^V.$$

Following Dobrushin, the family $\boldsymbol{Q}(P_\Lambda) = \{Q_V^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus V}, V \subsetneq \Lambda\}$ will be called the *conditional distribution* of the finite random field P_Λ.

Since by Theorem 4 distribution P_Λ can be represented in the Gibbs form (11) with transition energy function Δ_Λ defined by (12), we can write

$$\frac{Q_V^{\bar{x}}(x)}{Q_V^{\bar{x}}(u)} = \frac{P_\Lambda(x\bar{x})}{P_\Lambda(u\bar{x})} = \exp\{\Delta_\Lambda(x\bar{x}, u\bar{x})\} = \exp\{\Delta_V^{\bar{x}}(x, u)\}, \tag{14}$$

$x, u \in X^V$, where $\Delta_V^{\bar{x}}$ is defined by (3). Therefore, the following statement holds true.

Theorem 5 *For any finite random field P_Λ, the elements of its conditional* *distribution* $\boldsymbol{Q}(P_\Lambda)$ *have the Gibbs form:*

$$Q_V^{\bar{x}}(x) = \frac{\exp\{\Delta_V^{\bar{x}}(x,u)\}}{\sum\limits_{z\in X^V} \exp\{\Delta_V^{\bar{x}}(x,u)\}}, \qquad x \in X^V, \bar{x} \in X^{\Lambda\setminus V}, V \subsetneq \Lambda,$$

where $\Delta_V^{\bar{x}}$ *is the transition energy in* V *with boundary condition* $\bar{x} \in X^{\Lambda\setminus V}$ *and* $u \in X^V$ *is an arbitrary configuration.*

Using the additivity of transition energy, we prove the following result (for the proof based solely on the probabilistic arguments, see [13]).

Theorem 6 *Conditional distribution* $\boldsymbol{Q}(P_\Lambda)$ *restores its generating random field* P_Λ.

Proof. By Theorem 4, we have

$$P_\Lambda(x) = \frac{\exp\{\Delta_\Lambda(x,u)\}}{\sum\limits_{z\in X^\Lambda} \exp\{\Delta_\Lambda(z,u)\}} = \frac{1}{\sum\limits_{z\in X^\Lambda} \exp\{\Delta_\Lambda(z,x)\}},$$

$x, u \in X^\Lambda$. Further, for any fixed $V \subsetneq \Lambda$, using Proposition 1 and relation (14), we can write

$$\begin{aligned} P_\Lambda(x) &= \left(\sum_{z\in X^\Lambda} \exp\{\Delta_V^{z_{\Lambda\setminus V}}(z_V, x_V)\} + \Delta_{\Lambda\setminus V}^{x_V}(z_{\Lambda\setminus V}, x_{\Lambda\setminus V}) \right)^{-1} \\ &= \left(\sum_{z\in X^\Lambda} \frac{Q_V^{z_{\Lambda\setminus V}}(z_V)}{Q_V^{z_{\Lambda\setminus V}}(x_V)} \cdot \frac{Q_{\Lambda\setminus V}^{x_V}(z_{\Lambda\setminus V})}{Q_{\Lambda\setminus V}^{x_V}(x_{\Lambda\setminus V})} \right)^{-1} \end{aligned}$$

□

The obtained in Theorem 6 formula for restoration of P_Λ by $\boldsymbol{Q}(P_\Lambda)$ can be further simplified in the following way:

$$\begin{aligned} P_\Lambda(x) &= Q_{\Lambda\setminus V}^{x_V}(x_{\Lambda\setminus V}) \left(\sum_{z_{\Lambda\setminus V}\in X^{\Lambda\setminus V}} \frac{Q_{\Lambda\setminus V}^{x_V}(z_{\Lambda\setminus V})}{Q_V^{z_{\Lambda\setminus V}}(x_V)} \sum_{z_V\in X^V} Q_V^{z_{\Lambda\setminus V}}(z_V) \right)^{-1} \\ &= Q_{\Lambda\setminus V}^{x_V}(x_{\Lambda\setminus V}) \left(\sum_{z\in X^{\Lambda\setminus V}} \frac{Q_{\Lambda\setminus V}^{x_V}(z)}{Q_V^{z}(x_V)} \right)^{-1}, \end{aligned}$$

$x \in X^\Lambda$. Note also that since by the multiplication law of probability, for any $V \subsetneq \Lambda$ and $x \in X^\Lambda$,

$$P_\Lambda(x) = P_V(x_V) Q_{\Lambda\setminus V}^{x_V}(x_{\Lambda\setminus V}),$$

we see that

$$P_V(x) = \left(\sum_{z \in X^{\Lambda \setminus V}} \frac{Q^x_{\Lambda\setminus V}(z)}{Q^z_V(x)} \right)^{-1}, \qquad x \in X^V, V \subsetneq \Lambda.$$

The following theorem shows that for the restoration of the random field P_Λ, it is sufficient to consider its *one-point conditional distribution* only, that is, the system $\boldsymbol{Q}_1(P_\Lambda) = \{Q_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ of conditional probabilities on X^t under conditions $\bar{x} \in X^{\Lambda\setminus t}$, $t \in \Lambda$. Such conditional probabilities are called local characteristics and are usually considered in practice (see, for example, [10, 11, 19]).

Theorem 7 *One-point conditional distribution $\boldsymbol{Q}_1(P_\Lambda)$ restores its generating random field P_Λ.*

Proof. Let $\Lambda = \{t_1, t_2, \ldots, t_n\}$, $n = |\Lambda|$, be some enumeration of the points in Λ. Then for any configuration $x \in X^\Lambda$, by successively using Theorem 4, Corollary 1 and formula (14), we can write

$$\begin{aligned} P_\Lambda(x) &= \left(\sum_{z \in X^\Lambda} \exp\{\Delta_\Lambda(z,x)\} \right)^{-1} \\ &= \Bigg(\sum_{z \in X^\Lambda} \exp\Big\{ \Delta_{t_1}^{z_{\Lambda\setminus t_1}}(z_{t_1}, x_{t_1}) + \Delta_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2}, x_{t_2}) + \\ &\qquad\qquad + \ldots + \Delta_{t_n}^{x_{\Lambda\setminus t_n}}(z_{t_n}, x_{t_n}) \Big\} \Bigg)^{-1} \\ &= \left(\sum_{z \in X^\Lambda} \frac{Q_{t_1}^{z_{t_2}\ldots z_{t_n}}(z_{t_1})}{Q_{t_1}^{z_{t_2}\ldots z_{t_n}}(x_{t_1})} \cdot \frac{Q_{t_2}^{x_{t_1} z_{t_3}\ldots z_{t_n}}(z_{t_2})}{Q_{t_2}^{x_{t_1} z_{t_3}\ldots z_{t_n}}(x_{t_2})} \cdot \ldots \cdot \frac{Q_{t_n}^{x_{t_1}\ldots x_{t_{n-1}}}(z_{t_n})}{Q_{t_n}^{x_{t_1}\ldots x_{t_{n-1}}}(x_{t_n})} \right)^{-1}. \end{aligned}$$

□

Now let us consider an *a priory* given system $\mathbf{Q}_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ of probability distributions $q_t^{\bar{x}}$ on X^t parameterized by boundary conditions $\bar{x} \in X^{\Lambda\setminus t}$, $t \in \Lambda$. The following natural question arises: when for a given system $\mathbf{Q}_1$, there exists a finite random field P_Λ for which $\mathbf{Q}_1$ is its one-point conditional distribution, that is, $\boldsymbol{Q}_1(P_\Lambda) = \mathbf{Q}_1$. In other words, when a system $\mathbf{Q}_1$ forms a conditional distribution?

In [13], necessary and sufficient conditions were obtained under which a given system $\mathbf{Q}_1$ of distributions is a one-point conditional distribution of a finite random field P_Λ, and an explicit formula expressing P_Λ through the elements of this system was given. To obtain this result, we used the approach developed in [7], where a solution to Dobrushin's problem of describing a random field (in infinite volume) by a consistent system of one-point distributions was obtained. Below, we will show how this problem can be solved by using the fact that transition energy possess the additivity property.

Consider now an autonomously given system $\mathbf{Q}_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ of probability distributions $q_t^{\bar{x}}$ on X^t parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus t}$, $t \in \Lambda$. Since for each $t \in \Lambda$ and $\bar{x} \in X^{\Lambda \setminus t}$, function $q_t^{\bar{x}}$ is a probability distribution on X^t, by Theorem 1 the function

$$\delta_t^{\bar{x}}(x,u) = \ln \frac{q_t^{\bar{x}}(x)}{q_t^{\bar{x}}(u)}, \qquad x,u \in X^t, \tag{15}$$

is a transition energy in t. Thus, for the system $\mathbf{D}_1(\mathbf{Q}_1) = \{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ to be a one-point transition energy field, it is necessary and sufficient that the elements of $\mathbf{D}_1(\mathbf{Q}_1)$ satisfy (9), that is,

$$\ln \frac{q_t^{\bar{x}y}(x)}{q_t^{\bar{x}y}(u)} + \ln \frac{q_s^{\bar{x}u}(y)}{q_s^{\bar{x}u}(v)} = \ln \frac{q_s^{\bar{x}x}(y)}{q_s^{\bar{x}x}(v)} + \ln \frac{q_t^{\bar{x}v}(x)}{q_t^{\bar{x}v}(u)},$$

or, equivalently,

$$q_t^{\bar{x}y}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(u) q_s^{\bar{x}u}(y) = q_t^{\bar{x}y}(u) q_s^{\bar{x}u}(v) q_t^{\bar{x}v}(x) q_s^{\bar{x}x}(y) \tag{16}$$

for all $t, s \in \Lambda$ and $x, u \in X^t$, $y, v \in X^s$, $\bar{x} \in X^{\Lambda \setminus \{t,s\}}$.

A system $\mathbf{Q}_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ of probability distributions satisfying consistency conditions (16) is called a *1–specification* in Λ (see [13]).

The previous considerations show that the following statement holds true.

Theorem 8 *For a system $\mathbf{Q}_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ of functions on X^t parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus t}$, $t \in \Lambda$, to be a 1–specification in Λ it is necessary and sufficient that its elements have the Gibbs form*

$$q_t^{\bar{x}}(x) = \frac{\exp\{\delta_t^{\bar{x}}(x,u)\}}{\sum\limits_{z \in X^t} \exp\{\delta_t^{\bar{x}}(z,u)\}}, \qquad x \in X^t,$$

where $\mathbf{D}_1 = \{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ is a one-point transition energy field and $u \in X^t$.

Now we present the necessary and sufficient condition for a system $\mathbf{Q}_1$ to be a one-point conditional distribution of a unique finite random field P_Λ (the probabilistic proof of this result was given in [13]).

Theorem 9 *For a system $\mathbf{Q}_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda \setminus t}, t \in \Lambda\}$ of functions on X^t parameterized by boundary conditions $\bar{x} \in X^{\Lambda \setminus t}$, $t \in \Lambda$, to be a one-point conditional distribution of a unique finite random field P_Λ, it is necessary and sufficient that it be a 1–specification in Λ.*

Proof. Let P_Λ be a probability distribution on X^Λ and $\boldsymbol{Q}_1(P_\Lambda) = \{Q_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ be its one-point conditional distribution. Then, using (14) and Proposition 2, for any $t, s \in \Lambda$ and $x, u \in X^t$, $y, v \in X^s$, $\bar{x} \in X^{\Lambda\setminus\{t,s\}}$, we can write

$$\begin{aligned}\frac{Q_t^{\bar{x}y}(x)}{Q_t^{\bar{x}y}(u)} \cdot \frac{Q_s^{\bar{x}u}(y)}{Q_s^{\bar{x}u}(v)} &= \exp\{\Delta_t^{\bar{x}y}(x,u) + \Delta_s^{\bar{x}u}(y,v)\} \\ &= \exp\{\Delta_s^{\bar{x}x}(y,v) + \Delta_t^{\bar{x}v}(x,u)\} = \frac{Q_s^{\bar{x}x}(y)}{Q_s^{\bar{x}x}(v)} \cdot \frac{Q_t^{\bar{x}v}(x)}{Q_t^{\bar{x}v}(u)},\end{aligned}$$

that is, conditions (16) are satisfied.

Now, let $\mathbf{Q}_1 = \{q_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ be a 1–specification in Λ. Then the system $\mathbf{D}_1(\mathbf{Q}_1) = \{\delta_t^{\bar{x}}, \bar{x} \in X^{\Lambda\setminus t}, t \in \Lambda\}$ of functions defined by (15) is a one-point transition energy field. By Corollary 4, it uniquely determines the transition energy Δ_Λ in Λ, which, in its turn, uniquely determines the probability distribution P_Λ on X^Λ (see Theorem 4). Further, using (10) and (15), for any enumeration $\Lambda = \{t_1, t_2, ..., t_n\}$, $n = |\Lambda|$, of the points in Λ, we can write

$$\begin{aligned}P_\Lambda(x) &= \left(\sum_{z\in X^\Lambda} \exp\{\Delta_\Lambda(z,x)\}\right)^{-1} \\ &= \Bigg(\sum_{z\in X^\Lambda} \exp\Big\{\delta_{t_1}^{z_{\Lambda\setminus t_1}}(z_{t_1}, x_{t_1}) + \delta_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2}, x_{t_2}) + \\ &\qquad\qquad + \ldots + \delta_{t_n}^{x_{\Lambda\setminus t_n}}(z_{t_n}, x_{t_n})\Big\}\Bigg)^{-1} \\ &= \left(\sum_{z\in X^\Lambda} \frac{q_{t_1}^{z_{\Lambda\setminus t_1}}(z_{t_1})}{q_{t_1}^{z_{\Lambda\setminus t_1}}(x_{t_1})} \cdot \frac{q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2})}{q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(x_{t_2})} \cdot \ldots \cdot \frac{q_{t_n}^{x_{\Lambda\setminus t_n}}(z_{t_n})}{q_{t_n}^{x_{\Lambda\setminus t_n}}(x_{t_n})}\right)^{-1}.\end{aligned}$$

It remains to show that $\boldsymbol{Q}_1(P_\Lambda) = \mathbf{Q}_1$. First note that the the expression in parentheses on the right side of the obtained formula for P_Λ can be written in the following way:

$$\sum_{z\in X^\Lambda} \frac{q_{t_1}^{z_{\Lambda\setminus t_1}}(z_{t_1})}{q_{t_1}^{z_{\Lambda\setminus t_1}}(x_{t_1})} \cdot \frac{q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2})}{q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(x_{t_2})} \cdot \ldots \cdot \frac{q_{t_n}^{x_{\Lambda\setminus t_n}}(z_{t_n})}{q_{t_n}^{x_{\Lambda\setminus t_n}}(x_{t_n})} = \frac{1}{q_{t_n}^{x_{\Lambda\setminus\{t_n\}}}(x_{t_n})} f(x_{\Lambda\setminus t_n}),$$

where

$$f(x_{\Lambda\setminus t_n}) = \sum_{z\in X^{\Lambda\setminus t_1}} \frac{q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(z_{t_2}) \cdot \ldots \cdot q_{t_{n-1}}^{x_{\Lambda\setminus\{t_{n-1},t_n\}} z_{t_n}}(z_{t_{n-1}}) \cdot q_{t_n}^{x_{\Lambda\setminus t_n}}(z_{t_n})}{q_{t_1}^{z_{\Lambda\setminus t_1}}(x_{t_1}) \cdot q_{t_2}^{x_{t_1} z_{\Lambda\setminus\{t_1,t_2\}}}(x_{t_2}) \cdot \ldots \cdot q_{t_{n-1}}^{x_{\Lambda\setminus\{t_{n-1},t_n\}} z_{t_n}}(x_{t_{n-1}})}.$$

Further, by the consistency conditions (9) of the elements of $\mathbf{D}_1(\mathbf{Q}_1)$, values of P_Λ do not depend on the way of enumeration of points in Λ. Thus, for any

$t \in \Lambda$, assuming $t_n = t$ in the above formula, for any $x \in X^t$ and $\bar{x} \in X^{\Lambda \setminus t}$, we obtain

$$Q_t^{\bar{x}}(x) = \frac{P_\Lambda(x\bar{x})}{\sum\limits_{u \in X^t} P_\Lambda(u\bar{x})} = \frac{q_t^{\bar{x}}(x)}{\sum\limits_{u \in X^t} q_t^{\bar{x}}(u)} = q_t^{\bar{x}}(x).$$

$\square$

It is clear that after construction of P_Λ by 1–specification $\mathbf{Q}_1$, one can easily construct its conditional distribution $\boldsymbol{Q}(P_\Lambda)$. At the same time, using Theorem 3, it is possible to construct $\boldsymbol{Q}(P_\Lambda)$ directly starting from $\mathbf{Q}_1$.

Conclusion

In this paper, we showed that transition energy has the property of additivity. We demonstrated how this property can be applied to the well-known problem of describing a finite random field (joint probability distribution) by a set of consistent conditional distributions, as well as to finding consistency conditions under which a set of functions, parameterized by boundary conditions, forms a conditional distribution for a finite random field. Here we also used the fact that there is a deep connection between transition energy and probability: for a function to be a probability distribution, it is necessary and sufficient that it has a Gibbs form in terms of the corresponding transition energy.

Another application of the above results is in the method of correlation equations. For example, in [15, 16], it was shown how to determine finite-volume correlation functions in terms of transition energies. Under a physically justified supplementary requirement on the transition energy, we derived a recurrence relation for such correlation functions.

It should be noted that, unlike the transition energy, the Hamiltonian does not exhibit additivity. This is why working with the transition energy instead of the Hamiltonian is not only theoretically correct (since there is no one-to-one correspondence between probability and the Hamiltonian) but also much more convenient and allows for obtaining non-obvious results.

Acknowledgments. The authors are grateful to the anonymous reviewers for carefully reading of our paper and valuable comments. The work was supported by the Science Committee of the Republic of Armenia in the frames of the research project 21AG-1A045.

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Please, cite to this paper as published in
Armen. J. Math., V. **18**, N. 7(2026), pp. 1–19
https://doi.org/10.52737/18291163-2026.18.07-1-19