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Equicontinuity Regions and Kulkarni Limit Sets for Discrete Subgroups of Quaternionic Hyperbolic Isometry Groups Acting on Grassmannians

Vanlalruatkimi, S. Dutta and J. Ramdikpuia

Abstract. Let Γ be a discrete subgroup of $\mathrm{PSp}(n,1)$. We realize $\mathbb{H}^{n+1}$ as $\mathbb{C}^{2n+2}$ and study the induced action of Γ on $\mathbb{P}(\wedge^{2k+2}_{\mathbb{C}}\mathbb{C}^{2n+2})$ where $0 \leq k \leq n-1$, into which the quaternionic Grassmannian $\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$ embeds equivariantly. We determine all pseudo-projective limits of the action. Their images are the linear spans of the $(k+1)$-planes tangent to the boundary at points of the Chen–Greenberg limit set, and their kernels are the orthogonal complements of such spans. As a consequence, we describe the equicontinuity region explicitly, identify the k-Chen–Greenberg limit set with the set of tangent $(k+1)$-planes at limit points, and show that the equicontinuity region is contained in the Kulkarni domain of discontinuity. For non-elementary groups, we also give a lower bound for the Kulkarni limit set in terms of the k-Chen–Greenberg limit set.

Key Words: Quaternionic Hyperbolic Space, Chen–Greenberg Limit Set, Equicontinuity Region, Kulkarni Limit Set, Quaternionic Grassmannian, Pseudo-Projective Transformations
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1 Introduction

Limit sets are the basic objects in the study of Kleinian groups. For a discrete group of Möbius transformations of the Riemann sphere, the limit set is a closed invariant set, and the group acts properly discontinuously on its complement. In higher dimensions, and for groups acting on spaces that

are not boundaries of hyperbolic spaces, there is no single natural notion of limit set. Seade and Verjovsky [17] began the study of discrete groups of projective transformations of $\mathbb{CP}^n$, the *complex Kleinian groups*; see the monograph [5]. Two notions have proved most useful there. The first is the *equicontinuity region* $\mathrm{Eq}(G)$, the set of points near which the group forms a normal family. The second is the *Kulkarni limit set* $\Lambda_{\mathrm{Kul}}(G)$ [14], built from points with infinite isotropy, cluster points of orbits, and cluster points of images of compact sets; its complement $\Omega_{\mathrm{Kul}}(G)$ is an open set on which G acts properly discontinuously. Both sets are described effectively by *pseudo-projective maps*, i.e., limits of sequences of distinct group elements in the compactification $\mathbb{P}(M_N(\mathbb{C}))$ of $\mathrm{PSL}(N,\mathbb{C})$: by [4, Proposition 2.5], $\mathrm{Eq}(G)$ is the complement of the union of the kernels of these limits.

A rich source of examples comes from hyperbolic geometry. Let Γ be a discrete subgroup of $\mathrm{PU}(1,n)$, the holomorphic isometry group of complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}} \subset \mathbb{CP}^n$ [12]. Its Chen–Greenberg limit set $\Lambda(\Gamma)$ is the set of accumulation points in $\partial\mathbf{H}^n_{\mathbb{C}}$ of an orbit in $\mathbf{H}^n_{\mathbb{C}}$ [7]. Cano and Seade [6] showed that the complement of the equicontinuity region of Γ acting on $\mathbb{CP}^n$ is the union of the complex hyperplanes $p^\perp$ tangent to $\partial\mathbf{H}^n_{\mathbb{C}}$ at the points $p \in \Lambda(\Gamma)$. For non-elementary Γ, Cano, Liu and López [3] proved that this union is also the Kulkarni limit set, and that its complement is the largest open set on which Γ acts properly discontinuously; for $n = 2$, this is due to Navarrete [15]. Cyclic subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ were treated in detail in [4].

The quaternionic case is still developing. Let $\mathrm{PSp}(n,1)$ be the isometry group of quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}} \subset \mathbb{P}^n_{\mathbb{H}}$ [7, 13]. Cano and Castro Morales [2] proved a Cartan decomposition for $\mathrm{PSp}(n,1)$ and used it to show that, for a non-elementary discrete $\Gamma \subset \mathrm{PSp}(n,1)$, the set $\mathbb{P}^n_{\mathbb{H}} \setminus \bigcup_{p\in\Lambda(\Gamma)} p^\perp$ is the equicontinuity region of Γ and the largest open set on which Γ acts properly discontinuously. Kulkarni limit sets of cyclic subgroups of $\mathrm{PSL}(3,\mathbb{H})$ acting on $\mathbb{P}^2_{\mathbb{H}}$ were computed in [8], where quaternionic pseudo-projective transformations were also introduced. Kulkarni limit sets of cyclic quaternionic projective groups were further studied in [10], and quaternionic projective transformations were classified by their equicontinuity regions in [11].

Instead of points of the projective space one can consider the action on projective k-planes. Zúñiga [19] studied the action of a discrete $\Gamma \subset \mathrm{PU}(1,n)$ on the Grassmannian of k-planes in $\mathbb{CP}^n$, by passing through the Plücker embedding to the linear action of $\bigwedge^{k+1}\Gamma$ on $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$. She showed that $\bigwedge^{k+1}\Gamma$ is a discrete subgroup of a pseudo-unitary group $\mathrm{PU}(R,S)$, and that its equicontinuity region is the complement of the union of the projective subspaces spanned by the k-planes that pass through, or lie in the tangent hyperplane at, a point of $\Lambda(\Gamma)$. In [20], she introduced the *k-Chen–Greenberg limit set*, the closure of the union of the images of the pseudo-projective limits of $\bigwedge^{k+1}\Gamma$, and showed that the Kulkarni limit set

of $\bigwedge^{k+1}\Gamma$ is the union of the orthogonal complements of its components; for non-elementary Γ it coincides with the complement of the equicontinuity region.

The purpose of this paper is to carry out this programme for discrete subgroups of $\mathrm{PSp}(n,1)$. The main difficulty is that the Plücker embedding has no direct quaternionic analogue. Exterior powers of right $\mathbb{H}$-modules do not have the expected properties, and Alesker [1] showed that there is not even a map from the quaternionic Grassmannian to $\mathbb{HP}^\infty$ which behaves like a Plücker map at the level of the first Pontryagin class. We avoid this problem by passing to the complex realization $V_{\mathbb{C}} \cong \mathbb{C}^{2n+2}$ of $\mathbb{H}^{n+1}$. A quaternionic $(k+1)$-plane U becomes a complex $(2k+2)$-plane $U_{\mathbb{C}}$ invariant under the quaternionic structure J_q, and $U \mapsto U_{\mathbb{C}}$ followed by the ordinary complex Plücker embedding gives an injective equivariant map $\mathcal{P}_k\colon \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}) \to \mathbb{P}\big(\bigwedge_{\mathbb{C}}^{2k+2} V_{\mathbb{C}}\big)$. The group $\mathrm{PSp}(n,1)$ acts on the target through $\Phi_k = \bigwedge_{\mathbb{C}}^{2k+2}$, which preserves a Hermitian form h_k induced from the complex part of the quaternionic Hermitian form.

The main tools are the Cartan decomposition of [2] and a weight decomposition of $\Phi_k(a_t)$, where a_t is the Cartan subgroup. Since the complex realization of a_t has the eigenvalue e^t with multiplicity two, $\Phi_k(a_t)$ has (for $n \geq 2$) five weights e^{jt}, $-2 \leq j \leq 2$, rather than three as in the complex case. This determines all pseudo-projective limits of $\Phi_k(\Gamma)$: their images and kernels are the spans $\langle H_p^k \rangle$ of the Plücker points of the k-planes tangent to $\partial\mathbf{H}^n_{\mathbb{H}}$ at limit points p, and the orthogonal complements of such spans. The equicontinuity region then follows from the kernels by a second-order perturbation argument, and the Kulkarni sets from proper discontinuity.

Let $h_{\mathbb{C}}$ be the complex part of the quaternionic Hermitian form; it has signature $(2n,2)$. Put $W_k := \bigwedge_{\mathbb{C}}^{2k+2} V_{\mathbb{C}}$, and let h_k be the Hermitian form on W_k induced by $h_{\mathbb{C}}$; orthogonal complements in W_k are taken with respect to h_k. For $U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$, let $\mathcal{P}_k(U) \in \mathbb{P}(W_k)$ be its complexified Plücker point. For $p \in \partial\mathbf{H}^n_{\mathbb{H}}$, let

$$H_p^k := \{U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}) : p \subset U \subset p^\perp\}$$

be the set of projective k-planes tangent to $\partial\mathbf{H}^n_{\mathbb{H}}$ at p, and let $\langle H_p^k \rangle \subset W_k$ be the complex linear span of $\mathcal{P}_k(H_p^k)$. For a pseudo-projective map $\rho = [T]$, we write $\mathfrak{Im}(\rho) = \mathbb{P}(\mathfrak{Im}T)$ and $\mathrm{Ker}(\rho) = \mathbb{P}(\mathrm{Ker}T)$, and $\mathrm{Lim}(G)$ is the set of limits of sequences of distinct elements of G (Section 2.6).

The notion of the k-Chen–Greenberg limit set was introduced in [20] for discrete subgroups of $\mathrm{PU}(1,n)$. We use the following quaternionic version.

Definition 1 *Let $\Gamma \subset \mathrm{PSp}(n,1)$ be a discrete subgroup and $0 \leq k \leq n-1$. The* k-Chen–Greenberg limit set *of Γ is*

$$L_k(\Gamma) := \overline{\bigcup_{\rho \in \mathrm{Lim}(\Phi_k(\Gamma))} \mathfrak{Im}(\rho)} \subset \mathbb{P}(W_k),$$

and its Grassmannian version is $\Lambda_k(\Gamma) := \mathcal{P}_k^{-1}\big(L_k(\Gamma)\big) \subset \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$.

The main result of the current work is the following statement.

Theorem 1 *Let* $\Gamma \subset \mathrm{PSp}(n,1)$ *be a discrete subgroup with Chen–Greenberg limit set* $\Lambda(\Gamma)$*, and let* $0 \leq k \leq n-1$*. Then* $\Phi_k(\Gamma)$ *is a discrete subgroup of* $\mathrm{PU}(h_k)$*, and the following hold.*

(a) (Limit maps) *Let* $\rho = \lim \Phi_k(\gamma_m) \in \mathrm{Lim}(\Phi_k(\Gamma))$*, and put* $p^+ := \lim \gamma_m(o)$ *and* $p^- := \lim \gamma_m^{-1}(o)$ *for some* $o \in \mathbf{H}^n_{\mathbb{H}}$*. Then* $p^\pm \in \Lambda(\Gamma)$ *and*

$$\mathfrak{Im}(\rho) = \mathbb{P}\big(\langle H^k_{p^+}\rangle\big), \qquad \mathrm{Ker}(\rho) = \mathbb{P}\big(\langle H^k_{p^-}\rangle^\perp\big),$$

where $\dim_{\mathbb{C}}\langle H^k_p\rangle = \binom{2n-2}{2k}$ *for every* p*. Conversely, every* $p \in \Lambda(\Gamma)$ *arises as* p^+ *for some element of* $\mathrm{Lim}(\Phi_k(\Gamma))$ *and as* p^- *for another.*

(b) (k-Chen–Greenberg limit set)

$$L_k(\Gamma) = \bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\big(\langle H^k_p\rangle\big) \qquad \textit{and} \qquad \Lambda_k(\Gamma) = \bigcup_{p\in\Lambda(\Gamma)} H^k_p.$$

(c) (Equicontinuity region)

$$\mathrm{Eq}\big(\Phi_k(\Gamma)\big) = \mathbb{P}(W_k) \setminus \bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\big(\langle H^k_p\rangle^\perp\big).$$

(d) (Kulkarni) $\mathrm{Eq}\big(\Phi_k(\Gamma)\big) \subseteq \Omega_{\mathrm{Kul}}\big(\Phi_k(\Gamma)\big)$*; equivalently,*

$$\Lambda_{\mathrm{Kul}}\big(\Phi_k(\Gamma)\big) \subseteq \bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\big(\langle H^k_p\rangle^\perp\big).$$

If, moreover, Γ *is non-elementary or infinite cyclic, then*

$$\bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\big(\langle H^k_p\rangle\big) = L_k(\Gamma) \subseteq \Lambda_{\mathrm{Kul}}\big(\Phi_k(\Gamma)\big) \subseteq \bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\big(\langle H^k_p\rangle^\perp\big).$$

The unions in (b), (c) *and* (d) *are closed.*

Corollary 1 *Under the hypotheses of Theorem 1, for* $U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$*, we have* $\mathcal{P}_k(U) \in \mathrm{Eq}(\Phi_k(\Gamma))$ *if and only if* $p \not\subset U$ *and* $U \not\subset p^\perp$ *for every* $p \in \Lambda(\Gamma)$*. The set*

$$\mathcal{O}_k(\Gamma) := \{U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}) : p \not\subset U \text{ and } U \not\subset p^\perp \text{ for all } p \in \Lambda(\Gamma)\}$$

is open and Γ*-invariant.* Γ *acts properly discontinuously on it, and it is contained in the equicontinuity region of the action of* Γ *on* $\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$.

For $k = 0$, Theorem 1(a) contains the λ-lemma of [2], and Corollary 1 recovers the region $\mathbb{P}^n_{\mathbb{H}} \setminus \bigcup_{p\in\Lambda(\Gamma)} p^\perp$ of [2, Theorem 1.2].

Remark 1 *Three features distinguish the quaternionic case from the complex one [19, 20].*
(i) The limit maps have rank $\binom{2n-2}{2k}$*. No non-elementary hypothesis is needed in Theorem 1(a)–(c).*
(ii) The space $\langle H^k_p\rangle^\perp$ *is strictly larger than the span of the Plücker points of the* k*-planes through* p *or contained in* $p^\perp$ *(Remark 4). Therefore, part (c) is not the literal analogue of [19, Theorem 4.14], which would be false here; only on the Grassmannian do the two descriptions agree (Corollary 1).*
(iii) In the complex case, equality holds in (d) [20, Theorem 1.1]. Here $\Phi_k(a_t)$ *has five weights instead of three, and the argument of [20] does not carry over (Remark 7). We do not know whether the upper bound in (d) is an equality in general. The lower bound in (d) comes from the recurrence of* $\Phi_k(g)$ *on* $\mathbb{P}\langle H^k_p\rangle$ *for limit points* p *fixed by elements* g *of infinite order (Proposition 8).*

The rest of the paper is organised as follows. Section 2 collects the preliminaries: quaternionic hyperbolic space, the complex realization, the Plücker map $\mathcal{P}_k$ and the induced form h_k, and pseudo-projective maps. Section 3 computes all limit maps of $\Phi_k(\Gamma)$ and proves Theorem 1(a),(b). Section 4 proves Theorem 1(c) and Corollary 1, and Section 5 proves Theorem 1(d) and the lower bound for the Kulkarni limit set.

2 Preliminaries

In this section, we fix some notation and recall the necessary background that will be used throughout this paper.

Let $\mathbb{H} := \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ denote the division algebra of Hamilton's quaternions, where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$. If $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then its conjugate is $\bar{q} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$, real part of q is $\mathfrak{Re}(q) = a_0$ and imaginary part is $\mathfrak{Im}(q) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. For an elaborate discussion on the theory of quaternions and matrices over the quaternions, see [18, 16].

2.1 Quaternions and right-module conventions

Throughout the paper, $\mathbb{H}^{n+1}$ is considered as a right quaternionic vector space [16, Sect. 3.1] of dimension $(n+1)$. We write $V := \mathbb{H}^{n+1}$. Thus, scalar multiplication is written on the right, and $\mathbb{H}$-linear maps $T(vq) = T(v)q$ are given by left multiplication by matrices in $M_{n+1}(\mathbb{H})$. We write $\mathrm{GL}(n,\mathbb{H})$ as the set of all matrices with non-zero Dieudonné determinant, $\mathrm{SL}(n,\mathbb{H})$ as the set of all matrices with Dieudonné determinant 1 and $\mathrm{PSL}(n,\mathbb{H}) =$

$\mathrm{SL}(n,\mathbb{H})/\{\pm I_n\}$. Consider the quaternion (right) space $\mathbb{H}^{n+1}$. The quaternion projective space $\mathbb{P}^n_{\mathbb{H}}$ is formed by taking the quotient of $\mathbb{H}^{n+1}\setminus\{0\}$ under the equivalence relation $\sim$ in $\mathbb{H}^{n+1}\setminus\{0\}$, where $z\sim w$ if $z=w\,\alpha$ for some non-zero quaternion α. Let $\mathbb{P}:\mathbb{H}^{n+1}\setminus\{0\}\longrightarrow\mathbb{P}^n_{\mathbb{H}}$ be the corresponding quotient map. A non-empty set $W\subseteq\mathbb{P}^n_{\mathbb{H}}$ is said to be a projective subspace of dimension k if there exists a $(k+1)$-dimensional $\mathbb{H}$-linear subspace $\widetilde{W}$ of $\mathbb{H}^{n+1}$ such that $\mathbb{P}(\widetilde{W}\setminus\{0\})=W$. The projective subspaces of dimension $(n-1)$ are called hyperplanes and the quaternion projective subspaces of dimension 1 are called lines (cf. [18, 16]). Denote by $[z]$ the points in $\mathbb{P}^n_{\mathbb{H}}$. Note that $[z]=[z\alpha]$ for all $z\in\mathbb{H}^{n+1}$ and $\alpha\in\mathbb{H}^*=\mathbb{H}\setminus\{0\}$. We define a hyperplane $H(\alpha)$ in $\mathbb{P}^n_{\mathbb{H}}$ using the quaternionic Hermitian pairing,

$$H(\alpha)=\{[z]\in\mathbb{P}^n_{\mathbb{H}}:\langle z,\alpha\rangle=0\}$$

This definition is strictly invariant under right scalar multiplication. We fix the quaternionic Hermitian form of signature $(n,1)$ (see [13] for details) as

$$\langle v,w\rangle=w^*Jv,\ \text{where } J=\begin{bmatrix}0&0&1\\0&I_{n-1}&0\\1&0&0\end{bmatrix}.$$

It satisfies $\langle vq,w\rangle=\langle v,w\rangle q$, $\langle v,wq\rangle=\bar{q}\langle v,w\rangle$ and $\langle w,v\rangle=\overline{\langle v,w\rangle}$. For a quaternionic subspace $U\subset\mathbb{H}^{n+1}$, define

$$U^{\perp}=\{v\in\mathbb{H}^{n+1}:\langle v,u\rangle=0\text{ for all }u\in U\}.$$

Hence, a projective hyperplane is written as $\mathbb{P}(U^{\perp})$, rather than by a coordinate equation, so that it is independent of the choice of quaternionic representative.

For a right $\mathbb{H}$-linear map T, we use

$$\mathrm{Ker}T=\{v:T(v)=0\},\qquad \mathrm{Im}\,T=T(\mathbb{H}^{n+1}),$$

and for $T\neq 0$,

$$\mathbb{P}(T)([v])=[T(v)].$$

2.2 Quaternionic hyperbolic space and discrete groups

Let V_-,V_0,V_+ be the sets of vectors v with $\langle v,v\rangle$ negative, zero or positive, respectively. Then $\mathbf{H}^n_{\mathbb{H}}=\mathbb{P}(V_-)$ and $\partial\mathbf{H}^n_{\mathbb{H}}=\mathbb{P}(V_0\setminus\{0\})$.

Let $\mathrm{Sp}(n,1)=\{g\in\mathrm{GL}(n+1,\mathbb{H}):\ g^*Jg=J\}$ and $\mathrm{PSp}(n,1)=\mathrm{Sp}(n,1)/\{\pm I\}$, which acts on $\mathbf{H}^n_{\mathbb{H}}$ by isometries; for $n\geq 2$ it is the full isometry group. Let

$$o=[e_1-e_{n+1}]\in\mathbf{H}^n_{\mathbb{H}},\quad K=\mathrm{Sp}(n,1)\cap\mathrm{Sp}(n+1)\cong\mathrm{Sp}(n)\times\mathrm{Sp}(1),$$

where $\mathrm{Sp}(n)=\{g\in\mathrm{GL}(n,\mathbb{H}):\ g^*g=gg^*=I\}$; K is the stabilizer of o and a maximal compact group.

Lemma 1 *Let $p \neq q$ be points of $\partial\mathbf{H}^n_{\mathbb{H}}$.*

(i) $p \subset p^\perp$, and the form is positive semidefinite on $p^\perp$ with radical p. In particular, $\mathbb{P}(p^\perp) \cap \overline{\mathbf{H}^n_{\mathbb{H}}} = \{p\}$, i.e., $\mathbb{P}(p^\perp)$ is the tangent hyperplane to $\partial\mathbf{H}^n_{\mathbb{H}}$ at p.

(ii) $q \not\subset p^\perp$. Consequently, $V = p \oplus (p^\perp \cap q^\perp) \oplus q$, and the form is positive definite on $p^\perp \cap q^\perp$, which has dimension $n-1$.

(iii) There is $g \in \mathrm{Sp}(n,1)$ with $g(e_1\mathbb{H}) = p$ and $g(e_{n+1}\mathbb{H}) = q$. In particular $\mathrm{Sp}(n,1)$ acts transitively on ordered pairs of distinct points of $\partial\mathbf{H}^n_{\mathbb{H}}$.

Proof. We use two elementary facts. First, $\langle v, v\rangle \in \mathbb{R}$ for every $v \in V$, since $\langle v, v\rangle = \overline{\langle v, v\rangle}$. Second, for right subspaces $S, T \subset V$ we have $\dim_{\mathbb{H}}(S+T) = \dim_{\mathbb{H}} S + \dim_{\mathbb{H}} T - \dim_{\mathbb{H}}(S \cap T)$, as over any division ring. Write $p = \tilde{p}\mathbb{H}$ and $q = \tilde{q}\mathbb{H}$ with $\langle \tilde{p}, \tilde{p}\rangle = \langle \tilde{q}, \tilde{q}\rangle = 0$. Since $\langle v, \tilde{p}a\rangle = \bar{a}\langle v, \tilde{p}\rangle$, we have $p^\perp = \{v : \langle v, \tilde{p}\rangle = 0\}$, and similarly for q.

Step 0. If $S \subset V$ is a right subspace with $\langle s, s\rangle \leq 0$ for all $s \in S$, then $\dim_{\mathbb{H}} S \leq 1$. Put $f_\pm = (e_1 \pm e_{n+1})/\sqrt{2}$. From the definition of J,

$$\begin{aligned} &\langle f_+, f_+\rangle = 1, \quad \langle f_-, f_-\rangle = -1, \quad \langle f_+, f_-\rangle = 0, \\ &\langle e_i, e_j\rangle = \delta_{ij}, \quad \langle e_i, f_\pm\rangle = 0 \qquad (2 \leq i, j \leq n). \end{aligned}$$

Let $P = f_+\mathbb{H} + e_2\mathbb{H} + \cdots + e_n\mathbb{H}$; then $\dim_{\mathbb{H}} P = n$. For $v = f_+a + \sum_{i=2}^n e_i b_i \in P$, the cross terms vanish and $\langle v, v\rangle = |a|^2 + \sum_i |b_i|^2$; thus, the form is positive definite on P. Hence, $S \cap P = 0$, and $\dim_{\mathbb{H}} S \leq (n+1) - n = 1$.

Step 1. If $v \in p^\perp$ and $a, b \in \mathbb{H}$, then $\langle \tilde{p}a + vb, \tilde{p}a + vb\rangle = |b|^2\langle v, v\rangle$. Indeed, $\langle \tilde{p}a, \tilde{p}a\rangle = \bar{a}\langle \tilde{p}, \tilde{p}\rangle a = 0$, $\langle vb, \tilde{p}a\rangle = \bar{a}\langle v, \tilde{p}\rangle b = 0$, $\langle \tilde{p}a, vb\rangle = \overline{\langle vb, \tilde{p}a\rangle} = 0$, and $\langle vb, vb\rangle = \bar{b}\langle v, v\rangle b = |b|^2\langle v, v\rangle$ because $\langle v, v\rangle$ is real.

Proof of (i). Since $\langle \tilde{p}a, \tilde{p}\rangle = \langle \tilde{p}, \tilde{p}\rangle a = 0$, we have $p \subset p^\perp$. The map $v \mapsto \langle v, \tilde{p}\rangle$ is right $\mathbb{H}$-linear and nonzero (because J is invertible and $\tilde{p} \neq 0$), hence, surjective onto $\mathbb{H}$. Therefore, $\dim_{\mathbb{H}} p^\perp = n$.

Let $v \in p^\perp \setminus p$. Then $S = \tilde{p}\mathbb{H} + v\mathbb{H}$ has dimension 2, and by Step 1, $\langle s, s\rangle = |b|^2\langle v, v\rangle$ for $s = \tilde{p}a + vb \in S$. If $\langle v, v\rangle \leq 0$, this contradicts Step 0; hence, $\langle v, v\rangle > 0$. Thus, the form is positive semidefinite on $p^\perp$, and its null vectors in $p^\perp$ are exactly the vectors of p.

Let $R = \{v \in p^\perp : \langle v, w\rangle = 0 \text{ for all } w \in p^\perp\}$ be the radical. Every $v \in R$ is null; hence, $R \subset p$. Conversely, for $w \in p^\perp$, we have $\langle \tilde{p}a, w\rangle = \langle \tilde{p}, w\rangle a = \overline{\langle w, \tilde{p}\rangle}\, a = 0$; thus, $p \subset R$. Hence, $R = p$.

For the last assertion, note that $\{[v] : \langle v, v\rangle \leq 0\}$ is well defined, because $\langle vq, vq\rangle = |q|^2\langle v, v\rangle$. It is closed in $\mathbb{P}^n_{\mathbb{H}}$, since its preimage in $V \setminus \{0\}$ is closed, and it contains $\mathbf{H}^n_{\mathbb{H}}$. Therefore, by the above, $\mathbb{P}(p^\perp) \cap \overline{\mathbf{H}^n_{\mathbb{H}}} \subset \{p\}$. Conversely, choose $u \in V$ with $\langle u, \tilde{p}\rangle = 1$. For real t,

$$\langle \tilde{p} + ut, \tilde{p} + ut\rangle = 2t + t^2\langle u, u\rangle < 0 \quad \text{for small } t < 0,$$

and $[\tilde{p} + ut] \to p$ as $t \to 0$. Hence $p \in \overline{\mathbf{H}^n_{\mathbb{H}}}$.

Proof of (ii). If $q \subset p^\perp$, then $\tilde{q}$ is a null vector in $p^\perp$; hence, $\tilde{q} \in p$ by (i) and $q = p$, a contradiction. Hence $\langle \tilde{q}, \tilde{p} \rangle \neq 0$, and also $\langle \tilde{p}, \tilde{q} \rangle = \overline{\langle \tilde{q}, \tilde{p} \rangle} \neq 0$.

Put $W = p^\perp \cap q^\perp$. Since $q \subset q^\perp$ but $q \not\subset p^\perp$, the hyperplanes $p^\perp$ and $q^\perp$ are distinct. Therefore, $p^\perp + q^\perp = V$ and $\dim_{\mathbb{H}} W = n + n - (n+1) = n - 1$.

The sum $p + W + q$ is direct. Suppose $\tilde{p}a + w + \tilde{q}b = 0$ with $w \in W$ and $a, b \in \mathbb{H}$. Pairing with $\tilde{p}$ gives $\langle \tilde{q}, \tilde{p} \rangle b = 0$, which gives $b = 0$. Pairing with $\tilde{q}$ gives $\langle \tilde{p}, \tilde{q} \rangle a = 0$, which gives $a = 0$. Hence, $w = 0$. Counting dimensions, $V = p \oplus W \oplus q$.

Finally, let $w \in W$. By (i), $\langle w, w \rangle \geq 0$, with equality only if $w \in p$. But $p \cap q^\perp = 0$, since $\langle \tilde{p}a, \tilde{q} \rangle = \langle \tilde{p}, \tilde{q} \rangle a$. Therefore, the form is positive definite on W.

Proof of (iii). Replacing $\tilde{q}$ by $\tilde{q} \langle \tilde{q}, \tilde{p} \rangle^{-1}$, we may assume $\langle \tilde{q}, \tilde{p} \rangle = 1$, and hence, $\langle \tilde{p}, \tilde{q} \rangle = 1$.

Since the form is positive definite on W, the Gram–Schmidt process gives a basis $w_2, \ldots, w_n$ of W with $\langle w_i, w_j \rangle = \delta_{ij}$. It works over $\mathbb{H}$: normalize by the real scalar $\langle v, v \rangle^{-1/2}$, and replace v by $v - \sum_i w_i \langle v, w_i \rangle$, which is orthogonal to the w_i already chosen.

Let $g \in M_{n+1}(\mathbb{H})$ have columns $c_1 = \tilde{p}$, $c_i = w_i$ $(2 \leq i \leq n)$ and $c_{n+1} = \tilde{q}$. The (i, j)-entry of $g^* J g$ is $c_i^* J c_j = \langle c_j, c_i \rangle$. By the choices made and since $W \subset p^\perp \cap q^\perp$, this matrix is exactly J. Hence, $g \in \mathrm{Sp}(n, 1)$; it is invertible by (ii). Moreover, $g(e_1 \mathbb{H}) = p$ and $g(e_{n+1} \mathbb{H}) = q$.

If (p', q') is another ordered pair and g' the corresponding element, then $g' g^{-1}$ maps (p, q) to (p', q'). This proves the transitivity. □

Non-trivial elements of $\mathrm{PSp}(n, 1)$ are classified as (see [13, 9])

1. loxodromic if they fix no point of $\mathbf{H}^n_{\mathbb{H}}$ and exactly two points of $\partial \mathbf{H}^n_{\mathbb{H}}$;
2. parabolic if they fix no point of $\mathbf{H}^n_{\mathbb{H}}$ and exactly one point of $\partial \mathbf{H}^n_{\mathbb{H}}$;
3. elliptic if they fix at least one point of $\mathbf{H}^n_{\mathbb{H}}$.

Definition 2 *Let $\Gamma \subset \mathrm{PSp}(n, 1)$ be a subgroup and $x \in \mathbf{H}^n_{\mathbb{H}}$. The* Chen–Greenberg limit set *of Γ is $\Lambda(\Gamma) = \overline{\Gamma \cdot x} \cap \partial \mathbf{H}^n_{\mathbb{H}}$, the closure being taken in $\mathbb{P}^n_{\mathbb{H}}$ [7]. We call Γ* elementary *if it has a finite orbit in $\overline{\mathbf{H}^n_{\mathbb{H}}}$, and* non-elementary *otherwise.*

Proposition 1 *Let $\Gamma \subset \mathrm{PSp}(n, 1)$ be a subgroup.*

(i) $\Lambda(\Gamma)$ is closed, Γ-invariant and independent of x. More precisely, if $\gamma_m(x) \to q \in \partial \mathbf{H}^n_{\mathbb{H}}$ for some $x \in \mathbf{H}^n_{\mathbb{H}}$, then $\gamma_m(x') \to q$ for every $x' \in \mathbf{H}^n_{\mathbb{H}}$.

(ii) If Γ is discrete, then $\Lambda(\Gamma) = \emptyset$ if and only if Γ is finite.

(iii) If Γ is non-elementary, then $\overline{\Gamma \cdot p} = \Lambda(\Gamma)$ for every $p \in \Lambda(\Gamma)$.

Proof. (i) It is [7, Lemmas 4.3.1, 4.3.2(a)]; closedness is clear from the definition.

(ii) If $\Lambda(\Gamma) = \emptyset$, then Γ fixes a point of $\mathbf{H}^n_{\mathbb{H}}$ [7, Lemma 4.3.5(a)]; therefore, it lies in a conjugate of the compact group K and, being discrete, is finite; the converse is clear.

(iii) If Γ is non-elementary, then $\overline{\Gamma \cdot p}$ is a closed invariant subset of $\partial\mathbf{H}^n_{\mathbb{H}}$ with more than one point; hence, $\Lambda(\Gamma) \subseteq \overline{\Gamma \cdot p}$ by [7, Lemma 4.3.3] and $\overline{\Gamma \cdot p} \subseteq \Lambda(\Gamma)$ by (i). $\square$

For discrete Γ, it is classical that $\Lambda(\Gamma)$ is empty, has one or two points, or is infinite, and that Γ has a finite orbit in $\overline{\mathbf{H}^n_{\mathbb{H}}}$ exactly in the first three cases [7] (see [6, Theorem 4.5] for the complex case). Thus, Definition 2 agrees with the convention of [6, Definition 4.6] and [3, §1.3], where Γ is called elementary when $\Lambda(\Gamma)$ has at most two points. One direction is immediate: if $\Lambda(\Gamma) = \emptyset$, then Γ is finite by Proposition 1(ii); if $\Lambda(\Gamma)$ has one or two points, it is a finite Γ-invariant set. Cartan decomposition in $\mathrm{Sp}(n,1)$ is already discussed in [2, Theorem 1.1]. We briefly describe it here.

Let $K = \mathrm{Sp}(n+1) \cap \mathrm{Sp}(n,1)$ be a maximal compact subgroup of $\mathrm{Sp}(n,1)$. By the Cartan decomposition, every $g \in \mathrm{Sp}(n,1)$ can be written as $g = k_1 a_t k_2$, where $k_1, k_2 \in K$, $t \geq 0$, and

$$a_t = \begin{pmatrix} e^t & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}.$$

We denote t by $\lambda(g)$ and call it the Cartan projection of g. Thus, $g = k_1 a_{\lambda(g)} k_2$.

Proposition 2 *For $t \in \mathbb{R}$, let $a_t = \mathrm{diag}(e^t, 1, \ldots, 1, e^{-t})$; then $a_t \in \mathrm{Sp}(n,1)$. Let $\|\cdot\|$ be the operator norm for $||v||^2 = v^*v$.*

- *(i) K is the stabilizer of o in $\mathrm{Sp}(n,1)$. It is compact and isomorphic to $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$.*
- *(ii) Every $g \in \mathrm{Sp}(n,1)$ can be written as $g = k_1 a_{\lambda(g)} k_2$ with $k_1, k_2 \in K$ and $\lambda(g) \geq 0$. Here $\lambda(g) = \log \|g\|$; in particular $\lambda(g)$ is unique and depends continuously on g.*
- *(iii) A sequence (γ_m) in $\mathrm{Sp}(n,1)$ leaves every compact set if and only if $\lambda(\gamma_m) \to \infty$.*
- *(iv) $a_t(o) \to [e_1]$ and $a_{-t}(o) \to [e_{n+1}]$ as $t \to \infty$.*

Proof. The existence of the decomposition in (ii) is [2, Theorem 1.1]. The group called $\mathrm{Sp}(1,n)$ there preserves the same form $\langle z, w\rangle = w^*Hz$ with $H = J$ (the signature is written as $(1,n)$ in [2]); therefore, it is our $\mathrm{Sp}(n,1)$. Also, $\mathrm{Sp}(n+1) \cap \mathrm{Sp}(1,n)$ is our K and $\mu(g)$ is our $a_{\lambda(g)}$. We prove the remaining assertions.

Let $f_\pm = (e_1 \pm e_{n+1})/\sqrt{2}$, $N = f_-\mathbb{H}$ and $P = f_+\mathbb{H} + e_2\mathbb{H} + \cdots + e_n\mathbb{H}$. The basis $\mathcal{B} = (f_-, f_+, e_2, \ldots, e_n)$ is orthonormal for the standard product w^*v. By Step 0 in the proof of Lemma 1, the Gram matrix of $\langle\cdot,\cdot\rangle$ in $\mathcal{B}$ is $\mathrm{diag}(-1, 1, \ldots, 1)$. Moreover, N and P are the (-1)- and $(+1)$-eigenspaces of J, $P = N^\perp$, and $o = [f_-]$ since $e_1 - e_{n+1} = \sqrt{2}\, f_-$.

(i) Let $k \in K$. From $k^{-1} = k^*$ and $k^*Jk = J$ we get $Jk = kJ$; hence, k preserves N; in particular $k(o) = o$.

Conversely, let $g \in \mathrm{Sp}(n, 1)$ with $g(o) = o$. Then $gf_- = f_-c$ with $|c| = 1$, and g preserves $N^\perp = P$. Therefore, g is block diagonal with respect to $V = N \oplus P$ and preserves the form on each block. On N and on P the form agrees, up to the sign on N, with the standard product. Hence $g^*g = I$ and $g \in K$.

Thus $K = \mathrm{Stab}(o) = \mathrm{U}(N) \times \mathrm{U}(P) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(n)$, which is closed and bounded, hence compact.

(ii) Since $K \subset \mathrm{Sp}(n + 1)$ preserves $|\cdot|$, any decomposition $g = k_1 a_t k_2$ with $t \geq 0$ gives $\|g\| = \|a_t\| = e^t$. Hence, $t = \log \|g\|$ is unique and depends continuously on g.

(iii) By (ii), λ is continuous, hence bounded on compact sets. Therefore, if $\lambda(\gamma_m) \to \infty$, the sequence leaves every compact set. Conversely, if $\lambda(\gamma_m) \not\to \infty$, a subsequence satisfies $\lambda(\gamma_m) \leq T$ for some T. It then lies in the compact set $K \cdot \{a_t : 0 \leq t \leq T\} \cdot K$.

(iv) We have $a_t(e_1 - e_{n+1}) = e^t e_1 - e^{-t} e_{n+1} = \sqrt{2}\,(f_- \cosh t + f_+ \sinh t)$, and thus $a_t(o) = [f_- + f_+ \tanh t]$. Hence, $a_t(o) \to [f_- + f_+] = [e_1]$ and $a_{-t}(o) = [f_- - f_+ \tanh t] \to [f_- - f_+] = [e_{n+1}]$ as $t \to \infty$. □

2.3 The complex realization

We now describe the complex realization of V. Write $v \in V$ uniquely as $v = a + \mathbf{j}b$ with $a, b \in \mathbb{C}^{n+1}$, and put $\chi(v) = (a, b) \in \mathbb{C}^{2n+2} =: V_\mathbb{C}$. Define

$$J_q(a, b) = (-\bar{b}, \bar{a}), \quad M_\mathbb{C} = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \text{ for } M = A + \mathbf{j}B \in M_{n+1}(\mathbb{H}),$$

$$h_\mathbb{C}(x, y) = y^* \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} x.$$

For a right subspace $U \subset V$ put $U_\mathbb{C} = \chi(U)$, and for $g \in \mathrm{Sp}(n, 1)$, write $g_\mathbb{C}$ for its complex matrix.

Lemma 2 *(i) $\chi(vc) = \chi(v)c$ for $c \in \mathbb{C}$, $\chi(v\mathbf{j}) = J_q\chi(v)$ and $\chi(Mv) = M_\mathbb{C}\chi(v)$. The map J_q is antilinear with $J_q^2 = -1$, and $M \mapsto M_\mathbb{C}$ is an injective ring homomorphism with $(M^*)_\mathbb{C} = (M_\mathbb{C})^*$.*

(ii) For $v, w \in V$, we have

$$\langle v, w\rangle = h_\mathbb{C}(\chi v, \chi w) + \mathbf{j}\, h_\mathbb{C}(\chi v, \chi(w\mathbf{j})). \tag{1}$$

The form $h_{\mathbb{C}}$ has signature $(2n,2)$, and $\{g_{\mathbb{C}} : g \in \mathrm{Sp}(n,1)\} = \{T \in \mathrm{U}(h_{\mathbb{C}}) : TJ_q = J_qT\}$.

(iii) $U \mapsto U_{\mathbb{C}}$ is a bijection from right $\mathbb{H}$-subspaces of V onto J_q-invariant complex subspaces of $V_{\mathbb{C}}$. It satisfies $\dim_{\mathbb{C}} U_{\mathbb{C}} = 2\dim_{\mathbb{H}} U$, $(U^{\perp})_{\mathbb{C}} = (U_{\mathbb{C}})^{\perp_{h_{\mathbb{C}}}}$ and $(gU)_{\mathbb{C}} = g_{\mathbb{C}}U_{\mathbb{C}}$.

(iv) $(a_t)_{\mathbb{C}}$ has singular values e^t (twice), 1 ($2n-2$ times) and e^{-t} (twice), and $k_{\mathbb{C}}$ is unitary for every $k \in K$.

Proof. We use that every $q \in \mathbb{H}$ is uniquely representable as $q = z + \mathbf{j}w$ with $z, w \in \mathbb{C}$, that $c\mathbf{j} = \mathbf{j}\bar{c}$ for $c \in \mathbb{C}$ (entrywise for complex vectors and matrices), and that $\mathbf{j}\bar{c} = -\mathbf{j}c$.

(i) For $v = a + \mathbf{j}b$ and $c \in \mathbb{C}$, we have $vc = ac + \mathbf{j}(bc)$ and $v\mathbf{j} = a\mathbf{j} + \mathbf{j}b\mathbf{j} = -\bar{b} + \mathbf{j}\bar{a}$. These give the first two identities. For $M = A + \mathbf{j}B$,

$$Mv = Aa + A\mathbf{j}b + \mathbf{j}Ba + \mathbf{j}B\mathbf{j}b = (Aa - \bar{B}b) + \mathbf{j}(Ba + \bar{A}b),$$

i.e., $\chi(Mv) = M_{\mathbb{C}}\chi(v)$. Clearly, $J_q(xc) = J_q(x)\bar{c}$ and $J_q^2 = -1$. Since χ is bijective, $\chi(Mv) = M_{\mathbb{C}}\chi(v)$ shows that $M \mapsto M_{\mathbb{C}}$ is additive, multiplicative and injective. Finally, $(\mathbf{j}B)^* = -\mathbf{j}B^T$; hence, $M^* = A^* + \mathbf{j}(-B^T)$ and $(M^*)_{\mathbb{C}} = \begin{pmatrix} A^* & B^* \\ -B^T & A^T \end{pmatrix} = (M_{\mathbb{C}})^*$. (Equivalently, $M_{\mathbb{C}}$ is conjugate to the complex representation $\omega(M)$ of [16, Sect. 3.4] by a fixed signed permutation matrix, and these properties are [16, Sect. 3.4(i),(ii),(iv)].)

(ii) Write $w = c + \mathbf{j}d$. Then $w^* = c^* - \mathbf{j}d^T$, and the rules above give

$$w^*Jv = (c^*Ja + d^*Jb) + \mathbf{j}\,(c^TJb - d^TJa).$$

The first bracket is $h_{\mathbb{C}}(\chi v, \chi w)$. Since $\chi(w\mathbf{j}) = (-\bar{d}, \bar{c})$ by (i), the second is $h_{\mathbb{C}}(\chi v, \chi(w\mathbf{j}))$. This proves (1). As J has signature $(n,1)$, $\mathrm{diag}(J,J)$ has signature $(2n,2)$.

If $g \in \mathrm{Sp}(n,1)$, then $g_{\mathbb{C}}$ commutes with J_q by (i) and preserves $h_{\mathbb{C}}$ by (1). Conversely, let $T \in \mathrm{U}(h_{\mathbb{C}})$ commute with J_q. Then $g = \chi^{-1}T\chi$ is additive and commutes with right multiplication by $\mathbb{C}$ and by $\mathbf{j}$, hence by $\mathbb{H} = \mathbb{C}+\mathbf{j}\mathbb{C}$. Therefore, g is $\mathbb{H}$-linear and $g_{\mathbb{C}} = T$. Since $\chi(g(w)\mathbf{j}) = J_qT\chi(w) = T\chi(w\mathbf{j})$, identity (1) shows that g preserves $\langle\cdot,\cdot\rangle$, i.e., $g \in \mathrm{Sp}(n,1)$.

(iii) By (i), $U_{\mathbb{C}}$ is a complex subspace and is J_q-invariant. Conversely, if $W \subset V_{\mathbb{C}}$ is a J_q-invariant complex subspace, then $\chi^{-1}(W)$ is closed under right multiplication by $\mathbb{C}$ and by $\mathbf{j}$, hence is a right $\mathbb{H}$-subspace.

If $u_1, \dots, u_m$ is an $\mathbb{H}$-basis of U, then $u_1, \dots, u_m, u_1\mathbf{j}, \dots, u_m\mathbf{j}$ is a $\mathbb{C}$-basis of U. Indeed, $u(z + \mathbf{j}w) = uz + (u\mathbf{j})w$, and $\sum_i u_iz_i + \sum_i (u_i\mathbf{j})w_i = \sum_i u_i(z_i + \mathbf{j}w_i)$. Hence, $\dim_{\mathbb{C}} U_{\mathbb{C}} = 2m$ (cf. [16, Proposition 3.4.2]).

Next, $v \in U^{\perp}$ if and only if $\langle v, u\rangle = 0$ for all $u \in U$. By (ii), and since $U\mathbf{j} = U$, this is equivalent to $h_{\mathbb{C}}(\chi v, \chi u) = 0$ for all $u \in U$, i.e., $\chi v \in (U_{\mathbb{C}})^{\perp_{h_{\mathbb{C}}}}$. Finally, $(gU)_{\mathbb{C}} = g_{\mathbb{C}}U_{\mathbb{C}}$ by (i).

(iv) $(a_t)_{\mathbb{C}} = \mathrm{diag}(a_t, a_t)$, and for $k \in K$, we have $(k_{\mathbb{C}})^*k_{\mathbb{C}} = (k^*k)_{\mathbb{C}} = I$ by (i). □

We write $v = a + \mathbf{j}b$ rather than $v = a + b\mathbf{j}$ because $(a + b\mathbf{j})c = ac + b\bar{c}\mathbf{j}$ for $c \in \mathbb{C}$; with that convention the second coordinate would transform antilinearly.

2.4 Complexified exterior powers and the Plücker map

Our construction parallels that of Zúñiga [20, 19] for discrete subgroups of $\mathrm{PU}(1,n)$ acting on $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$. The difference is that a quaternionic $(k+1)$-plane U is recorded through the complex $(2k+2)$-plane $U_{\mathbb{C}}$; therefore, we work in $\bigwedge_{\mathbb{C}}^{2k+2} V_{\mathbb{C}}$ rather than in an exterior power over $\mathbb{H}$ (compare with Remark 2).

Definition 3 *For $0 \le k \le n-1$, let $\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$ be the set of right $\mathbb{H}$-subspaces of dimension $k+1$ (equivalently, projective k-planes of $\mathbb{P}^n_{\mathbb{H}}$). Put*

$$W_k = \textstyle\bigwedge_{\mathbb{C}}^{2k+2} V_{\mathbb{C}}, \qquad N_k = \dim_{\mathbb{C}} W_k = \binom{2n+2}{2k+2}, \qquad \Phi_k(g) = \textstyle\bigwedge^{2k+2} g_{\mathbb{C}}.$$

Since $\Phi_k(-I) = I$, Φ_k is defined on $\mathrm{PSp}(n,1)$. For $U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$ with basis $u_1, \dots, u_{k+1}$, put

$$\mathrm{vol}(U_{\mathbb{C}}) = \chi(u_1) \wedge \cdots \wedge \chi(u_{k+1}) \wedge \chi(u_1\mathbf{j}) \wedge \cdots \wedge \chi(u_{k+1}\mathbf{j}),$$
$$\mathcal{P}_k(U) = [\mathrm{vol}(U_{\mathbb{C}})] \in \mathbb{P}(W_k).$$

Lemma 3 *(i) $\mathcal{P}_k(U)$ does not depend on the choice of basis of U.*
(ii) $\mathcal{P}_k$ is injective.
(iii) $\mathcal{P}_k$ is continuous with closed image, hence a homeomorphism onto a compact subset of $\mathbb{P}(W_k)$.
(iv) $\Phi_k(g)\,\mathcal{P}_k(U) = \mathcal{P}_k(gU)$ for all $g \in \mathrm{PSp}(n,1)$.
(v) $\sigma = \bigwedge^{2k+2} J_q$ is an antilinear involution of W_k commuting with $\Phi_k(\mathrm{PSp}(n,1))$, and $\sigma(\mathrm{vol}U_{\mathbb{C}}) = \mathrm{vol}U_{\mathbb{C}}$.

Proof. (i) By the proof of Lemma 2(iii), the vectors $\chi(u_1), \dots, \chi(u_{k+1})$, $\chi(u_1\mathbf{j}), \dots, \chi(u_{k+1}\mathbf{j})$ form a $\mathbb{C}$-basis of $U_{\mathbb{C}}$. Therefore, $\mathrm{vol}(U_{\mathbb{C}}) \neq 0$, and another $\mathbb{H}$-basis of U multiplies $\mathrm{vol}(U_{\mathbb{C}})$ by the determinant of the corresponding change of $\mathbb{C}$-basis of $U_{\mathbb{C}}$.

(ii) For linearly independent $x_1, \dots, x_r$, we have $\{x : x \wedge x_1 \wedge \cdots \wedge x_r = 0\} = \mathrm{span}_{\mathbb{C}}(x_1, \dots, x_r)$. In particular, $U_{\mathbb{C}} = \{x \in V_{\mathbb{C}} : x \wedge \mathrm{vol}(U_{\mathbb{C}}) = 0\}$. Hence, $\mathcal{P}_k(U)$ determines $U_{\mathbb{C}}$, and $U_{\mathbb{C}}$ determines U by Lemma 2(iii).

(iii) We give $\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$ the quotient topology from the set of $(n+1) \times (k+1)$ quaternionic matrices of rank $k+1$. By Gram–Schmidt, it is the image of the compact set of matrices with orthonormal columns, hence compact. The map $(u_1, \dots, u_{k+1}) \mapsto [\mathrm{vol}(U_{\mathbb{C}})]$ is continuous (its coordinates are polynomials in the entries of the $\chi(u_i)$ and $\chi(u_i\mathbf{j})$); therefore, $\mathcal{P}_k$ is continuous. A continuous injection from a compact space into the Hausdorff space $\mathbb{P}(W_k)$ is a homeomorphism onto its image, which is compact.

(iv) By Lemma 2(i), $g_{\mathbb{C}}\chi(u_i) = \chi(gu_i)$ and $g_{\mathbb{C}}\chi(u_i\mathbf{j}) = \chi((gu_i)\mathbf{j})$. Since $gu_1, \dots, gu_{k+1}$ is a basis of gU, we get $\Phi_k(g)\mathrm{vol}(U_{\mathbb{C}}) = \mathrm{vol}((gU)_{\mathbb{C}})$.

(v) The map $(x_1, \dots, x_r) \mapsto J_q x_1 \wedge \cdots \wedge J_q x_r$ is alternating and additive in each variable, with $J_q(xc) = J_q(x)\bar{c}$. It therefore induces an antilinear map σ of W_k ($r = 2k+2$). We have $\sigma^2 = \bigwedge^r(J_q^2) = (-1)^r = 1$, and σ commutes with $\Phi_k(g)$ because $g_{\mathbb{C}}J_q = J_q g_{\mathbb{C}}$. Finally, $J_q\chi(u) = \chi(u\mathbf{j})$ and $J_q\chi(u\mathbf{j}) = -\chi(u)$. Moving the block $\chi(u_1\mathbf{j}) \wedge \cdots \wedge \chi(u_{k+1}\mathbf{j})$ to the front costs the sign $(-1)^{(k+1)^2}$; hence, $\sigma(\mathrm{vol}U_{\mathbb{C}}) = (-1)^{k+1}(-1)^{(k+1)^2}\mathrm{vol}U_{\mathbb{C}} = \mathrm{vol}U_{\mathbb{C}}$. □

Remark 2 *Alesker [1, Theorem 1] proved that for $m \geq 2$ and $N \geq m+2$, there is no continuous map $\eta\colon \mathrm{Gr}_m(\mathbb{H}^N) \to \mathbb{P}^\infty_{\mathbb{H}}$ such that η^* maps the first Pontryagin class of the tautological bundle over $\mathbb{P}^\infty_{\mathbb{H}}$ to that of the tautological bundle over $\mathrm{Gr}_m(\mathbb{H}^N)$; hence, a quaternionic Plücker map $U \mapsto \bigwedge^m U$ does not exist, even up to homotopy. With $m = k+1$ and $N = n+1$, this is exactly the range $1 \leq k \leq n-2$; for $k = 0$ and $k = n-1$, the maps $U \mapsto U$ and $U \mapsto U^\perp$ take values in $\mathbb{P}^n_{\mathbb{H}}$.*

The map $\mathcal{P}_k$ does not contradict this. It takes values in the complex projective space $\mathbb{P}(W_k)$: it is the ordinary complex Plücker embedding $U_{\mathbb{C}} \mapsto [\mathrm{vol}(U_{\mathbb{C}})]$, restricted to J_q-invariant subspaces, and by Lemma 3(v) its image even lies in the real points $\mathbb{P}(W_k^\sigma)$. It pulls back the tautological line bundle to $\det_{\mathbb{C}}(U_{\mathbb{C}})$, which is topologically trivial because $H^2(\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}); \mathbb{Z}) = 0$ (all Schubert cells have real dimension divisible by 4). Therefore, $\mathcal{P}_k$ carries no characteristic-class information; we use it only as a $\mathrm{PSp}(n,1)$-equivariant embedding. No exterior power of right $\mathbb{H}$-modules is used in this paper.

2.5 The induced Hermitian form and discreteness

Lemma 4 *The formula*

$$h_k(x_1 \wedge \cdots \wedge x_{2k+2},\, y_1 \wedge \cdots \wedge y_{2k+2}) = \det\big(h_{\mathbb{C}}(x_i, y_j)\big) \tag{2}$$

(an ordinary complex determinant) defines a nondegenerate Hermitian form on W_k with $\Phi_k(\mathrm{Sp}(n,1)) \subset \mathrm{U}(h_k)$. Its signature (negative, positive) is

$$\left(2\binom{2n}{2k+1},\ \binom{2n}{2k+2} + \binom{2n}{2k}\right). \tag{3}$$

Proof. Put $r = 2k+2$. The right-hand side of (2) is alternating and linear in $(x_1, \dots, x_r)$, and alternating and conjugate-linear in $(y_1, \dots, y_r)$. Therefore, it defines a sesquilinear form h_k on W_k. Since

$$\det\big(h_{\mathbb{C}}(y_i, x_j)\big) = \det\big(\overline{h_{\mathbb{C}}(x_j, y_i)}\big) = \overline{\det\big(h_{\mathbb{C}}(x_i, y_j)\big)},$$

the form h_k is Hermitian. If $g \in \mathrm{Sp}(n,1)$, then $g_{\mathbb{C}} \in \mathrm{U}(h_{\mathbb{C}})$ by Lemma 2(ii); hence, $\det\big(h_{\mathbb{C}}(g_{\mathbb{C}}x_i, g_{\mathbb{C}}y_j)\big) = \det\big(h_{\mathbb{C}}(x_i, y_j)\big)$, i.e., $\Phi_k(g) \in \mathrm{U}(h_k)$.

By Lemma 2(ii) there is a basis $b_1, \dots, b_{2n+2}$ of $V_{\mathbb{C}}$ with $h_{\mathbb{C}}(b_i, b_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = -1$ for exactly two indices. For increasing multi-indices I, J of length r, the matrix $\big(h_{\mathbb{C}}(b_i, b_j)\big)_{i\in I, j\in J}$ has a zero row if $I \neq J$. Hence,

$$h_k(b_I, b_J) = 0 \ (I \neq J), \qquad h_k(b_I, b_I) = \prod_{i\in I} \varepsilon_i = (-1)^{m(I)},$$

where $m(I)$ is the number of negative indices in I. Therefore, h_k is non-degenerate. This proves (3): it is negative on the $2\binom{2n}{2k+1}$ vectors b_I with $m(I) = 1$, and positive on the $\binom{2n}{2k+2} + \binom{2n}{2k}$ vectors with $m(I) \in \{0, 2\}$. □

Lemma 5 *For $0 \leq k \leq n-1$, the homomorphism $\Phi_k \colon \mathrm{PSp}(n,1) \to \mathrm{PU}(h_k)$ is injective, continuous and proper. Consequently, for every discrete subgroup $\Gamma \subset \mathrm{PSp}(n,1)$, $\Phi_k(\Gamma)$ is a discrete subgroup of $\mathrm{PU}(h_k)$ and $\Phi_k|_\Gamma$ is an isomorphism onto it.*

Proof. Put $r = 2k+2$; then $2 \leq r \leq 2n$. By Lemma 4 and $\Phi_k(-I) = I$, Φ_k is a well-defined continuous homomorphism into $\mathrm{PU}(h_k)$.

Injectivity. Suppose $\Phi_k(g)$ is scalar. Let $X \subset V_{\mathbb{C}}$ be any r-dimensional subspace, with volume vector $\mathrm{vol}(X)$. Then $\bigwedge^r g_{\mathbb{C}} \mathrm{vol}(X) = \mathrm{vol}(g_{\mathbb{C}} X)$ is a multiple of $\mathrm{vol}(X)$; hence, $g_{\mathbb{C}} X = X$ as in Lemma 3(ii). Since $r < 2n+2$, every complex line is an intersection of r-dimensional subspaces; therefore, $g_{\mathbb{C}}$ preserves every line and $g_{\mathbb{C}} = \lambda I$ with $\lambda \in \mathbb{C}^*$. Now $g_{\mathbb{C}} J_q = J_q g_{\mathbb{C}}$ gives $\lambda = \bar{\lambda}$, and $g_{\mathbb{C}} \in \mathrm{U}(h_{\mathbb{C}})$ gives $\lambda^2 = 1$. Thus, $g = \pm I$.

Properness. For an invertible h, put $\kappa(h) = s_{\max}(h)/s_{\min}(h)$, the ratio of its largest and smallest singular values. It is invariant under scalars. A closed subset of $\mathrm{PGL}(W_k)$ is compact if and only if κ is bounded on it since the lifts of norm 1 with $s_{\min} \geq 1/C$ form a compact subset of $\mathrm{GL}(W_k)$.

Write $g = k_1 a_t k_2$ with $t = \lambda(g)$ (Proposition 2). Then $\Phi_k(k_i) = \bigwedge^r (k_i)_{\mathbb{C}}$ is unitary by Lemma 2(iv). Moreover $\Phi_k(a_t)$ is diagonal in the standard basis of W_k, with entries $e^{(a-c)t}$, where a and c count the factors with $(a_t)_{\mathbb{C}}$-eigenvalue e^t and e^{-t}. Since $2 \leq r \leq 2n$, both $(a,c) = (2,0)$ and $(a,c) = (0,2)$ occur. Hence, $\kappa(\Phi_k(g)) = e^{4\lambda(g)}$.

Therefore, if $C \subset \mathrm{PU}(h_k)$ is compact, then $\lambda \leq T$ on the closed set $\Phi_k^{-1}(C)$ for some T. By Proposition 2(ii), this set lies in the image of the compact set $K \cdot \{a_t : 0 \leq t \leq T\} \cdot K$; hence, it is compact.

Discreteness. Let $\gamma_m \in \Gamma$ be distinct with $\Phi_k(\gamma_m)$ convergent in $\mathrm{PU}(h_k)$. By properness, a subsequence of (γ_m) converges in $\mathrm{PSp}(n,1)$. This is impossible for distinct elements of the discrete (hence closed) subgroup Γ. Since Φ_k is injective, $\Phi_k|_\Gamma$ is an isomorphism onto $\Phi_k(\Gamma)$. □

2.6 Pseudo-projective maps, equicontinuity and Kulkarni sets

In this subsection, $N \geq 2$ and $G \subset \mathrm{PSL}(N, \mathbb{C})$ acts on $\mathbb{P}^{N-1} = \mathbb{P}(\mathbb{C}^N)$. We apply it to $\mathbb{C}^N = W_k$ and $G = \Phi_k(\Gamma)$.

Definition 4 *For a nonzero $T \in M_N(\mathbb{C})$, the* pseudo-projective map $\rho = [T]$ *is* $\mathbb{P}^{N-1} \setminus \mathrm{Ker}(\rho) \to \mathbb{P}^{N-1}$, $[x] \mapsto [Tx]$, *where* $\mathrm{Ker}(\rho) = \mathbb{P}(\mathrm{Ker}T)$*; we write* $\mathfrak{Im}(\rho) = \mathbb{P}(\mathfrak{Im}T)$. *Let* $\mathbb{QP}(N,\mathbb{C}) = \mathbb{P}(M_N(\mathbb{C})) \supset \mathrm{PSL}(N,\mathbb{C})$. *For* $G \subset \mathrm{PSL}(N,\mathbb{C})$, *let* $\mathrm{Lim}(G)$ *be the set of* $\rho \in \mathbb{QP}(N,\mathbb{C})$ *with* $\rho = \lim g_m$ *for some sequence of* distinct *elements* $g_m \in G$.

Lemma 6 *(i)* $\mathbb{QP}(N,\mathbb{C})$ *is compact; hence, every sequence of distinct elements of* $\mathrm{PSL}(N,\mathbb{C})$ *has a subsequence converging in* $\mathbb{QP}(N,\mathbb{C})$.
(ii) If $g_m \to \rho$ *in* $\mathbb{QP}(N,\mathbb{C})$, *then* $g_m \to \rho$ *uniformly on compact subsets of* $\mathbb{P}^{N-1} \setminus \mathrm{Ker}(\rho)$.
(iii) If G *is discrete, then every* $\rho \in \mathrm{Lim}(G)$ *is singular, i.e.,* $\mathrm{Ker}(\rho) \neq \emptyset$.

Proof. (i) and (ii) are [4, Proposition 1.1]; see also [5, Proposition 7.4.1].

(iii) $\mathrm{PSL}(N,\mathbb{C}) = \{[T] : \det T \neq 0\}$ is open in $\mathbb{QP}(N,\mathbb{C})$, and its group topology is the subspace topology. If $\rho = \lim g_m$ with distinct $g_m \in G$ were invertible, the g_m would converge in $\mathrm{PSL}(N,\mathbb{C})$. This is impossible for a discrete subgroup. Hence, ρ is singular. □

Definition 5 *Let G act by homeomorphisms on a compact metric space X. A family $\mathcal{F}$ of continuous maps $U \to X$ is* normal *if every sequence in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets of U. The* equicontinuity region $\mathrm{Eq}(G)$ *is the set of $x \in X$ having an open neighbourhood U such that $\{g|_U : g \in G\}$ is normal.*

We use this for $X = \mathbb{P}(W_k)$ and for $X = \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$ (Corollary 1).

Definition 6 ([14]) *Let G act on a compact metric space X. A point x is a* cluster point *of a family $\{A_\beta\}_{\beta\in B}$ of subsets of X if every neighbourhood of x meets A_β for infinitely many β. Let*

(a) $L_0(G)$ be the closure of the set of points with infinite isotropy group;
(b) $L_1(G)$ be the closure of the set of cluster points of orbits $G \cdot x$ with $x \in X \setminus L_0(G)$;
(c) $L_2(G)$ be the closure of the set of cluster points of $\{g(C)\}_{g\in G}$, where C runs over the compact subsets of $X \setminus (L_0(G) \cup L_1(G))$.

The Kulkarni limit set *is* $\Lambda_{\mathrm{Kul}}(G) = L_0(G) \cup L_1(G) \cup L_2(G)$, *and the* Kulkarni domain of discontinuity *is* $\Omega_{\mathrm{Kul}}(G) = X \setminus \Lambda_{\mathrm{Kul}}(G)$.

G acts properly discontinuously on $\Omega_{\mathrm{Kul}}(G)$ [14].

Proposition 3 ([4, Proposition 2.5]) *For every subgroup $G \subset \mathrm{PSL}(N,\mathbb{C})$,*

$$\mathrm{Eq}(G) = \mathbb{P}^{N-1} \setminus \bigcup_{\rho\in\mathrm{Lim}(G)} \mathrm{Ker}(\rho);$$

in particular, this union is closed.

This is reproved in Section 4 (proof of Theorem 1(c)).

Pseudo-projective transformations go back to Myrberg; see [5] for the complex case and [8, §2.3, Lemma 2.8] for their quaternionic analogue on $\mathbb{P}^2_{\mathbb{H}}$, where the Kulkarni sets of cyclic subgroups of $\mathrm{PSL}(3,\mathbb{H})$ are computed. We do not need the quaternionic version: all dynamics below takes place in the complex projective space $\mathbb{P}(W_k)$, where Lemma 6 is the classical statement.

3 Limits of the exterior-power action

In this section, we determine all pseudo-projective limits of $\Phi_k(\Gamma)$ and prove parts (a) and (b) of Theorem 1. Everything reduces to an explicit computation for the diagonal elements a_t, which is then transported by the transitivity results of Section 2. Throughout, $0 \le k \le n-1$ and $r = 2k+2$.

3.1 Weights of $\Phi_k(a_t)$

Let $\varepsilon_1, \dots, \varepsilon_{2n+2}$ be the standard basis of $V_{\mathbb{C}} = \mathbb{C}^{2n+2}$. Then $\chi(e_i) = \varepsilon_i$ and $\chi(e_i j) = \varepsilon_{n+1+i}$. Put

$$E_1 = \mathrm{span}(\varepsilon_1, \varepsilon_{n+2}) = \chi(e_1\mathbb{H}), \qquad E_{-1} = \mathrm{span}(\varepsilon_{n+1}, \varepsilon_{2n+2}) = \chi(e_{n+1}\mathbb{H}),$$

and let E_0 be the span of the remaining $2n-2$ basis vectors. Put $\mathrm{vol}(E_1) = \varepsilon_1 \wedge \varepsilon_{n+2}$ and $\mathrm{vol}(E_{-1}) = \varepsilon_{n+1} \wedge \varepsilon_{2n+2}$. For an increasing multi-index I of length r, put $\varepsilon_I = \bigwedge_{i\in I} \varepsilon_i$. Let $a(I)$ and $c(I)$ be the numbers of indices of I in $\{1, n+2\}$ and in $\{n+1, 2n+2\}$, respectively. For $j \in \{-2, \dots, 2\}$, let $W_j \subset W_k$ be spanned by the ε_I with $a(I) - c(I) = j$, and let $P_{\max}$ be the projection of W_k onto W_2 along $\bigoplus_{j\le 1} W_j$.

Lemma 7 *(i) $W_k = \bigoplus_{j=-2}^{2} W_j$, and $\Phi_k(a_t)$ acts on W_j by e^{jt}. Moreover, $W_{\pm 2} = \mathrm{vol}(E_{\pm 1}) \wedge \bigwedge^{2k} E_0$, and both have dimension $\binom{2n-2}{2k}$.*

(ii) $e^{-2t}\Phi_k(a_t) \to P_{\max}$ as $t \to \infty$. The kernel $\bigoplus_{j\le 1} W_j$ of $P_{\max}$ contains every ε_I with $a(I) = 2$ and $c(I) \ge 1$.

(iii) $h_k(W_i, W_j) = 0$ unless $i + j = 0$, and h_k pairs W_j and W_{-j} nondegenerately. Hence, $W_j^{\perp} = \bigoplus_{i\ne -j} W_i$; in particular, $\mathrm{Ker} P_{\max} = W_{-2}^{\perp}$.

Proof. (i) By Lemma 2(iv), $(a_t)_{\mathbb{C}} = \mathrm{diag}(a_t, a_t)$ acts by e^t, 1 and e^{-t} on E_1, E_0 and E_{-1}; hence, $\Phi_k(a_t)\varepsilon_I = e^{(a(I)-c(I))t}\varepsilon_I$. We have $a(I) - c(I) = 2$ exactly when I contains both indices of E_1, none of E_{-1}, and $2k$ of the $2n-2$ indices of E_0. This is possible since $2k \le 2n-2$, and gives the description of W_2 and its dimension; W_{-2} is analogous.

(ii) This follows from (i) since $e^{(j-2)t} \to 0$ for $j < 2$.

(iii) Let π be the involution of $\{1, \dots, 2n+2\}$ exchanging $1 \leftrightarrow n+1$ and $n+2 \leftrightarrow 2n+2$ and fixing the other indices. Since $h_{\mathbb{C}}(x, y) = y^* \mathrm{diag}(J, J)x$,

we have $h_{\mathbb{C}}(\varepsilon_i, \varepsilon_l) = \delta_{l,\pi(i)}$. Hence, $h_k(\varepsilon_I, \varepsilon_L) = \det\left(\delta_{l,\pi(i)}\right)_{i\in I, l\in L}$, which is 0 unless $L = \pi(I)$, and ± 1 if $L = \pi(I)$. Since $a(\pi(I)) = c(I)$ and $c(\pi(I)) = a(I)$, the map $I \mapsto \pi(I)$ matches the basis of W_j with that of W_{-j}, and all claims follow. $\square$

Note that, by part (ii), the kernel of $P_{\max}$ also contains wedges that contain both expanding directions together with a contracting one.

3.2 Tangent planes and their spans

Recall that for $p \in \partial\mathbf{H}^n_{\mathbb{H}}$, $H^k_p = \{U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}) : p \subset U \subset p^\perp\}$, and $\langle H^k_p\rangle$ is the complex span of $\mathcal{P}_k(H^k_p)$ in W_k. By Lemma 1(i), the form is positive semidefinite on $p^\perp$ with radical p. Therefore, $\mathbb{P}(U)$ meets $\overline{\mathbf{H}^n_{\mathbb{H}}}$ only in p for $U \in H^k_p$; these are the projective k-planes tangent to $\partial\mathbf{H}^n_{\mathbb{H}}$ at p. The map $U \mapsto U/p$ identifies H^k_p with the Grassmannian of k-dimensional subspaces of $p^\perp/p \cong \mathbb{H}^{n-1}$. We have $H^0_p = \{p\}$, $H^{n-1}_p = \{p^\perp\}$, and H^1_p is the space $\pi_\beta(p)$ of [2, Lemma 5.1]. Write $p_{\mathbb{C}}$ for $(p)_{\mathbb{C}}$, and $\mathrm{vol}(p_{\mathbb{C}}) = \chi(\tilde p) \wedge \chi(\tilde p j)$ for any $\tilde p \in p \setminus \{0\}$.

Lemma 8 *Let $p \in \partial\mathbf{H}^n_{\mathbb{H}}$ and $g \in \mathrm{Sp}(n,1)$.*

(i) $\langle H^k_p\rangle = \mathrm{vol}(p_{\mathbb{C}}) \wedge \bigwedge^{2k}(p^\perp)_{\mathbb{C}}$, and $\dim_{\mathbb{C}}\langle H^k_p\rangle = \binom{2n-2}{2k}$. In particular, $\langle H^k_{[e_1]}\rangle = W_2$ and $\langle H^k_{[e_{n+1}]}\rangle = W_{-2}$.

(ii) For $U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$, $\mathcal{P}_k(U) \in \mathbb{P}\langle H^k_p\rangle$ if and only if $U \in H^k_p$.

(iii) $\Phi_k(g)\langle H^k_p\rangle = \langle H^k_{gp}\rangle$, and the map $p \mapsto \langle H^k_p\rangle$ from $\partial\mathbf{H}^n_{\mathbb{H}}$ to the Grassmannian of $\binom{2n-2}{2k}$-dimensional subspaces of W_k is continuous.

Proof. (i) Put $A = \mathrm{vol}(p_{\mathbb{C}}) \wedge \bigwedge^{2k}(p^\perp)_{\mathbb{C}}$.

If $U \in H^k_p$, choose an $\mathbb{H}$-basis $\tilde p, u_2, \dots, u_{k+1}$ of U. Then $\mathrm{vol}(U_{\mathbb{C}}) = \pm\mathrm{vol}(p_{\mathbb{C}}) \wedge \chi(u_2) \wedge \dots \wedge \chi(u_{k+1}j) \in A$; hence, $\langle H^k_p\rangle \subset A$.

For the converse, pick $q \in \partial\mathbf{H}^n_{\mathbb{H}} \setminus \{p\}$ and put $W' = p^\perp \cap q^\perp$. By Lemma 1(ii), $p^\perp = p \oplus W'$ with $\dim_{\mathbb{H}} W' = n-1$. Since $\mathrm{vol}(p_{\mathbb{C}}) \wedge p_{\mathbb{C}} = 0$, the map $\xi \mapsto \mathrm{vol}(p_{\mathbb{C}}) \wedge \xi$ is an isomorphism from $\bigwedge^{2k} W'_{\mathbb{C}}$ onto A; in particular, $\dim A = \binom{2n-2}{2k}$. For a k-dimensional subspace $U' \subset W'$, we have $p \oplus U' \in H^k_p$ and $\mathrm{vol}((p \oplus U')_{\mathbb{C}}) = \pm\mathrm{vol}(p_{\mathbb{C}}) \wedge \mathrm{vol}(U'_{\mathbb{C}})$. Therefore, it suffices to show that the span S of the vectors $\mathrm{vol}(U'_{\mathbb{C}})$ is all of $\bigwedge^{2k} W'_{\mathbb{C}}$.

The subspace $W'_{\mathbb{C}}$ is J_q-invariant. Let X be a complex-linear map of $W'_{\mathbb{C}}$ commuting with J_q. Then $\exp(sX)$ ($s \in \mathbb{R}$) also commutes with J_q; hence, it is χ-conjugate to an $\mathbb{H}$-linear automorphism of W' (cf. the proof of Lemma 2(ii)). By Lemma 2(i) it maps each $U'_{\mathbb{C}}$ to some $U''_{\mathbb{C}}$ of the same kind; therefore, $\bigwedge^{2k}\exp(sX)$ preserves S. Differentiating at $s = 0$, S is invariant under the induced derivation D_X of $\bigwedge^{2k} W'_{\mathbb{C}}$. Every complex-linear Y on $W'_{\mathbb{C}}$ is $X_1 + iX_2$ with such X_1, X_2, namely, $X_1 = \frac{1}{2}(Y + J_q^{-1} Y J_q)$ and

$iX_2 = \frac{1}{2}(Y - J_q^{-1} Y J_q)$; here X_2 commutes with J_q because J_q is antilinear. Therefore, S is invariant under $D_Y = D_{X_1} + iD_{X_2}$ for all Y, hence under $\mathrm{GL}(W'_{\mathbb{C}})$. This group is transitive on the $2k$-dimensional subspaces of $W'_{\mathbb{C}}$; hence, S contains every decomposable vector. Decomposable vectors span $\bigwedge^{2k} W'_{\mathbb{C}}$.

For $p = [e_1]$, we have $p_{\mathbb{C}} = E_1$ and $(p^\perp)_{\mathbb{C}} = E_1 \oplus E_0$; thus, $A = W_2$; similarly for $[e_{n+1}]$.

(ii) If $\mathrm{vol}(U_{\mathbb{C}}) \in A$, then $x \wedge \mathrm{vol}(U_{\mathbb{C}}) = 0$ for $x \in p_{\mathbb{C}}$; hence, $p_{\mathbb{C}} \subset U_{\mathbb{C}}$ as in the proof of Lemma 3(ii). Also, $\mathrm{vol}(U_{\mathbb{C}}) \in \bigwedge^r (p^\perp)_{\mathbb{C}}$; hence, $U_{\mathbb{C}} \subset (p^\perp)_{\mathbb{C}}$ since a nonzero decomposable vector lying in $\bigwedge^r Y$ has its span in Y. By Lemma 2(iii), $p \subset U \subset p^\perp$. The converse is contained in (i).

(iii) Since g preserves the form, $g(p^\perp) = (gp)^\perp$; hence, $gH_p^k = H_{gp}^k$, and Lemma 3(iv) gives $\Phi_k(g)\langle H_p^k\rangle = \langle H_{gp}^k\rangle$.

For continuity, use the notation of the proof of Proposition 2. A null line has the form $[f_- + w]$ with $w \in P$ and $\langle w, w\rangle = 1$. Since $\mathrm{U}(P) \subset K$ is transitive on the unit sphere of P and $[e_1] = [f_- + f_+]$, the continuous map $K \to \partial\mathbf{H}^n_{\mathbb{H}}$, $h \mapsto h[e_1]$, is surjective; as K is compact, it is a quotient map. By (i) and the equivariance, $p \mapsto \langle H_p^k\rangle$ is induced by the continuous map $h \mapsto \Phi_k(h)W_2$, hence, continuous. □

Lemma 9 *Let $p \neq q$ be points of $\partial\mathbf{H}^n_{\mathbb{H}}$.*

(i) $\langle H_p^k\rangle$ *is totally isotropic for* h_k.
(ii) $\langle H_p^k\rangle^\perp = p_{\mathbb{C}} \wedge \bigwedge^{2k+1} V_{\mathbb{C}} + \bigwedge^{2k+1}(p^\perp)_{\mathbb{C}} \wedge V_{\mathbb{C}}$.
(iii) h_k *pairs* $\langle H_p^k\rangle$ *and* $\langle H_q^k\rangle$ *nondegenerately. Hence,* $\langle H_p^k\rangle \cap \langle H_q^k\rangle = 0$, $\langle H_p^k\rangle \cap \langle H_q^k\rangle^\perp = 0$ *and* $\langle H_p^k\rangle^\perp \neq \langle H_q^k\rangle^\perp$.
(iv) For $U \in \mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$, $\mathcal{P}_k(U) \in \mathbb{P}\big(\langle H_q^k\rangle^\perp\big)$ *if and only if* $q \subset U$ *or* $U \subset q^\perp$.

Proof. By Lemma 1(iii), there is $g \in \mathrm{Sp}(n,1)$ with $g[e_1] = p$ and $g[e_{n+1}] = q$. By Lemma 8(i),(iii), $\langle H_p^k\rangle = \Phi_k(g)W_2$ and $\langle H_q^k\rangle = \Phi_k(g)W_{-2}$. Since $\Phi_k(g) \in \mathrm{U}(h_k)$ (Lemma 4) and all statements are $\mathrm{Sp}(n,1)$-equivariant, we may assume $p = [e_1]$ and $q = [e_{n+1}]$. Then (i) and (iii) are Lemma 7(iii).

(ii) By Lemma 7(iii), $W_2^\perp$ is spanned by the ε_I with $(a(I), c(I)) \neq (0,2)$, i.e., with $a(I) \geq 1$ or $c(I) \leq 1$. These span $E_1 \wedge \bigwedge^{r-1} V_{\mathbb{C}}$ and $\bigwedge^{r-1}(E_1 \oplus E_0) \wedge V_{\mathbb{C}}$, respectively.

(iv) By Lemma 7(iii), $\langle H_q^k\rangle^\perp = W_{-2}^\perp = \bigoplus_{i\neq 2} W_i$. Therefore, we must show that the W_2-component of $\mathrm{vol}(U_{\mathbb{C}})$ vanishes if and only if $q \subset U$ or $U \subset q^\perp$. Note that $q_{\mathbb{C}} = E_{-1}$ and $(q^\perp)_{\mathbb{C}} = E_0 \oplus E_{-1}$.

If $U \subset q^\perp$, no factor of $\mathrm{vol}(U_{\mathbb{C}})$ has an E_1-component. If $q \subset U$, then $\mathrm{vol}(U_{\mathbb{C}})$ is divisible by $\mathrm{vol}(E_{-1})$. In both cases the W_2-component is 0.

Otherwise $U \cap q^\perp$ has dimension k and meets q trivially. Choose an $\mathbb{H}$-basis $u_1, \ldots, u_k$ of $U \cap q^\perp$ and $u_{k+1} \in U \setminus q^\perp$. Let π_1 and π_0 be the

projections of $V_{\mathbb{C}}$ onto E_1 and E_0 along the other two summands. Only the factors $\chi(u_{k+1})$ and $\chi(u_{k+1}j)$ have nonzero E_1-components. Hence, the W_2-component of $\mathrm{vol}(U_{\mathbb{C}})$ is

$$\pm\,\pi_1\chi(u_{k+1}) \wedge \pi_1\chi(u_{k+1}j) \wedge \pi_0\chi(u_1) \wedge \cdots \wedge \pi_0\chi(u_k j).$$

The first two factors are independent, because $(u_{k+1}\mathbb{H})_{\mathbb{C}} \cap (q^\perp)_{\mathbb{C}} = 0$ and $\ker \pi_1 = (q^\perp)_{\mathbb{C}}$. The others are independent, because π_0 is injective on $(U \cap q^\perp)_{\mathbb{C}}$: its kernel in $(q^\perp)_{\mathbb{C}}$ is $q_{\mathbb{C}}$. Therefore, the component is nonzero. □

Part (iii) relies on the condition $U \subset p^\perp$ in the definition of H^k_p.

3.3 All limit maps

Proposition 4 *Let $\Gamma \subset \mathrm{PSp}(n,1)$ be discrete and let (γ_m) be a sequence of distinct elements of Γ. After passing to a subsequence:*

- *(i) $\gamma_m = k_m a_{t_m} l_m$ with $k_m, l_m \in K$, $t_m = \lambda(\gamma_m) \to \infty$, $k_m \to k$ and $l_m \to l$;*
- *(ii) $e^{-2t_m}\Phi_k(\gamma_m) \to T := \Phi_k(k)\, P_{\max}\, \Phi_k(l) \neq 0$; hence, $\Phi_k(\gamma_m) \to \rho = [T]$ in $\mathbb{QP}(N_k, \mathbb{C})$;*
- *(iii) $\mathfrak{Im}(\rho) = \mathbb{P}\langle H^k_{p^+}\rangle$ and $\mathrm{Ker}(\rho) = \mathbb{P}\big(\langle H^k_{p^-}\rangle^\perp\big)$, where $p^+ = k[e_1] = \lim \gamma_m(x)$ and $p^- = l^{-1}[e_{n+1}] = \lim \gamma_m^{-1}(x)$ for every $x \in \mathbf{H}^n_{\mathbb{H}}$; both points lie in $\Lambda(\Gamma)$.*

Every element of $\mathrm{Lim}(\Phi_k(\Gamma))$ arises in this way. Conversely, every $p \in \Lambda(\Gamma)$ is the point p^+ of some element of $\mathrm{Lim}(\Phi_k(\Gamma))$, and the point p^- of another. In particular, Theorem 1(a) holds.

Proof. A discrete subgroup is closed; therefore, it meets every compact set in a finite set. Hence, (γ_m) leaves every compact set, and $\lambda(\gamma_m) \to \infty$ by Proposition 2(iii). Write $\gamma_m = k_m a_{t_m} l_m$ as in Proposition 2(ii) for lifts to $\mathrm{Sp}(n,1)$. Since K is compact, we may assume $k_m \to k$ and $l_m \to l$. This gives (i).

(ii) By Lemma 7(ii), $e^{-2t_m}\Phi_k(\gamma_m) = \Phi_k(k_m)\big(e^{-2t_m}\Phi_k(a_{t_m})\big)\Phi_k(l_m) \to T$, and $T \neq 0$ since $P_{\max} \neq 0$.

(iii) By Lemma 8(i),(iii), $\mathfrak{Im}T = \Phi_k(k)W_2 = \langle H^k_{k[e_1]}\rangle$. By Lemma 7(iii) and since $\Phi_k(l^{-1}) \in \mathrm{U}(h_k)$,

$$\begin{aligned}\mathrm{Ker}T = \Phi_k(l^{-1})\mathrm{Ker}P_{\max} &= \Phi_k(l^{-1})\big(W_{-2}^\perp\big)\\ &= \big(\Phi_k(l^{-1})W_{-2}\big)^\perp = \langle H^k_{l^{-1}[e_{n+1}]}\rangle^\perp.\end{aligned}$$

By Proposition 2(iv), $\gamma_m(o) = k_m a_{t_m}(o) \to k[e_1]$ and $\gamma_m^{-1}(o) = l_m^{-1} a_{-t_m}(o) \to l^{-1}[e_{n+1}]$. These limits lie in $\Lambda(\Gamma)$, and by Proposition 1(i) the same limits are obtained from every $x \in \mathbf{H}^n_{\mathbb{H}}$.

If $\rho = \lim \Phi_k(\gamma_m)$ with $\Phi_k(\gamma_m)$ distinct, then the γ_m are distinct, and the above applies to a subsequence with the same limit ρ.

Conversely, let $p \in \Lambda(\Gamma)$ and choose $\gamma_m \in \Gamma$ with $\gamma_m(o) \to p$. Since $p \notin \mathbf{H}^n_{\mathbb{H}}$, the points $\gamma_m(o)$ take infinitely many values; hence, we may assume the γ_m distinct. Applying the above to (γ_m) gives $p^+ = p$. Applying it to (γ_m^{-1}) gives $p^- = \lim \gamma_m(o) = p$. $\square$

The limit ρ depends on (k, l), not only on $p^{\pm}$; only its image and kernel are determined by p^+ and p^-, respectively. The image is a single point if and only if $\binom{2n-2}{2k} = 1$, i.e., $k \in \{0, n-1\}$. For $k = 0$, Lemma 9(iv) shows that $\mathcal{P}_0^{-1}(\mathrm{Ker}\rho) = \mathbb{P}\big((p^-)^\perp\big)$. Therefore, Proposition 4 contains the λ-lemma [2, Lemma 5.2] for the action on $\mathbb{P}^n_{\mathbb{H}}$. In the terminology of [3, Definition 3.1], a sequence as in (i), whose Cartan factors k_m, l_m converge and whose Cartan projections t_m tend to ∞, *tends simply to infinity*, and the first step of the proof shows that every sequence of distinct elements of Γ has such a subsequence. For every k, Proposition 4 is the analogue for $\Phi_k(\Gamma)$ of the λ-lemma [3, Proposition 3.6] for discrete subgroups of $\mathrm{PU}(1, n)$.

3.4 The k-Chen–Greenberg limit set

Proposition 5 *Let $\Gamma \subset \mathrm{PSp}(n, 1)$ be discrete.*

(i) $L_k(\Gamma) = \bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\langle H^k_p\rangle$. *This union and the union* $\bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\big(\langle H^k_p\rangle^\perp\big)$ *are closed.*

(ii) $\Lambda_k(\Gamma) = \bigcup_{p\in\Lambda(\Gamma)} H^k_p$. *In particular,* $\Lambda_0(\Gamma) = \Lambda(\Gamma)$, *and* $\Lambda_{n-1}(\Gamma) = \{p^\perp : p \in \Lambda(\Gamma)\}$ *is the set of tangent hyperplanes at limit points.*

(iii) If Γ is non-elementary, then for every $p \in \Lambda(\Gamma)$

$$L_k(\Gamma) = \overline{\bigcup_{\gamma\in\Gamma} \Phi_k(\gamma)\, \mathbb{P}\langle H^k_p\rangle} \qquad \text{and} \qquad \Lambda_k(\Gamma) = \overline{\Gamma \cdot H^k_p}.$$

In particular, Theorem 1(b) holds.

Proof. (i) By Proposition 4, the images of the elements of $\mathrm{Lim}(\Phi_k(\Gamma))$ are exactly the spaces $\mathbb{P}\langle H^k_p\rangle$ with $p \in \Lambda(\Gamma)$.

For closedness, let $y_m \in \mathbb{P}(Z_{p_m})$ with $p_m \in \Lambda(\Gamma)$ and $y_m \to y$, where $Z_p = \langle H^k_p\rangle$ or $Z_p = \langle H^k_p\rangle^\perp$. Passing to a subsequence, $p_m \to p \in \Lambda(\Gamma)$ since $\Lambda(\Gamma)$ is compact. By Lemma 8(iii), $Z_{p_m} \to Z_p$; taking h_k-orthogonal complements is continuous because h_k is nondegenerate. Hence, $y \in \mathbb{P}(Z_p)$. Therefore, the closure in Definition 1 is not needed.

(ii) By (i) and Lemma 8(ii), $\Lambda_k(\Gamma) = \mathcal{P}_k^{-1}(L_k(\Gamma)) = \bigcup_{p\in\Lambda(\Gamma)} H^k_p$. For $k = 0$ and $k = n-1$, use $H^0_p = \{p\}$ and $H^{n-1}_p = \{p^\perp\}$.

(iii) By Lemma 8(iii), $\Phi_k(\gamma)\mathbb{P}\langle H^k_p\rangle = \mathbb{P}\langle H^k_{\gamma p}\rangle$ and $\gamma H^k_p = H^k_{\gamma p}$. Therefore, both right-hand sides are contained in the (closed) left-hand sides.

Conversely, let $q \in \Lambda(\Gamma)$. By Proposition 1(iii) there are $\gamma_m \in \Gamma$ with $\gamma_m p \to q$. By Lemma 8(iii), $\langle H^k_{\gamma_m p}\rangle \to \langle H^k_q\rangle$; hence, $\mathbb{P}\langle H^k_q\rangle$ lies in the first closure. Choose $h_m \in K$ with $h_m \to 1$ and $h_m q = \gamma_m p$, using local sections of the orbit map $K \to \partial\mathbf{H}^n_{\mathbb{H}}$, $h \mapsto hq$. Then for $U \in H^k_q$, we have $h_m U \in H^k_{\gamma_m p}$ and $h_m U \to U$; therefore, H^k_q lies in the second closure. $\square$

Remark 3 *The action of* Γ *on* $L_k(\Gamma)$ *is in general not minimal. Indeed, for* $1 \le k \le n-2$ *and* $\Lambda(\Gamma) \neq \emptyset$*, the set* $\mathcal{P}_k(\Lambda_k(\Gamma)) = L_k(\Gamma) \cap \mathcal{P}_k(\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}))$ *is a nonempty, closed,* Γ*-invariant and proper subset of* $L_k(\Gamma)$*. It is proper because, by Lemma 8(ii), its intersection with* $\mathbb{P}\langle H^k_p\rangle$ *is* $\mathcal{P}_k(H^k_p)$*. This is the smooth image of a manifold of real dimension* $4k(n-1-k)$*, which is smaller than* $\dim_{\mathbb{R}} \mathbb{P}\langle H^k_p\rangle = 2\binom{2n-2}{2k} - 2$*.*

Remark 4 *In the complex case [20, Proposition 3.9],* $\langle H^k_p\rangle^\perp$ *is spanned by the Plücker points of the* k*-planes through* p *and of those contained in* $p^\perp$*. This fails here: these Plücker points lie in* $\mathrm{vol}(p_{\mathbb{C}}) \wedge \bigwedge^{2k} V_{\mathbb{C}} + \bigwedge^{2k+2}(p^\perp)_{\mathbb{C}}$*, which is a proper subspace of* $\langle H^k_p\rangle^\perp$ *by Lemma 9(ii) (for* $n = 3$*,* $k = 1$ *it has dimension* 24*, against* 64*). Only the intersection of* $\mathbb{P}(\langle H^k_p\rangle^\perp)$ *with* $\mathcal{P}_k(\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}))$ *is described by Lemma 9(iv).*

4 The equicontinuity region

In this section, we prove Theorem 1(c) and Corollary 1. Throughout, $\Gamma \subset \mathrm{PSp}(n,1)$ is discrete and $0 \le k \le n-1$. By Lemma 5, $\Phi_k(\Gamma)$ is a discrete subgroup of $\mathrm{PSL}(N_k, \mathbb{C})$. Put

$$\mathcal{K}_k(\Gamma) = \bigcup_{p \in \Lambda(\Gamma)} \mathbb{P}\big(\langle H^k_p\rangle^\perp\big) \subset \mathbb{P}(W_k).$$

This set is closed by Proposition 5(i). It is invariant under $\Phi_k(\Gamma)$ by Lemma 8(iii), because $\Lambda(\Gamma)$ is Γ-invariant by Proposition 1(i), and each $\Phi_k(\gamma)$ preserves h_k by Lemma 4. By Proposition 4, it is the union of the kernels $\mathrm{Ker}(\rho)$ over all $\rho \in \mathrm{Lim}(\Phi_k(\Gamma))$.

4.1 Proper discontinuity

Lemma 10 *Let* $G \subset \mathrm{PSL}(N, \mathbb{C})$ *be discrete, and let* $\Omega \subset \mathbb{P}^{N-1}$ *be an open* G*-invariant set that is disjoint from* $\mathrm{Ker}(\rho)$ *for every* $\rho \in \mathrm{Lim}(G)$*. Then* G *acts properly discontinuously on* Ω*: for every compact* $C \subset \Omega$*, the set* $\{g \in G : gC \cap C \neq \emptyset\}$ *is finite.*

For $\Omega = \mathrm{Eq}(G)$, this is part of [4, Corollary 2.6]; we include the short proof, which uses only Lemma 6.

Proof. Suppose there are distinct $g_m \in G$ and $c_m \in C$ with $g_m c_m \in C$. By Lemma 6(i) we may assume that $g_m \to \rho$ and $g_m^{-1} \to \rho'$, where $\rho = [T]$ and $\rho' = [T']$ lie in $\mathrm{Lim}(G)$. We may also assume $c_m \to c \in C$ and $g_m c_m \to x \in C$.

Since $c, x \in \Omega$ lie outside $\mathrm{Ker}(\rho) \cup \mathrm{Ker}(\rho')$, we can choose a compact neighbourhood B of x disjoint from $\mathrm{Ker}(\rho')$, and an open neighbourhood U of c with $\overline{U} \cap \mathrm{Ker}(\rho) = \emptyset$. By Lemma 6(ii), $g_m \to \rho$ uniformly on $\overline{U}$ and $g_m^{-1} \to \rho'$ uniformly on B. In particular, $x = \rho(c)$. Shrinking U, we may assume $\rho(\overline{U}) \subset \mathrm{int}\, B$.

For $z \in U$, we then have $g_m z \to \rho(z) \in \mathrm{int}\, B$, hence, $z = g_m^{-1}(g_m z) \to \rho'(\rho(z))$. Therefore, $\rho' \circ \rho$ is the identity on the nonempty open set U. But $\rho(U) \subset \mathbb{P}(\mathfrak{Im}T)$, and by Lemma 6(iii), T is singular. Hence, $\rho'(\rho(U))$ lies in the proper projective subspace $\mathbb{P}(T'\mathfrak{Im}T)$. This is a contradiction. □

4.2 The equicontinuity region of $\Phi_k(\Gamma)$

Proposition 6 $\mathrm{Eq}\big(\Phi_k(\Gamma)\big) = \mathbb{P}(W_k) \setminus \mathcal{K}_k(\Gamma)$. *In particular, Theorem 1(c) holds.*

Proof. *Step 1: points off $\mathcal{K}_k(\Gamma)$ are in* Eq. Let $y \notin \mathcal{K}_k(\Gamma)$. Since $\mathcal{K}_k(\Gamma)$ is closed, y has an open neighbourhood U whose closure $\overline{U}$ is compact and disjoint from $\mathcal{K}_k(\Gamma)$.

Let (γ_m) be a sequence in Γ. If some element occurs infinitely often, a constant subsequence converges uniformly. Otherwise, we may assume the γ_m distinct. Then the $\Phi_k(\gamma_m)$ are distinct by Lemma 5, and by Lemma 6(i) we may assume that $\Phi_k(\gamma_m) \to \rho$ with $\rho \in \mathrm{Lim}(\Phi_k(\Gamma))$. Since $\mathrm{Ker}(\rho) \subset \mathcal{K}_k(\Gamma)$, Lemma 6(ii) gives uniform convergence on $\overline{U}$. Therefore, $\{\Phi_k(\gamma)|_U\}$ is normal and $y \in \mathrm{Eq}(\Phi_k(\Gamma))$.

Step 2: points of $\mathcal{K}_k(\Gamma)$ are not in Eq. Let $y = [\tilde{y}] \in \mathbb{P}(\langle H_p^k \rangle^\perp)$ with $p \in \Lambda(\Gamma)$. By Proposition 4, there are distinct $\gamma_m \in \Gamma$ with $\gamma_m = k_m a_{t_m} l_m$, $t_m \to \infty$, $k_m \to k$, $l_m \to l$, and $l^{-1}[e_{n+1}] = p$. Hence, $y \in \mathrm{Ker}(\rho)$ for $\rho = \lim \Phi_k(\gamma_m)$.

Write $T_m = e^{-2t_m}\Phi_k(\gamma_m) = \Phi_k(k_m) D_m \Phi_k(l_m)$ with $D_m = e^{-2t_m}\Phi_k(a_{t_m})$. By Proposition 4(ii), $\rho = [T]$ with $T = \Phi_k(k) P_{\max} \Phi_k(l)$; hence, $\mathrm{Ker}T = \Phi_k(l)^{-1}\mathrm{Ker}P_{\max}$. Hence, $z := \Phi_k(l)\tilde{y}$ is a nonzero vector in $\mathrm{Ker}P_{\max} = \bigoplus_{j\le 1} W_j$ (Lemma 7(ii)). Let z_j be its W_j-component, and put $j_0 = \max\{j : z_j \neq 0\} \le 1$. Fix $0 \neq u \in W_2$, and for $s \in \mathbb{C}$ put

$$\tilde{y}_m(s) = \Phi_k(l_m)^{-1}\big(z + s\, e^{(j_0-2)t_m} u\big) \longrightarrow \Phi_k(l)^{-1} z = \tilde{y}.$$

By Lemma 7(i), D_m acts on W_j by $e^{(j-2)t_m}$. Hence,

$$T_m \tilde{y}_m(s) = e^{(j_0-2)t_m}\, \Phi_k(k_m)\big(su + z_{j_0} + O(e^{-t_m})\big),$$

and therefore,

$$\Phi_k(\gamma_m)[\tilde{y}_m(s)] \longrightarrow \big[\Phi_k(k)(su + z_{j_0})\big].$$

Since $u \in W_2$ and $z_{j_0} \in W_{j_0}$ with $j_0 \neq 2$ are linearly independent, different values of s give different limits.

Suppose $y \in \mathrm{Eq}(\Phi_k(\Gamma))$. Then a subsequence of $(\Phi_k(\gamma_m))$ converges uniformly on a compact neighbourhood of y to a continuous map f. Along it, $\Phi_k(\gamma_m)[\tilde{y}_m(s)] \to f(y)$ for every s since $[\tilde{y}_m(s)] \to y$. As the above limits do not depend on the subsequence, this contradicts their being distinct.

Step 3. Steps 1 and 2 give $\mathrm{Eq}(\Phi_k(\Gamma)) = \mathbb{P}(W_k) \setminus \mathcal{K}_k(\Gamma)$. □

The proof does not use that Γ is non-elementary, and it also covers the cases $k \in \{0, n-1\}$, in which the limit maps have rank one. For $\Phi_k(\Gamma)$ it gives a direct proof of Proposition 3 [4, Proposition 2.5], together with an explicit description of the union of the kernels. It is the quaternionic counterpart of [19, Theorem 4.14]. Note, however, that $\langle H_p^k \rangle^\perp$ is strictly larger than the span of the Plücker points of the k-planes through p or inside $p^\perp$ (Remark 4); therefore, the literal analogue of [19, Theorem 4.14] would be false here.

Corollary 2 *$\Phi_k(\Gamma)$ acts properly discontinuously on $\mathrm{Eq}(\Phi_k(\Gamma))$. Moreover, for every compact set $C \subset \mathrm{Eq}(\Phi_k(\Gamma))$, the cluster points of the orbit $\Phi_k(\Gamma)C$ lie in $L_k(\Gamma)$.*

This is the analogue of [6, Theorem 1] (see also [6, Corollary 5.2]).

Proof. This is a special case of [4, Corollary 2.6]. By Proposition 6, $\mathrm{Eq}(\Phi_k(\Gamma)) = \mathbb{P}(W_k) \setminus \mathcal{K}_k(\Gamma)$. This set is open and invariant, and it is disjoint from every $\mathrm{Ker}(\rho)$, $\rho \in \mathrm{Lim}(\Phi_k(\Gamma))$, because these kernels lie in $\mathcal{K}_k(\Gamma)$. It remains to apply Lemma 10.

For the second statement, let y be a cluster point of $\Phi_k(\Gamma)C$. Then $y = \lim \Phi_k(\gamma_m)c_m$ for some distinct $\gamma_m \in \Gamma$ and $c_m \in C$. By Lemma 6(i) and Proposition 4, we may assume that $\Phi_k(\gamma_m) \to \rho \in \mathrm{Lim}(\Phi_k(\Gamma))$ and $c_m \to c \in C$. Since $\mathrm{Ker}(\rho) \subset \mathcal{K}_k(\Gamma)$ is disjoint from the compact set C, Lemma 6(ii) gives $\Phi_k(\gamma_m)c_m \to \rho(c)$. Hence, $y = \rho(c) \in \mathfrak{Im}(\rho) = \mathbb{P}\langle H_{p^+}^k \rangle$ with $p^+ \in \Lambda(\Gamma)$, and $y \in L_k(\Gamma)$ by Proposition 5(i). □

4.3 The quaternionic Grassmannian

Corollary 1 is the quaternionic counterpart of [19, Theorem 4.15], where equality with the equicontinuity region on the Grassmannian is stated in the complex case.

Proof of Corollary 1. By Proposition 6, $\mathcal{P}_k(U) \notin \mathrm{Eq}(\Phi_k(\Gamma))$ if and only if $\mathcal{P}_k(U) \in \mathbb{P}(\langle H_p^k \rangle^\perp)$ for some $p \in \Lambda(\Gamma)$. By Lemma 9(iv), this happens if and only if $p \subset U$ or $U \subset p^\perp$ for some $p \in \Lambda(\Gamma)$. Therefore, $\mathcal{O}_k(\Gamma) = \mathcal{P}_k^{-1}\big(\mathrm{Eq}(\Phi_k(\Gamma))\big)$. It is open because $\mathcal{P}_k$ is continuous, and Γ-invariant by Lemma 3(iv).

Proper discontinuity. Let $C \subset \mathcal{O}_k(\Gamma)$ be compact. If $\gamma C \cap C \neq \emptyset$, then $\Phi_k(\gamma)\mathcal{P}_k(C) \cap \mathcal{P}_k(C) \neq \emptyset$. By Corollary 2 this happens for only finitely many $\Phi_k(\gamma)$, hence, by injectivity of Φ_k (Lemma 5), for only finitely many γ.

Equicontinuity. Let $U \in \mathcal{O}_k(\Gamma)$, and let B be an open neighbourhood of $\mathcal{P}_k(U)$ on which $\{\Phi_k(\gamma)|_B\}$ is normal. Then $\mathcal{P}_k^{-1}(B)$ is an open neighbourhood of U, and on it $\gamma = \mathcal{P}_k^{-1} \circ \Phi_k(\gamma) \circ \mathcal{P}_k$.

Given (γ_m), pass to a subsequence with $\Phi_k(\gamma_m) \to f$ uniformly on compact subsets of B. On $\mathcal{P}_k^{-1}(B)$ the maps $\Phi_k(\gamma_m) \circ \mathcal{P}_k$ take values in the closed set $\mathcal{P}_k(\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1}))$; hence, $f \circ \mathcal{P}_k$ does too. Since $\mathcal{P}_k^{-1}$ is uniformly continuous on this compact set (Lemma 3(iii)), $\gamma_m \to \mathcal{P}_k^{-1} \circ f \circ \mathcal{P}_k$ uniformly on compact subsets of $\mathcal{P}_k^{-1}(B)$. Hence, U lies in the equicontinuity region of Γ on $\mathrm{Gr}_{k+1}(\mathbb{H}^{n+1})$. $\square$

Remark 5 *(i) For $k = 0$, $H_p^0 = \{p\}$ and $p \in p^\perp$; hence, $\mathcal{O}_0(\Gamma) = \mathbb{P}^n_{\mathbb{H}} \setminus \bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}(p^\perp)$. For non-elementary Γ, this is exactly the equicontinuity region of Γ on $\mathbb{P}^n_{\mathbb{H}}$, and the largest open set on which Γ acts properly discontinuously by [2, Theorem 1.2]. Therefore, Corollary 1 is sharp for $k = 0$. Moreover, $x \mapsto x^\perp$ maps $\mathcal{O}_0(\Gamma)$ onto $\mathcal{O}_{n-1}(\Gamma)$ for every Γ: for a line x and $U = x^\perp$, we have $p \subset U$ if and only if $x \subset p^\perp$, and $U \subset p^\perp$ if and only if $p = x$.*

(ii) Suppose $\Lambda(\Gamma) = \partial\mathbf{H}^n_{\mathbb{H}}$, e.g., for a lattice. Since $U \subset p^\perp$ if and only if $p \subset U^\perp$, $U \in \mathcal{O}_k(\Gamma)$ if and only if neither U nor $U^\perp$ contains a null line. A subspace S contains no null line if and only if the form is definite on S: if $u, v \in S$ with $\langle u, u\rangle > 0 > \langle v, v\rangle$, then $t \mapsto \langle u + vt, u + vt\rangle$ changes sign on $\mathbb{R}$. If U is negative definite, then $\dim_{\mathbb{H}} U = 1$ by Step 0 of the proof of Lemma 1. If U is positive definite, then $U^\perp$ has signature $(n - k - 1, 1)$; hence, it is definite only if $k = n - 1$. Hence, for $n \geq 2$, we get $\mathcal{O}_k(\Gamma) = \emptyset$ for $1 \leq k \leq n - 2$ and

$$\mathcal{O}_0(\Gamma) = \mathbf{H}^n_{\mathbb{H}}, \qquad \mathcal{O}_{n-1}(\Gamma) = \{x^\perp : x \in \mathbf{H}^n_{\mathbb{H}}\},$$

while for $n = 1$, we get $\mathcal{O}_0(\Gamma) = \mathbb{P}^1_{\mathbb{H}} \setminus \partial\mathbf{H}^1_{\mathbb{H}}$.

5 The Kulkarni limit set

In this section, we prove Theorem 1(d), give a lower bound for the Kulkarni limit set, and complete the proof of Theorem 1. As in Section 4, $\Gamma \subset \mathrm{PSp}(n, 1)$ is discrete, $0 \leq k \leq n - 1$, $G = \Phi_k(\Gamma)$ acts on $X = \mathbb{P}(W_k)$, and

$$\mathcal{K}_k(\Gamma) = \bigcup_{p\in\Lambda(\Gamma)} \mathbb{P}\big(\langle H_p^k\rangle^\perp\big), \qquad \text{so that} \qquad \mathrm{Eq}(G) = X \setminus \mathcal{K}_k(\Gamma)$$

by Proposition 6. $L_0(G)$, $L_1(G)$, $L_2(G)$ are the sets of Definition 6.

5.1 An upper bound

Proposition 7 $\Lambda_{\mathrm{Kul}}(G) \subseteq \mathcal{K}_k(\Gamma)$*; equivalently,* $\mathrm{Eq}(G) \subseteq \Omega_{\mathrm{Kul}}(G)$*. In particular, Theorem 1(d) holds.*

Proposition 7 is the quaternionic counterpart of one inclusion of [20, Theorem 1.1]. The step for L_2 below is a variant of the criterion [8, Lemma 2.9]; see also [4, Proposition 1.4(3)].

Proof. Put $\Omega = \mathrm{Eq}(G)$. It is open and G-invariant, and G acts properly discontinuously on it by Corollary 2. Since $\mathcal{K}_k(\Gamma) = X \setminus \Omega$ is closed, it is enough to show that the points with infinite isotropy and the cluster points in Definition 6(b),(c) lie in $\mathcal{K}_k(\Gamma)$.

If $x \in \Omega$, proper discontinuity applied to $C = \{x\}$ shows that the isotropy group of x is finite. Hence, $L_0(G) \subseteq \mathcal{K}_k(\Gamma)$.

Let y be a cluster point of an orbit $G\cdot x$. Then $g_m x \to y$ for some distinct $g_m \in G$. If $x \notin \Omega$, then $g_m x \notin \Omega$ for all m; hence, $y \notin \Omega$. If $x \in \Omega$ and $y \in \Omega$, then $C = \{x, y\} \cup \{g_m x : m \geq 1\}$ is a compact subset of Ω with $g_m x \in g_m C \cap C$ for all m, contradicting proper discontinuity. Therefore, $y \in \mathcal{K}_k(\Gamma)$ in all cases, and $L_1(G) \subseteq \mathcal{K}_k(\Gamma)$.

Let $C \subset X \setminus (L_0(G) \cup L_1(G))$ be compact, and let y be a cluster point of $\{g(C)\}_{g\in G}$. Then there are distinct $g_m \in G$ and $c_m \in C$ with $g_m c_m \to y$. Suppose $y \in \Omega$. By Lemma 6(i) we may assume that $g_m^{-1} \to \rho' \in \mathrm{Lim}(G)$ and $c_m \to c \in C$. Since $\mathrm{Ker}(\rho') \subset \mathcal{K}_k(\Gamma)$ by Proposition 4, y has a compact neighbourhood B disjoint from $\mathrm{Ker}(\rho')$. By Lemma 6(ii), $g_m^{-1} \to \rho'$ uniformly on B. As $g_m c_m \in B$ for large m, we get $c = \lim g_m^{-1}(g_m c_m) = \rho'(y) = \lim g_m^{-1} y$. Now $y \notin L_0(G)$, because $L_0(G) \subseteq \mathcal{K}_k(\Gamma)$, and the g_m^{-1} are distinct. Therefore, c is a cluster point of the orbit $G \cdot y$, i.e., $c \in L_1(G)$. This contradicts $c \in C$. Hence, $y \in \mathcal{K}_k(\Gamma)$, and $L_2(G) \subseteq \mathcal{K}_k(\Gamma)$. □

Remark 6 *The proof only uses that G is discrete and that* $\mathrm{Eq}(G)$ *is open, G-invariant and disjoint from* $\mathrm{Ker}(\rho)$ *for every $\rho \in \mathrm{Lim}(G)$ (via Lemma 10). The last property holds for every discrete $G \subset \mathrm{PSL}(N, \mathbb{C})$ by Proposition 3; therefore, the inclusion* $\mathrm{Eq}(G) \subseteq \Omega_{\mathrm{Kul}}(G)$ *holds for every discrete G; compare [4, Corollary 2.6], where the Kulkarni sets are formulated slightly differently (with the closure of the cluster points of all orbits in place of $L_0 \cup L_1$). By part (c), it gives the explicit upper bound $\Lambda_{\mathrm{Kul}}(\Phi_k(\Gamma)) \subseteq \mathcal{K}_k(\Gamma)$; the lower bound in Theorem 1(d) is Proposition 8 below.*

5.2 A lower bound

Let $\Lambda_\infty(\Gamma)$ be the set of points of $\Lambda(\Gamma)$ that are fixed by some element of Γ of infinite order, and let $\overline{\mathbf{H}^n_{\mathbb{H}}}$ be the closure of $\mathbf{H}^n_{\mathbb{H}}$ in $\mathbb{P}^n_{\mathbb{H}}$.

Lemma 11 *(i) If $g \in \Gamma$ has infinite order, then $\Lambda(\langle g\rangle)$ is a nonempty subset of $\Lambda_\infty(\Gamma)$.*

(ii) If Γ is infinite cyclic, then $\Lambda_\infty(\Gamma) = \Lambda(\Gamma)$.

(iii) If Γ is non-elementary, then Γ contains an element of infinite order, and $\Lambda_\infty(\Gamma)$ is dense in $\Lambda(\Gamma)$.

Proof. (i) The group $\langle g\rangle$ is infinite and discrete; hence, $\Lambda(\langle g\rangle) \neq \emptyset$ by Proposition 1(ii), and clearly, $\Lambda(\langle g\rangle) \subseteq \Lambda(\Gamma)$. If $q = \lim g^{m_i}(o)$, then $g(q) = \lim g^{m_i}(g(o)) = q$ by Proposition 1(i).

(ii) This follows from (i).

(iii) Γ is infinite, since finite groups have finite orbits. Choose distinct $\gamma_m \in \Gamma$ and apply Proposition 4 with $k = 0$. After passing to a subsequence, we get points $p^\pm \in \Lambda(\Gamma)$ with $\Phi_0(\gamma_m) \to \rho$, $\mathfrak{Im}(\rho) = \{\mathcal{P}_0(p^+)\}$ (as $H^0_p = \{p\}$) and $\mathrm{Ker}(\rho) = \mathbb{P}(\langle H^0_{p^-}\rangle^\perp)$. If $p^+ = p^-$, choose $h \in \Gamma$ with $hp^+ \neq p^+$; it exists because Γ has no finite orbit. Replacing γ_m by $h\gamma_m$ replaces ρ by $\Phi_0(h) \circ \rho$, which has image $\{\mathcal{P}_0(hp^+)\}$ (Lemma 3(iv)) and the same kernel. Therefore, we may assume $p^+ \neq p^-$.

Let B be a closed neighbourhood of p^+ in $\overline{\mathbf{H}^n_{\mathbb{H}}}$ not containing p^-. Then $B \cap \mathbb{P}((p^-)^\perp) = \emptyset$ by Lemma 1(i), and hence, $\mathcal{P}_0(B) \cap \mathrm{Ker}(\rho) = \emptyset$ by Lemma 9(iv). By Lemma 6(ii) and Lemma 3(iii),(iv), $\gamma_m \to p^+$ uniformly on B. Therefore, $g := \gamma_m$ satisfies $g(B) \subseteq \operatorname{int} B$ for some m, where the interior is taken in $\overline{\mathbf{H}^n_{\mathbb{H}}}$. Then $g^j(B) \subseteq g(B) \subseteq \operatorname{int} B$ for all $j \geq 1$. Since $\overline{\mathbf{H}^n_{\mathbb{H}}}$ is connected and $\emptyset \neq B \neq \overline{\mathbf{H}^n_{\mathbb{H}}}$ is closed, B is not open; hence, $\operatorname{int} B \neq B$. Hence, $g^j \neq 1$ for all $j \geq 1$.

Let $p_0 \in \Lambda(\langle g\rangle)$. For $\gamma \in \Gamma$, the point γp_0 is fixed by $\gamma g \gamma^{-1}$, which has infinite order; therefore, $\Gamma \cdot p_0 \subseteq \Lambda_\infty(\Gamma)$. By Proposition 1(iii), $\overline{\Gamma \cdot p_0} = \Lambda(\Gamma)$. □

Proposition 8 *$\mathbb{P}\langle H^k_p\rangle \subseteq L_0(G) \cup L_1(G)$ for every $p \in \overline{\Lambda_\infty(\Gamma)}$. Consequently, if Γ is non-elementary or infinite cyclic, then*

$$L_k(\Gamma) \subseteq L_0(G) \cup L_1(G) \subseteq \Lambda_{\mathrm{Kul}}(G) \subseteq \mathcal{K}_k(\Gamma).$$

Proof. *Step 1: $p \in \Lambda_\infty(\Gamma)$.* Let $g \in \Gamma$ have infinite order with $gp = p$; lift it to $\mathrm{Sp}(n,1)$. Then g preserves p and $p^\perp$; hence, $g_{\mathbb{C}}$ preserves $p_{\mathbb{C}}$ and $(p^\perp)_{\mathbb{C}}$ by Lemma 2(iii). For every $w \in V$, the number $\langle w, w\rangle$ is real; therefore, Lemma 2(ii) and $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ give $h_{\mathbb{C}}(\chi w, \chi w) = \langle w, w\rangle$. By Lemma 1(i), $h_{\mathbb{C}}$ is therefore positive semidefinite on $(p^\perp)_{\mathbb{C}}$ with null vectors exactly $p_{\mathbb{C}}$; by the Cauchy–Schwarz inequality these form its radical. Therefore, $h_{\mathbb{C}}$ induces a positive definite Hermitian form on $Q = (p^\perp)_{\mathbb{C}}/p_{\mathbb{C}}$, and the map $\bar{g}$ induced by $g_{\mathbb{C}}$ on Q is unitary for this form. Hence, $\bigwedge^{2k} \bar{g}$ is also unitary on $\bigwedge^{2k} Q$.

By Lemma 8(i), $\omega \mapsto \mathrm{vol}(p_{\mathbb{C}}) \wedge \omega$ maps $\bigwedge^{2k}(p^\perp)_{\mathbb{C}}$ onto $\langle H^k_p\rangle$. It vanishes on $p_{\mathbb{C}} \wedge \bigwedge^{2k-1}(p^\perp)_{\mathbb{C}}$; hence, it factors through $\bigwedge^{2k} Q$. Both spaces have

dimension $\binom{2n-2}{2k}$; therefore, we get an isomorphism $\psi\colon \bigwedge^{2k} Q \to \langle H_p^k\rangle$ (for $k=0$, $\psi(1)=\mathrm{vol}(p_{\mathbb{C}})$). Since $\bigwedge^2 g_{\mathbb{C}}\mathrm{vol}(p_{\mathbb{C}}) = \det(g_{\mathbb{C}}|_{p_{\mathbb{C}}})\mathrm{vol}(p_{\mathbb{C}})$,

$$\Phi_k(g)\circ\psi = \det(g_{\mathbb{C}}|_{p_{\mathbb{C}}})\,\psi\circ\bigwedge^{2k}\bar{g}.$$

The powers of the unitary map $\bigwedge^{2k}\bar{g}$ lie in a compact group. Therefore, there are $m_i\to\infty$ with $(\bigwedge^{2k}\bar{g})^{m_i}$ convergent. Passing to a subsequence, the differences $n_i = m_{i+1}-m_i$ are strictly increasing, and $(\bigwedge^{2k}\bar{g})^{n_i}\to I$. Hence, $\Phi_k(g)^{n_i}y \to y$ for every $y\in\mathbb{P}\langle H_p^k\rangle$. The elements $\Phi_k(g^{n_i})$ are distinct, because g has infinite order and Φ_k is injective (Lemma 5). Thus, y is a cluster point of $G\cdot y$. Therefore, either $y\in L_0(G)$, or $y\in L_1(G)$ by Definition 6(b).

Step 2: $p\in\overline{\Lambda_\infty(\Gamma)}$. Let $p_m\in\Lambda_\infty(\Gamma)$ with $p_m\to p$. By Lemma 8(iii), $\langle H_{p_m}^k\rangle\to\langle H_p^k\rangle$; hence, every point of $\mathbb{P}\langle H_p^k\rangle$ is a limit of points of $\mathbb{P}\langle H_{p_m}^k\rangle$. These lie in the closed set $L_0(G)\cup L_1(G)$ by Step 1.

Step 3. If Γ is non-elementary or infinite cyclic, then $\overline{\Lambda_\infty(\Gamma)} = \Lambda(\Gamma)$ by Lemma 11. Therefore, $L_k(\Gamma) = \bigcup_{p\in\Lambda(\Gamma)}\mathbb{P}\langle H_p^k\rangle \subseteq L_0(G)\cup L_1(G)$ by Proposition 5(i) and Steps 1–2. The last inclusion is Proposition 7. □

Remark 7 *If Γ is non-elementary or infinite cyclic, then Propositions 7 and 8 place $\Lambda_{\mathrm{Kul}}(\Phi_k(\Gamma))$ between the explicit sets $L_k(\Gamma)$ and $\mathcal{K}_k(\Gamma)$. Equality in Theorem 1(d) means $\Lambda_{\mathrm{Kul}}(\Phi_k(\Gamma))=\mathcal{K}_k(\Gamma)$. Since G acts properly discontinuously on $\Omega_{\mathrm{Kul}}(G)$ [14], this would follow if $\mathrm{Eq}(G)$ were the largest open subset of $\mathbb{P}(W_k)$ on which G acts properly discontinuously. For the action of a non-elementary Γ on $\mathbb{P}^n_{\mathbb{H}}$, the corresponding maximality is [2, Theorem 1.2] (see Remark 5(i)). We do not know whether it holds for $\Phi_k(\Gamma)$ on $\mathbb{P}(W_k)$. Some hypothesis on Γ is needed here: for the cyclic group generated by a loxodromic element of $\mathrm{PU}(1,n)$ acting on $\mathbb{P}^n_{\mathbb{C}}$ one has $\mathrm{Eq}=\Omega_{\mathrm{Kul}}$, but this set is not the largest open set on which the group acts properly discontinuously; there is no largest such set, and the maximal ones are described in [3, §2].*

For discrete subgroups of $\mathrm{PU}(1,n)$ acting on $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$, equality does hold [20, Theorem 1.1]. The proof there uses sequences whose Cartan maps are the identity [3]; the quaternionic analogue of this ingredient is [2, Theorem 5.6]. The argument does not transfer directly. Since $(a_t)_{\mathbb{C}}$ has the eigenvalue e^t with multiplicity two, $\Phi_k(a_t)$ has five weights (Lemma 7) instead of three. Let $\gamma_m = k_m a_{t_m} l_m$ with $p^+=p^-=p$, let $w_0=[\tilde{w}_0]\in\mathbb{P}(\langle H_p^k\rangle^\perp)$, and let j_0 be the top weight of $z=\Phi_k(l)\tilde{w}_0$, with W_{j_0}-component z_{j_0}. The computation in Step 2 of the proof of Proposition 6 shows that the accumulation points of $\Phi_k(\gamma_m)x_m$, $x_m\to w_0$, lie in $\mathbb{P}\big(\Phi_k(k)(\bigoplus_{j>j_0}W_j\oplus\mathbb{C}z_{j_0})\big)$. Since $\Phi_k(lk)$ maps $\bigoplus_{j\geq i}W_j$ onto $\bigoplus_{j\leq -i}W_j$, this set does not contain w_0 when $j_0=1$, which is the generic case.

5.3 Proof of Theorem 1

By Lemma 5, $\Phi_k(\Gamma)$ is a discrete subgroup of $\mathrm{PU}(h_k)$. Part (a) is Proposition 4, with the dimension of $\langle H_p^k\rangle$ given by Lemma 8(i). There the points $p^{\pm}$ are obtained along a subsequence; in fact, $\gamma_m(o) \to p^+$ and $\gamma_m^{-1}(o) \to p^-$ for the whole sequence. Indeed, every subsequence of (γ_m) has a further subsequence to which Proposition 4 applies, with the same limit ρ. Along it, $\gamma_m(o) \to q^+$ and $\gamma_m^{-1}(o) \to q^-$ with $\mathfrak{Im}(\rho) = \mathbb{P}\langle H_{q^+}^k\rangle$ and $\mathrm{Ker}(\rho) = \mathbb{P}(\langle H_{q^-}^k\rangle^{\perp})$. By Lemma 9(iii), the maps $p \mapsto \langle H_p^k\rangle$ and $p \mapsto \langle H_p^k\rangle^{\perp}$ are injective; therefore, $q^{\pm}$ do not depend on the subsequence. Part (b) is Proposition 5(i),(ii), part (c) is Proposition 6, and part (d) is Propositions 7 and 8. The unions $\bigcup_p \mathbb{P}\langle H_p^k\rangle$ and $\bigcup_p \mathbb{P}(\langle H_p^k\rangle^{\perp})$ are closed by Proposition 5(i), and $\Lambda_k(\Gamma) = \mathcal{P}_k^{-1}(L_k(\Gamma))$ is closed because $\mathcal{P}_k$ is continuous.

6 Comparison with the complex case

For a discrete $\Gamma \subset \mathrm{PU}(1,n)$ acting on $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$, the analogues of Theorem 1(a)–(c) and Corollary 1 were proved by Zúñiga [19, 20]. Our results follow the same pattern, but the passage through the complex realization changes the picture in three ways. First, the representation is $\bigwedge_{\mathbb{C}}^{2k+2}$ of $V_{\mathbb{C}} \cong \mathbb{C}^{2n+2}$, the form h_k has signature $\big(2\binom{2n}{2k+1}, \binom{2n}{2k+2} + \binom{2n}{2k}\big)$ (Lemma 4) instead of $\big(\binom{n}{k}, \binom{n}{k+1}\big)$, and the limit maps have rank $\binom{2n-2}{2k}$ instead of $\binom{n-1}{k}$; in both cases, they have rank one only for $k \in \{0, n-1\}$. Second, in the complex case, $\langle H_p^k\rangle^{\perp}$ is spanned by the Plücker points of the k-planes through p or inside $p^{\perp}$ [20, Proposition 3.9]; here this span is a proper subspace (Remark 4); hence, the complement of the equicontinuity region is strictly larger than the naive analogue suggests, although on the Grassmannian the two descriptions agree (Corollary 1). Third, $\Phi_k(a_t)$ has five weights instead of three (Lemma 7). In the complex case, the Kulkarni limit set equals the complement of the equicontinuity region [20, Theorem 1.1]; here we obtain the two-sided bound of Theorem 1(d), $L_k(\Gamma) \subseteq \Lambda_{\mathrm{Kul}}(\Phi_k(\Gamma)) \subseteq \mathcal{K}_k(\Gamma)$ for non-elementary or infinite cyclic Γ, but not equality, since the extra weights ± 1 break the argument of [20] (Remark 7). Whether equality holds in Theorem 1(d), and whether the equicontinuity region is the largest open set on which $\Phi_k(\Gamma)$ acts properly discontinuously, as it is for $k = 0$ on $\mathbb{P}^n_{\mathbb{H}}$ [2, Theorem 1.2], remain open for non-elementary Γ; for elementary Γ, such maximality can already fail in the complex case [3, §2].

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Vanlalruatkimi
Department of Mathematics and Computer Science,
Mizoram University,
Aizawl 796004, Mizoram, India.
aruatihmar39@gmail.com

Sandipan Dutta (corresponding author)
Department of Mathematics and Computer Science,
Mizoram University,
Aizawl 796004, Mizoram, India.
sandipandutta98@gmail.com, mzut330@mzu.edu.in

Jonathan Ramdikpuia
Department of Mathematics and Computer Science,
Mizoram University,
Aizawl 796004, Mizoram, India.
jonathanramdikpuia@gmail.com

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