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Non–linear functionals preserving normal distribution and their asymptotic normality

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Abstract. We introduce sufficiently wide classes of non–linear functionals preserving normal (Gaussian) distribution and establish various conditions under which a sequence of such functionals is asymptotically normal. As a consequence, we obtain a generalization and sharpening of known results on the central limit theorem for weighted sums (linear functionals) of independent random variables.

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Introduction

Limit theorems for sums of independent random variables form a complete theory which presents one of the main parts of the probability theory (see, for instance, [4, 11]). At the same time, the question of the validity of the central limit theorem (CLT) for wider classes of random processes is still very actual. There are two main directions for researches: the validity of the CLT for sums of dependent random variables, for example, processes with weakly dependent components ([2, 6, 10]), martingales ([5]), and the asymptotic normality of different classes of non–linear functionals on independent as well as dependent random variables.

The present paper is concerned with the second problem. Namely, we introduce sufficiently wide classes of non–linear functionals on random variables which have the following property: in the case when random variables are independent and standard normally (Gaussian) distributed, the distribution of the functional is standard Gaussian too, or, in our terminology, the functional preserves the normal distribution (Section 1). In the second section, we consider functionals, which preserve the normal distribution, on independent, but not necessarily Gaussian random variables, and obtain conditions under which a sequence of such functionals is asymptotically normal.

As a consequence, we also obtain a generalization and sharpening of known results on the CLT for weighted sums (linear functionals) of independent random variables (Section 3).

1 Functionals preserving the normal distribution

Denote by $f_n = f_n(x_1, x_2, ..., x_n)$ a real–valued functional defined in $\mathbb{R}^n$, $n \geq 2$. We say that the functional f_n *preserves* the normal distribution, if for random variables $\xi_1, \xi_2, ..., \xi_n$ which are independent and standard normally distributed, the random variable $f_n(\xi_1, \xi_2, ..., \xi_n)$ has the standard normal distribution $\mathcal{N}(0, 1)$ too.

Here are several simple examples of functionals preserving the normal distribution (see, for instance, [8] and [12]).

Example 1 $f_2(x_1, x_2) = \dfrac{x_1 + x_2}{\sqrt{2}}$.

Example 2 $f_2(x_1, x_2) = \dfrac{\sqrt{2} \cdot x_1 x_2}{\sqrt{x_1^2 + x_2^2}}$.

Example 3 $f_3(x_1, x_2, x_3) = \dfrac{x_1 x_2 + x_3}{\sqrt{1 + x_1^2}}$.

It is obvious that any superposition of functionals preserving the normal distribution is again a functional which preserves the normal distribution. This observation gives us an approach to the construction sequences f_n, $n \geq 2$, of such functionals. For instance, Example 1 gives rise to the following sequence of functionals preserving the normal distribution

$$f_n(x_1, x_2, ..., x_n) = f_2\left(x_1, f_{n-1}(x_2, ..., x_n)\right) =$$

$$= \frac{x_1}{\sqrt{2}} + \frac{x_2}{(\sqrt{2})^2} + \frac{x_3}{(\sqrt{2})^3} + ... + \frac{x_{n-2}}{(\sqrt{2})^{n-2}} + \frac{x_{n-1} + x_n}{(\sqrt{2})^{n-1}}, \quad n > 2.$$

If we consider the functional

$$\hat{f}_2(x_1, x_2) = \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}$$

similar to the one in the Example 2, we obtain a sequence

$$\hat{f}_n(x_1, x_2, ..., x_n) = \hat{f}_2(x_1, \hat{f}_{n-1}(x_2, ..., x_n)) = \frac{\prod_{j=1}^{n} x_j}{\sqrt{\sum_{j=1}^{n} \prod_{1 \leq i \leq n \,:\, i \neq j} x_i^2}}, \qquad n > 2,$$

of random variables normally distributed with parameters 0 and n^{-1}. Hence, functionals

$$f_n(x_1, x_2, ..., x_n) = \frac{\sqrt{n} \cdot \prod\limits_{j=1}^{n} x_j}{\sqrt{\sum\limits_{j=1}^{n} \prod\limits_{1 \le i \le n \,:\, i \ne j} x_i^2}}, \qquad n > 2,$$

preserve the normal distribution.

Starting from the functional from the third example, we obtain the following sequence of functionals preserving the normal distribution

$$f_n(x_1, x_2, ..., x_n) = f_3(x_1, x_2, f_{n-1}(x_1, x_3, x_4, ..., x_n)) =$$

$$= \frac{x_1 x_2}{\sqrt{1 + x_1^2}} + \frac{x_1 x_3}{1 + x_1^2} + \frac{x_1 x_4}{(1 + x_1^2)^{3/2}} + ... + \frac{x_1 x_{n-1}}{(1 + x_1^2)^{(n-2)/2}} + \frac{x_n}{(1 + x_1^2)^{(n-2)/2}},$$

$n > 3$.

Sequences of preserving the normal distribution functionals can be obtained in other ways. For instance, Example 1 is a particular case of linear functionals (weighted sums) with coefficients $\alpha_j^{(n)}$, $j = \overline{1, n}$, i.e.

$$f_n(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} \alpha_j^{(n)} x_j.$$

The classical case is obtained for $\alpha_j^{(n)} = n^{-1/2}$, $j = \overline{1, n}$.

For linear functionals the following statement is true.

Proposition 1 *The linear functional f_n with coefficients $\alpha_j^{(n)}$, $j = \overline{1, n}$, preserves the normal distribution if and only if*

$$\sum_{j=1}^{n} \left(\alpha_j^{(n)}\right)^2 = 1. \tag{1}$$

Proof. Let $\xi_1, \xi_2, ..., \xi_n$ be independent random variables with the standard normal distributions. Consider the characteristic function φ_n of the random variable $\sum\limits_{j=1}^{n} \alpha_j^{(n)} \xi_j$

$$\varphi_n(t) = E \exp\left\{ it \sum_{j=1}^{n} \alpha_j^{(n)} \xi_j \right\} = \prod_{j=1}^{n} E e^{it\alpha_j^{(n)} \xi_j}.$$

To prove the proposition we need to show that $\varphi_n(t) = e^{-t^2/2}$, $t \in \mathbb{R}$. Since $\alpha_j^{(n)}\xi_j \sim \mathscr{N}(0, \alpha_j^{(n)})$, we have

$$E \exp\left\{it\alpha_j^{(n)}\xi_j\right\} = \exp\left\{-\frac{1}{2}\left(\alpha_j^{(n)}t\right)^2\right\}.$$

Hence

$$\varphi_n(t) = \prod_{j=1}^{n} e^{-(\alpha_j^{(n)}t)^2/2} = \exp\left\{-\frac{t^2}{2}\sum_{j=1}^{n}\left(\alpha_j^{(n)}\right)^2\right\}.$$

From here it follows that $\varphi_n(t) = e^{-t^2/2}$ if and only if $\sum_{j=1}^{n}\left(\alpha_j^{(n)}\right)^2 = 1$. □

Next proposition shows how one can construct linear functionals preserving the normal distribution using a superposition.

Proposition 2 *Consider the functional*

$$f_2(x_1, x_2) = \alpha x_1 + \beta x_2, \qquad \alpha, \beta \in \mathbb{R}\backslash\{0, 1\},$$

and the superposition

$$f_n(x_1, x_2, ..., x_n) = f_2\left(x_1, f_{n-1}(x_2, x_3, ..., x_n)\right), \qquad n > 2. \tag{2}$$

1. For any $n \geq 2$, f_n is the linear functional with coefficients $\alpha_j^{(n)}$ of the form

$$\alpha_n^{(n)} = \beta^{n-1}, \qquad \alpha_j^{(n)} = \alpha\beta^{j-1}, \quad 1 \leq j < n-1. \tag{3}$$

Conversely, any linear functional f_n with coefficients given by (3) *can be represented as the superposition* (2) *where f_2 has coefficients α and β.*

2. The linear functional (2) *preserves the normal distribution if and only if the coefficients α and β of the functional f_2 are such that $\alpha^2 + \beta^2 = 1$.*

Proof. It is easy to see that

$$f_2\left(x_1, f_{n-1}(x_2, x_3, ..., x_n)\right) = \sum_{j=1}^{n}\alpha_j^{(n)}x_j,$$

where $\alpha_j^{(n)}$, $1 \leq j \leq n$, are some coefficients. Let us show that these coefficients have the form (3). Indeed, for $n = 3$ we have

$$f_3(x_1, x_2, x_3) = f_2(x_1, f_2(x_2, x_3)) = \alpha x_1 + \alpha\beta x_2 + \beta^2 x_3$$

Assume now that the statement is true for the functional f_{n-1}. Then

$$f_n(x_1, x_2, ..., x_n) = f_2\left(x_1, f_{n-1}(x_2, x_3, ..., x_n)\right) = \alpha x_1 + \beta \sum_{j=1}^{n-1} \alpha_j^{(n-1)} x_{j+1} =$$

$$= \alpha x_1 + \beta \left(\sum_{j=1}^{n-2} \alpha \beta^{j-1} x_{j+1} + \beta^{n-2} x_n \right) = \sum_{j=1}^{n-1} \alpha \beta^{j-1} x_j + \beta^{n-1} x_n,$$

and hence coefficients of f_n have the form (3). Conversely, for any linear functional f_n with coefficients given by (3) we can write

$$f_n(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} \alpha_j^{(n)} x_j = \sum_{j=1}^{n-1} \alpha \beta^{j-1} x_j + \beta^{n-1} x_n =$$

$$= \alpha x_1 + \beta \left(\sum_{j=1}^{n-2} \alpha \beta^{j-1} x_{j+1} + \beta^{n-2} x_n \right) = f_2\left(x_1, f_{n-1}(x_2, x_3, ..., x_n)\right).$$

To prove the second statement of the proposition note that for any $n \geq 2$ we have

$$\sum_{j=1}^{n} \left(\alpha_j^{(n)}\right)^2 = \sum_{j=1}^{n-1} \alpha^2 \left(\beta^{j-1}\right)^2 + \left(\beta^{n-1}\right)^2 = \alpha^2 \cdot \frac{1 - \left(\beta^2\right)^{n-1}}{1 - \beta^2} + \left(\beta^2\right)^{n-1}.$$

If $\alpha^2 + \beta^2 = 1$ then the right–hand side of the expression above equals to 1, and hence for f_n condition (1) holds. On the other hand, if linear functionals f_n with coefficients (3) preserve the normal distribution then condition (1) is fulfilled for $n = 2$ as well, and hence $\alpha^2 + \beta^2 = 1$. □

Functional f_3 from the Example 3 can be generalized in the following way

Example 4 $f_{n+1}(x_0, x_1, x_2, ..., x_n) = \dfrac{x_0 x_1 + x_2 + ... + x_n}{\sqrt{n - 1 + x_0^2}}, \ n \geq 2.$

We can write

$$f_{n+1}(x_0, x_1, x_2, ..., x_n) = \frac{x_0 x_1 + x_2 + ... + x_n}{\sqrt{n - 1 + x_0^2}} =$$

$$= \frac{x_0}{\sqrt{n - 1 + x_0^2}} \cdot x_1 + \sum_{j=2}^{n} \frac{1}{\sqrt{n - 1 + x_0^2}} \cdot x_j.$$

Note that for any $x_0 \in \mathbb{R}$

$$\left(\frac{x_0}{\sqrt{n - 1 + x_0^2}} \right)^2 + \sum_{j=2}^{n} \left(\frac{1}{\sqrt{n - 1 + x_0^2}} \right)^2 = \frac{x_0^2}{n - 1 + x_0^2} + \frac{n - 1}{n - 1 + x_0^2} = 1.$$

Below we prove a general result, from which, particularly, one can see that considered functionals preserve the normal distribution.

The main class of non–linear functionals considered in the paper is defined as follows. Let $\alpha_j^{(n)}(x)$, $j=\overline{1,n}$, $x\in\mathbb{R}$, be a set of non–zero functions, and put

$$f_{n+1}(x_0,x_1,x_2,...,x_n)=f_n^{x_0}(x_1,x_2,...,x_n)=\sum_{j=1}^{n}\alpha_j^{(n)}(x_0)x_j. \tag{4}$$

In the next theorem, a condition for such functional to preserve the normal distribution is introduced.

Theorem 1 *The functional* (4) *preserves the normal distribution if*

$$\sum_{j=1}^{n}\left(\alpha_j^{(n)}(z)\right)^2=1 \qquad \textit{for any } z\in\mathbb{R}. \tag{5}$$

Proof. Let random variables $\xi_0,\xi_1,\xi_2,...,\xi_n$ be independent and standard normally distributed. We will show that

$$E\left(\sum_{j=1}^{n}\alpha_j^{(n)}(\xi_0)\xi_j\right)^{2k-1}=0,\quad E\left(\sum_{j=1}^{n}\alpha_j^{(n)}(\xi_0)\xi_j\right)^{2k}=\frac{(2k)!}{2^k k!},\quad k=1,2,...$$

Since the normal distribution is uniquely determined by its moments, from here it will follow that $\sum\limits_{j=1}^{n}\alpha_j^{(n)}(\xi_0)\xi_j\sim\mathscr{N}(0,1)$.

First consider odd moments. Taking into account the independence of $\xi_0,\xi_1,\xi_2,...,\xi_n$, we can write

$$E\left(\sum_{j=1}^{n}\alpha_j^{(n)}(\xi_0)\xi_j\right)^{2k-1}=$$

$$=E\left(\sum_{\substack{0\le m_1,m_2,...,m_n\le 2k-1:\\ m_1+m_2+...+m_n=2k-1}}\frac{(2k-1)!}{m_1!m_2!\cdot...\cdot m_n!}\prod_{i=1}^{n}\left(\alpha_i^{(n)}(\xi_0)\right)^{m_i}\xi_i^{m_i}\right)=$$

$$=\sum_{\substack{0\le m_1,...,m_n\le 2k-1:\\ m_1+...+m_n=2k-1}}\frac{(2k-1)!}{m_1!m_2!\cdot...\cdot m_n!}\prod_{i=1}^{n}E\xi_i^{m_i}\cdot E\left(\prod_{i=1}^{n}\left(\alpha_i^{(n)}(\xi_0)\right)^{m_i}\right).$$

It remains to note that since $m_1+m_2+...+m_n=2k-1$, for any numbers $0\le m_1,...,m_n\le 2k-1$ there exists i, $1\le i\le n$, such that m_i is odd, and hence $E\xi_i^{m_i}=0$.

Now consider even moments. We have

$$E\left(\sum_{j=1}^{n}\alpha_j^{(n)}(\xi_0)\xi_j\right)^{2k} =$$

$$= \sum_{\substack{0\le m_1,m_2,...,m_n\le 2k:\\ m_1+m_2+...+m_n=2k}} \frac{(2k)!}{m_1!m_2!\cdot\ldots\cdot m_n!}\prod_{i=1}^{n}E\xi_i^{m_i}\cdot E\left(\prod_{i=1}^{n}\left(\alpha_i^{(n)}(\xi_0)\right)^{m_i}\right).$$

If there is at least one odd number among $m_1, m_2, ..., m_n$, then $\prod_{i=1}^{n} E\xi_i^{m_i} = 0$. Hence only those summands remain in the sum for which all numbers $m_1, m_2, ..., m_n$ are even. Therefore we can write

$$E\left(\sum_{j=1}^{n}\alpha_j^{(n)}(\xi_0)\xi_j\right)^{2k} =$$

$$= \sum_{\substack{0\le m_1,...,m_n\le k:\\ m_1+...+m_n=k}} \frac{(2k)!}{(2m_1)!\cdot\ldots\cdot(2m_n)!}\prod_{i=1}^{n}E\xi_i^{2m_i}\cdot E\left(\prod_{i=1}^{n}\left(\alpha_i^{(n)}(\xi_0)\right)^{2m_i}\right) =$$

$$= \sum_{\substack{0\le m_1,...,m_n\le k:\\ m_1+...+m_n=k}} \frac{(2k)!}{(2m_1)!\cdot\ldots\cdot(2m_n)!}\prod_{i=1}^{n}\frac{(2m_i)!}{2^{m_i}m_i!}\cdot E\left(\prod_{i=1}^{n}\left(\alpha_i^{(n)}(\xi_0)\right)^{2m_i}\right) =$$

$$= \frac{(2k)!}{2^k k!}E\left(\sum_{\substack{0\le m_1,...,m_n\le k:\\ m_1+...+m_n=k}} \frac{k!}{m_1!\cdot\ldots\cdot m_n!}\cdot\prod_{i=1}^{n}\left(\left(\alpha_i^{(n)}(\xi_0)\right)^2\right)^{m_i}\right) =$$

$$= \frac{(2k)!}{2^k k!}E\left(\sum_{i=1}^{n}\left(\alpha_i^{(n)}(\xi_0)\right)^2\right)^{k}.$$

It remains to note that due to the condition (5), $E\left(\sum_{i=1}^{n}\left(\alpha_i^{(n)}(\xi_0)\right)^2\right)^{k} = 1$ for any k. □

It is not difficult to see that in the proof of the theorem above we do not use any condition on the distribution of ξ_0. Hence the result of the previous theorem can be generalized in the following form.

Theorem 2 *Let $\zeta, \xi_1, \xi_2, ..., \xi_n$ be independent random variables, and let $\xi_j \sim \mathscr{N}(0,1)$, $j = \overline{1,n}$. The random variable*

$$f_n^{\zeta}(\xi_1, \xi_2, ..., \xi_n) = \sum_{i=1}^{n}\alpha_i^{(n)}(\zeta)\xi_i$$

has the standard normal distribution if condition (5) *is fulfilled.*

Example 4 permits the following generalization.

Example 5 *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function, and put*

$$\alpha_n^{(n)}(z) = \frac{g(z)}{\sqrt{n-1+g^2(z)}}, \qquad \alpha_j^{(n)}(z) = \frac{1}{\sqrt{n-1+g^2(z)}},\ 1 \le j \le n-1.$$

Then functionals

$$f_n^z(x_1, x_2, ..., x_n) = \sum_{j=1}^{n-1} \frac{x_j}{\sqrt{n-1+g^2(z)}} + \frac{x_n \cdot g(z)}{\sqrt{n-1+g^2(z)}}, \qquad n \ge 2,$$

preserve the normal distribution.

Indeed, for any fixed $n \ge 2$

$$\sum_{j=1}^{n} \left(\alpha_j^{(n)}(z)\right)^2 = \frac{n-1}{n-1+g^2(z)} + \frac{g^2(z)}{n-1+g^2(z)} = 1.$$

Hence the condition (5) is fulfilled for any $z \in \mathbb{R}$.

Further generalization of this example one can find in [9].

2 Asymptotic normality of sequences of functionals preserving the normal distribution

In this section we consider functionals f_n on independent random variables $\eta_1, \eta_2, ..., \eta_n$ which are not necessarily Gaussian. We say that a sequence of functionals $f_n(\eta_1, \eta_2, ..., \eta_n)$, $n \ge 2$, is asymptotically normal if their distributions converge to the standard normal one as $n \to \infty$.

Conditions under which a sequence of preserving the normal distribution functionals (4) is asymptotically normal are given in the following technical lemma.

Lemma 1 *Let functions $\alpha_j^{(n)}$, $j = \overline{1, n}$, satisfying condition* (5), *be such that for independent random variables $\zeta, \xi_1, ..., \xi_n$, where $\xi_j \sim \mathcal{N}(0,1)$, and any $\varepsilon > 0$,*

$$\sum_{j=1}^{n} E\left(\left(\alpha_j^{(n)}(\zeta)\xi_j\right)^2 I\left(\left|\alpha_j^{(n)}(\zeta)\xi_j\right| > \varepsilon\right)\right) \to 0 \qquad \text{as } n \to \infty.$$

Then for any independent random variables $\eta_1, ..., \eta_n$ which are independent of ζ and such that $E\eta_j = 0$, $D\eta_j = 1$, $j = \overline{1, n}$, and for any $\varepsilon > 0$

$$\sum_{j=1}^{n} E\left(\left(\alpha_j^{(n)}(\zeta)\eta_j\right)^2 I\left(\left|\alpha_j^{(n)}(\zeta)\eta_j\right| > \varepsilon\right)\right) \to 0 \qquad \text{as } n \to \infty,$$

the sequence of functionals

$$f_n^{\zeta}(\eta_1, \eta_2, ..., \eta_n) = \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j, \qquad n \geq 2,$$

is asymptotically normal.

Proof. We need to show that

$$\left|E \exp\left\{it \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j\right\} - e^{-t^2/2}\right| \to 0 \qquad \text{as } n \to \infty.$$

Let the standard normally distributed random variables $\xi_1, ..., \xi_n$ be independent of $\eta_1, ..., \eta_n$. Since the functionals under consideration f_n^{ζ} preserve the normal distribution we have

$$E \exp\left\{it \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\xi_j\right\} = e^{-t^2/2}, \qquad n \geq 2.$$

Hence we can write

$$\left|E \exp\left\{it \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j\right\} - e^{-t^2/2}\right| =$$

$$= \left|E \exp\left\{it \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j\right\} - E \exp\left\{it \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\xi_j\right\}\right|.$$

Denote

$$\zeta_1^{(n)} = f_n^{\zeta}(0, \xi_2, ..., \xi_n) = \sum_{j=2}^{n} \alpha_j^{(n)}(\zeta)\xi_j,$$

$$\zeta_n^{(n)} = f_n^{\zeta}(\eta_1, ..., \eta_{n-1}, 0) = \sum_{j=1}^{n-1} \alpha_j^{(n)}(\zeta)\eta_j,$$

and for any k, $1 < k < n$,

$$\zeta_k^{(n)} = f_n^{\zeta}(\eta_1, ..., \eta_{k-1}, 0, \xi_{k+1}, ..., \xi_n) = \sum_{j=1}^{k-1} \alpha_j^{(n)}(\zeta)\eta_j + \sum_{j=k+1}^{n} \alpha_j^{(n)}(\zeta)\xi_j.$$

Then

$$\left| E\exp\left\{ it\sum_{j=1}^{n}\alpha_j^{(n)}(\zeta)\eta_j \right\} - E\exp\left\{ it\sum_{j=1}^{n}\alpha_j^{(n)}(\zeta)\xi_j \right\} \right| =$$

$$= \left| \sum_{k=1}^{n} \left(E\exp\left\{ it(\zeta_k^{(n)} + \alpha_k^{(n)}(\zeta)\eta_k) \right\} - E\exp\left\{ it(\zeta_k^{(n)} + \alpha_k^{(n)}(\zeta)\xi_k) \right\} \right) \right| \leq$$

$$\leq \sum_{k=1}^{n} \left| Ee^{it\zeta_k^{(n)}} \cdot e^{it\alpha_k^{(n)}(\zeta)\eta_k} - Ee^{it\zeta_k^{(n)}} \cdot e^{it\alpha_k^{(n)}(\zeta)\xi_k} \right|.$$

Since for any k

$$Ee^{it\zeta_k^{(n)}}\alpha_k^{(n)}(\zeta)\eta_k = Ee^{it\zeta_k^{(n)}}\alpha_k^{(n)}(\zeta)\cdot E\eta_k = 0 =$$

$$= Ee^{it\zeta_k^{(n)}}\alpha_k^{(n)}(\zeta)\cdot E\xi_k = Ee^{it\zeta_k^{(n)}}\alpha_k^{(n)}(\zeta)\xi_k,$$

$$Ee^{it\zeta_k^{(n)}}\left(\alpha_k^{(n)}(\zeta)\eta_k\right)^2 = Ee^{it\zeta_k^{(n)}}\left(\alpha_k^{(n)}(\zeta)\right)^2\cdot E\eta_k^2 =$$

$$= Ee^{it\zeta_k^{(n)}}\left(\alpha_k^{(n)}(\zeta)\right)^2\cdot E\xi_k^2 = Ee^{it\zeta_k^{(n)}}\left(\alpha_k^{(n)}(\zeta)\xi_k\right)^2,$$

and $\left| e^{it\zeta_k^{(n)}} \right| \leq 1$, further we can write

$$\sum_{k=1}^{n} \left| Ee^{it\zeta_k^{(n)}} \cdot e^{it\alpha_k^{(n)}(\zeta)\eta_k} - Ee^{it\zeta_k^{(n)}} \cdot e^{it\alpha_k^{(n)}(\zeta)\xi_k} \right| =$$

$$= \sum_{k=1}^{n} \left| Ee^{it\zeta_k^{(n)}} \cdot \left(e^{it\alpha_k^{(n)}(\zeta)\eta_k} - 1 - it\alpha_k^{(n)}(\zeta)\eta_k - \frac{(it)^2}{2}(\alpha_k^{(n)}(\zeta)\eta_k)^2 \right) - \right.$$

$$\left. - Ee^{it\zeta_k^{(n)}} \cdot \left(e^{it\alpha_k^{(n)}(\zeta)\xi_k} - 1 - it\alpha_k^{(n)}(\zeta)\xi_k - \frac{(it)^2}{2}(\alpha_k^{(n)}(\zeta)\xi_k)^2 \right) \right| \leq$$

$$\leq \sum_{k=1}^{n} E\left| e^{it\alpha_k^{(n)}(\zeta)\eta_k} - 1 - it\alpha_k^{(n)}(\zeta)\eta_k - \frac{(it)^2}{2}(\alpha_k^{(n)}(\zeta)\eta_k)^2 \right| +$$

$$+ \sum_{k=1}^{n} E\left| e^{it\alpha_k^{(n)}(\zeta)\xi_k} - 1 - it\alpha_k^{(n)}(\zeta)\xi_k - \frac{(it)^2}{2}(\alpha_k^{(n)}(\zeta)\xi_k)^2 \right|.$$

Using the well-known estimation

$$\left|e^{it}-\sum_{k=0}^{N}\frac{(it)^k}{k!}\right|\le\min\left\{\frac{2\left|t\right|^N}{N!},\frac{\left|t\right|^{N+1}}{(N+1)!}\right\},\qquad N=0,1,2,... \tag{6}$$

for any $\varepsilon>0$ we obtain

$$\sum_{k=1}^{n}E\left|e^{it\alpha_k^{(n)}(\zeta)\eta_k}-1-it\alpha_k^{(n)}(\zeta)\eta_k-\frac{(it)^2}{2}(\alpha_k^{(n)}(\zeta)\eta_k)^2\right|\le$$

$$\le\frac{\left|t\right|^3}{3!}\sum_{k:\left|\alpha_k^{(n)}(\zeta)\eta_k\right|\le\varepsilon}E\left|\alpha_k^{(n)}(\zeta)\eta_k\right|^3+\frac{2t^2}{2!}\sum_{k:\left|\alpha_k^{(n)}(\zeta)\eta_k\right|>\varepsilon}E\left|\alpha_k^{(n)}(\zeta)\eta_k\right|^2\le$$

$$\le\frac{\left|t\right|^3}{6}\varepsilon\sum_{k=1}^{n}E\left(\alpha_k^{(n)}(\zeta)\right)^2E\eta_k^2+t^2\sum_{k=1}^{n}E\left(\left(\alpha_k^{(n)}(\zeta)\eta_k\right)^2I\left(\left|\alpha_k^{(n)}(\zeta)\eta_k\right|>\varepsilon\right)\right)=$$

$$=\frac{\left|t\right|^3}{6}\varepsilon+t^2\sum_{k=1}^{n}E\left(\left(\alpha_k^{(n)}(\zeta)\eta_k\right)^2I\left(\left|\alpha_k^{(n)}(\zeta)\eta_k\right|>\varepsilon\right)\right).$$

Similarly,

$$\sum_{k=1}^{n}E\left|e^{it\alpha_k^{(n)}(\zeta)\xi_k}-1-it\alpha_k^{(n)}(\zeta)\xi_k-\frac{(it)^2}{2}(\alpha_k^{(n)}(\zeta)\xi_k)^2\right|\le$$

$$\le\frac{\left|t\right|^3}{6}\varepsilon+t^2\sum_{k=1}^{n}E\left(\left(\alpha_k^{(n)}(\zeta)\xi_k\right)^2I\left(\left|\alpha_k^{(n)}(\zeta)\xi_k\right|>\varepsilon\right)\right).$$

Hence

$$\left|E\exp\left\{it\sum_{j=1}^{n}\alpha_j^{(n)}(\zeta)\eta_j\right\}-e^{-t^2/2}\right|\le\frac{\left|t\right|^3}{3}\varepsilon+$$

$$+t^2\sum_{k=1}^{n}E\left(\left(\alpha_k^{(n)}(\zeta)\eta_k\right)^2I\left(\left|\alpha_k^{(n)}(\zeta)\eta_k\right|>\varepsilon\right)\right)+$$

$$+t^2\sum_{k=1}^{n}E\left(\left(\alpha_k^{(n)}(\zeta)\xi_k\right)^2I\left(\left|\alpha_k^{(n)}(\zeta)\xi_k\right|>\varepsilon\right)\right),$$

which completes the proof. □

An application of the Lemma 1 brings to the following result.

Theorem 3 *Let functions* $\alpha_j^{(n)}$, $j=\overline{1,n}$, *satisfying condition* (5), *be such that*

$$\max_{1\le j\le n}\sup_{z\in\mathbb{R}}\left|\alpha_j^{(n)}(z)\right|\to 0 \qquad \text{as } n\to\infty.$$

Then for any independent random variables $\zeta,\eta_1,...,\eta_n$ *such that* $E\eta_j=0$, $D\eta_j=1$, $j=\overline{1,n}$, *and for some* $\delta>0$

$$\sup_{1\le j\le n} E\left|\eta_j\right|^{2+\delta}=C_\delta<\infty,$$

the sequence of functionals

$$f_n^{\zeta}(\eta_1,\eta_2,...,\eta_n)=\sum_{j=1}^{n}\alpha_j^{(n)}(\zeta)\eta_j, \qquad n\ge 2,$$

is asymptotically normal.

Proof. We need to check that conditions of the Lemma 1 are fulfilled. For any $\varepsilon>0$ we have

$$E\left|\alpha_j^{(n)}(\zeta)\eta_j\right|^{2+\delta}\ge E\left(\left|\alpha_j^{(n)}(\zeta)\eta_j\right|^{2+\delta} I\left(\left|\alpha_j^{(n)}(\zeta)\eta_j\right|>\varepsilon\right)\right)>$$

$$>\varepsilon^{\delta}E\left(\left(\alpha_j^{(n)}(\zeta)\eta_j\right)^2 I\left(\left|\alpha_j^{(n)}(\zeta)\eta_j\right|>\varepsilon\right)\right),$$

and hence

$$E\left(\left(\alpha_j^{(n)}(\zeta)\eta_j\right)^2 I\left(\left|\alpha_j^{(n)}(\zeta)\eta_j\right|>\varepsilon\right)\right)\le$$

$$\le\frac{1}{\varepsilon^{\delta}}E\left|\alpha_j^{(n)}(\zeta)\right|^{2+\delta}E\left|\eta_j\right|^{2+\delta}\le\frac{C_\delta}{\varepsilon^{\delta}}E\left|\alpha_j^{(n)}(\zeta)\right|^{2+\delta},$$

$1\le j\le n$. Then

$$\sum_{j=1}^{n}E\left(\left(\alpha_j^{(n)}(\zeta)\eta_j\right)^2 I\left(\left|\alpha_j^{(n)}(\zeta)\eta_j\right|>\varepsilon\right)\right)\le\frac{C_\delta}{\varepsilon^{\delta}}\sum_{j=1}^{n}E\left|\alpha_j^{(n)}(\zeta)\right|^{2+\delta}\le$$

$$\le\frac{C_\delta}{\varepsilon^{\delta}}\max_{1\le j\le n}\sup_{z\in\mathbb{R}}\left|\alpha_j^{(n)}(z)\right|^{\delta}E\sum_{j=1}^{n}\left(\alpha_j^{(n)}(\zeta)\right)^2=\frac{C_\delta}{\varepsilon^{\delta}}\left(\max_{1\le j\le n}\sup_{z\in\mathbb{R}}\left|\alpha_j^{(n)}(z)\right|\right)^{\delta}\to 0$$

as $n\to\infty$.

Similarly, for independent standard normally distributed random variables $\xi_1,\xi_2,...,\xi_n$, which are independent of ζ, we have

$$E\left(\left(\alpha_j^{(n)}(\zeta)\xi_j\right)^2 I\left(\left|\alpha_j^{(n)}(\zeta)\xi_j\right|>\varepsilon\right)\right)\le\frac{3^{(2+\delta)/4}}{\varepsilon^{\delta}}E\left|\alpha_j^{(n)}(\zeta)\right|^{2+\delta},$$

and hence

$$\sum_{j=1}^{n} E\left(\left(\alpha_j^{(n)}(\zeta)\xi_j\right)^2 I\left(\left|\alpha_j^{(n)}(\zeta)\xi_j\right| > \varepsilon\right)\right) \leq$$

$$\leq \frac{3^{(2+\delta)/4}}{\varepsilon^\delta}\left(\max_{1\leq j\leq n}\sup_{z\in\mathbb{R}}\left|\alpha_j^{(n)}(z)\right|\right)^\delta \to 0$$

as $n \to \infty$. □

Another type of conditions on the functions $\alpha_j^{(n)}(z)$, $z \in \mathbb{R}$, $j = \overline{1,n}$, under which the sequence of functionals given by (4) is asymptotically normal, is introduced in the following two theorems.

Theorem 4 *Let random variable ζ and functions $\alpha_j^{(n)}$, $j = \overline{1,n}$, be such that*

$$\sum_{j=1}^{n} E\left|\alpha_j^{(n)}(\zeta) - \frac{1}{\sqrt{n}}\right| \to 0 \qquad \text{as } n \to \infty. \tag{7}$$

Let $\eta_1, \eta_2, ..., \eta_n$ be independent random variables with finite second moments which are independent of ζ too and for which the CLT holds. Then the sequence of functionals

$$f_n^\zeta(\eta_1, \eta_2, ..., \eta_n) = \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j, \qquad n \geq 2,$$

is asymptotically normal.

Proof. For any $n \geq 2$ we can write

$$f_n^\zeta(\eta_1, \eta_2, ..., \eta_n) = \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\eta_j + \sum_{j=1}^{n}\left(\alpha_j^{(n)}(\zeta) - \frac{1}{\sqrt{n}}\right)\eta_j.$$

Since for random variables $\eta_1, \eta_2, ..., \eta_n$ the CLT is valid, the first summand on the right–hand side of the obtained expression is asymptotically normal. Therefore to complete the proof we need to show that the second summand tends to 0 in probability.

Note that $E|\eta_j| \leq \left(E\eta_j^2\right)^{1/2} = C < \infty$ for any $j = \overline{1,n}$. Hence due to the Chebychev inequality for any $\varepsilon > 0$ we have

$$P\left(\left|\sum_{j=1}^{n}\left(\alpha_j^{(n)}(\zeta) - \frac{1}{\sqrt{n}}\right)\eta_j\right| > \varepsilon\right) \leq \frac{1}{\varepsilon}E\left|\sum_{j=1}^{n}\left(\alpha_j^{(n)}(\zeta) - \frac{1}{\sqrt{n}}\right)\eta_j\right| \leq$$

$$\leq \frac{1}{\varepsilon}\sum_{j=1}^{n} E\left|\alpha_j^{(n)}(\zeta) - \frac{1}{\sqrt{n}}\right| E|\eta_i| \leq \frac{C}{\varepsilon}\sum_{j=1}^{n} E\left|\alpha_j^{(n)}(\zeta) - \frac{1}{\sqrt{n}}\right| \to 0$$

as $n \to \infty$. □

Example 5 *(continuation).* Let us show that functions $\alpha_j^{(n)}$, $j = \overline{1,n}$, considered in the Example 5 satisfy the condition (7). Indeed, let function g be such that $Eg^2(\zeta) < \infty$. For any fixed n we have

$$\sum_{j=1}^{n} E\left|\alpha_j^{(n)}(\zeta) - \frac{1}{\sqrt{n}}\right| =$$

$$= E\left|\frac{g(\zeta)}{\sqrt{n-1+g^2(\zeta)}} - \frac{1}{\sqrt{n}}\right| + (n-1)E\left|\frac{1}{\sqrt{n-1+g^2(\zeta)}} - \frac{1}{\sqrt{n}}\right|.$$

For the first summand in the right–hand side of this expression we obtain

$$E\left|\frac{g(\zeta)}{\sqrt{n-1+g^2(\zeta)}} - \frac{1}{\sqrt{n}}\right| \le \frac{E|g(\zeta)|}{\sqrt{n-1}} + \frac{1}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty,$$

since $E|g(\zeta)| \le \left(Eg^2(\zeta)\right)^{1/2} < \infty$. For the second summand we can write

$$(n-1)E\left|\frac{1}{\sqrt{n-1+g^2(\zeta)}} - \frac{1}{\sqrt{n}}\right| = (n-1)E\left|\frac{\sqrt{n} - \sqrt{n-1+g^2(\zeta)}}{\sqrt{n}\cdot\sqrt{n-1+g^2(\zeta)}}\right| =$$

$$= (n-1)E\left|\frac{n-(n-1+g^2(\zeta))}{\left(\sqrt{n}+\sqrt{n-1+g^2(\zeta)}\right)\sqrt{n}\cdot\sqrt{n-1+g^2(\zeta)}}\right| \le$$

$$\le \frac{n-1}{n}\cdot\frac{E|1-g^2(\zeta)|}{\sqrt{n-1}} \to 0$$

as $n \to \infty$.

Theorem 5 *Let functions $\alpha_j^{(n)}$, $j = \overline{1,n}$, satisfy the condition* (5) *and let random variable ζ be such that*

$$\max_{1\le j\le n} E\left|\alpha_j^{(n)}(\zeta)\right| \to 0 \qquad \text{as } n \to 0$$

and for some δ, $0 < \delta < 1/2$,

$$\max_{n\ge 1}\sum_{j=1}^{n} E^{\frac{1}{2(1+\delta)}}\left(\alpha_j^{(n)}(\zeta)\right)^2 \le C < \infty$$

Let $\eta_1, \eta_2, ..., \eta_n$ be independent random variables which are independent of ζ too and such that $E\eta_j = 0$, $D\eta_j = 1$, $j = \overline{1,n}$. Then the sequence of functionals

$$f_n^{\zeta}(\eta_1, \eta_2, ..., \eta_n) = \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j, \qquad n \ge 2,$$

is asymptotically normal.

Proof. Let $\xi_1, \xi_2, ..., \xi_n$ be independent standard normally distributed random variables which are also independent of $\zeta, \eta_1, \eta_2, ..., \eta_n$. We have

$$f_n^{\zeta}(\eta_1, \eta_2, ..., \eta_n) = \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\eta_j = \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)\xi_j + \sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)(\eta_j - \xi_j).$$

Since the condition (5) is fulfilled, due to the Theorem 2, $\sum\limits_{j=1}^{n} \alpha_j^{(n)}(\zeta)\xi_j \sim \mathscr{N}(0,1)$. Hence to prove the theorem we need to show that the last summand in the expression above converges in probability to 0 as $n \to \infty$.

Successively applying Chebyshev and Minkowski inequalities, for any $\varepsilon > 0$ we can write

$$P\left(\left|\sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)(\eta_j - \xi_j)\right| > \varepsilon\right) = P\left(\left|\sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)(\eta_j - \xi_j)\right|^{1+\delta} > \varepsilon^{1+\delta}\right) \leq$$

$$\leq \frac{1}{\varepsilon^{1+\delta}} E\left|\sum_{j=1}^{n} \alpha_j^{(n)}(\zeta)(\eta_j - \xi_j)\right|^{1+\delta} \leq \frac{1}{\varepsilon^{1+\delta}}\left(\sum_{j=1}^{n} E^{\frac{1}{1+\delta}}\left|\alpha_j^{(n)}(\zeta)(\eta_j - \xi_j)\right|^{1+\delta}\right)^{1+\delta}.$$

Since random variables ζ, η_j and ξ_j are independent, for any $j = \overline{1,n}$ we have

$$E\left|\alpha_j^{(n)}(\zeta)(\eta_j - \xi_j)\right|^{1+\delta} = E\left|\alpha_j^{(n)}(\zeta)\right|^{1+\delta} \cdot E\left|\eta_j - \xi_j\right|^{1+\delta}.$$

Here

$$E\left|\eta_j - \xi_j\right|^{1+\delta} \leq \left(E(\eta_j - \xi_j)^2\right)^{\frac{1+\delta}{2}} = 2^{\frac{1+\delta}{2}},$$

by virtue of Hölder's inequality

$$E\left|\alpha_j^{(n)}(\zeta)\right|^{1+\delta} = E\left(\left|\alpha_j^{(n)}(\zeta)\right| \cdot \left|\alpha_j^{(n)}(\zeta)\right|^{\delta}\right) \leq E^{\frac{1}{2}}\left(\alpha_j^{(n)}(\zeta)\right)^2 \cdot E^{\frac{1}{2}}\left|\alpha_j^{(n)}(\zeta)\right|^{2\delta},$$

and

$$E\left|\alpha_j^{(n)}(\zeta)\right|^{2\delta} \leq E^{2\delta}\left|\alpha_j^{(n)}(\zeta)\right|$$

since $0 < 2\delta < 1$. Hence

$$\sum_{j=1}^{n} E^{\frac{1}{1+\delta}}\left|\alpha_j^{(n)}(\zeta)(\eta_j - \xi_j)\right|^{1+\delta} \leq \sqrt{2}\sum_{j=1}^{n} E^{\frac{1}{2(1+\delta)}}\left(\alpha_j^{(n)}(\zeta)\right)^2 E^{\frac{\delta}{1+\delta}}\left|\alpha_j^{(n)}(\zeta)\right| \leq$$

$$\leq \sqrt{2}\max_{1\leq j\leq n} E\left|\alpha_j^{(n)}(\zeta)\right|^{\frac{\delta}{1+\delta}} \sum_{j=1}^{n} E^{\frac{1}{2(1+\delta)}}\left(\alpha_j^{(n)}(\zeta)\right)^2 \leq C\sqrt{2}\left(\max_{1\leq j\leq n} E\left|\alpha_j^{(n)}(\zeta)\right|\right)^{\frac{\delta}{1+\delta}}.$$

Finally we obtain

$$P\left(\left|\sum_{j=1}^{n}\alpha_j^{(n)}(\zeta)(\eta_j-\xi_j)\right|>\varepsilon\right)\leq\left(\frac{C\sqrt{2}}{\varepsilon}\right)^{1+\delta}\left(\max_{1\leq j\leq n}E\left|\alpha_j^{(n)}(\zeta)\right|\right)^{\delta}\to 0$$

as $n\to\infty$. □

3 Remarks on asymptotic normality of sequences of linear functionals

In this section we consider linear functionals

$$f_n(\eta_1,\eta_2,...,\eta_n)=\sum_{j=1}^{n}\alpha_j^{(n)}\eta_j,\qquad n\geq 2.$$

Conditions under which a sequence of such functionals (weighted sums) is asymptotically normal were obtained in several works. For example, Weber in [13] introduced conditions on the 4-th power of the coefficients $\alpha_j^{(n)}$ as well as existence of $E\eta_j^p$ for $p>4$. Fisher ([3]) and Kevei ([7]) used the appropriate rate of convergence to 0 for the coefficients. It is also worth noting that in the mentioned papers only identically distributed random variables $\eta_1,\eta_2,...,\eta_n$ were considered.

Below we formulate the CLT for linear functionals on independent but not necessarily identically distributed random variables under the condition of the finiteness of their $(2+\delta)$- th moments for some $\delta>0$ and the condition on squares of corresponding coefficients.

Theorem 6 *Let coefficients $\alpha_j^{(n)}$, $j=\overline{1,n}$, of linear functionals f_n satisfy the condition* (1) *and be such that*

$$\max_{1\leq j\leq n}\left|\alpha_j^{(n)}\right|\to 0\quad \textit{as } n\to\infty.$$

Let $\eta_1,\eta_2,...,\eta_n$ be independent random variables such that $E\eta_j=0$, $E\eta_j^2=1$, $j=\overline{1,n}$, and for some $\delta>0$

$$\sup_{1\leq j\leq n}E\left|\eta_j\right|^{2+\delta}=C_\delta<\infty.$$

Then the sequence of linear functionals $f_n(\eta_1,\eta_2,...,\eta_n)$, $n\geq 2$, is asymptotically normal.

This theorem can be obtained as a direct corollary of the Theorem 3 for $\alpha_j^{(n)}(z)\equiv\alpha_j^{(n)}$, $1\leq j\leq n$. On the other hand, it can be easily proved by the use of the following general lemma.

Lemma 2 *Let the coefficients* $\alpha_j^{(n)}$, $j = \overline{1,n}$, *of the linear functionals* f_n *satisfy the condition* (1) *and be such that for independent random variables* $\eta_1, \eta_2, ..., \eta_n$ *with* $E\eta_j = 0$, $D\eta_j = 1$, $j = \overline{1,n}$, *and any* $\varepsilon > 0$

$$\sum_{j=1}^{n} E\left(\left(\alpha_j^{(n)}\eta_j\right)^2 I\left(\left|\alpha_j^{(n)}\eta_j\right| > \varepsilon\right)\right) \to 0 \qquad \text{as } n \to \infty.$$

Then the sequence of linear functionals $f_n(\eta_1, \eta_2, ..., \eta_n)$, $n \geq 2$, *is asymptotically normal.*

The proof of the Lemma 2 can be obtained as a corollary of the CLT for a series scheme of random variables independent in each series (see, for instance, Theorem 7.2 in [1]). An application of the method used in the proof of Lemma 1 allows us to give quite simple proof of this result.

Proof of the Lemma 2 First let us note that using mathematical induction, it is not difficult to show that for any two finite sequences of complex numbers $a_1, a_2, ..., a_N$ and $b_1, b_2, ..., b_N$ such that

$$|a_k| \leq 1, \qquad |b_k| \leq 1, \qquad 1 \leq k \leq N,$$

the following estimation is valid

$$\left|\prod_{k=1}^{N} a_k - \prod_{k=1}^{N} b_k\right| \leq \sum_{k=1}^{N} |a_k - b_k|. \tag{8}$$

Let $\xi_1, \xi_2, ..., \xi_n$ be independent standard normally distributed random variables which are independent of $\eta_1, \eta_2, ..., \eta_n$ as well. Then $\sum_{j=1}^{n} \alpha_j^{(n)}\xi_j \sim \mathcal{N}(0,1)$, and hence with the use of (8) we can write

$$\left|Ee^{it\sum_{j=1}^{n}\alpha_j^{(n)}\eta_j} - e^{-t^2/2}\right| = \left|\prod_{j=1}^{n} Ee^{it\alpha_j^{(n)}\eta_j} - \prod_{j=1}^{n} Ee^{it\alpha_j^{(n)}\xi_j}\right| \leq$$

$$\leq \sum_{j=1}^{n}\left|Ee^{it\alpha_j^{(n)}\eta_j} - Ee^{it\alpha_j^{(n)}\xi_j}\right|.$$

Since for any j

$$E\alpha_j^{(n)}\eta_j = E\alpha_j^{(n)}\xi_j = 0,$$

$$E(\alpha_j^{(n)}\eta_j)^2 = E(\alpha_j^{(n)}\xi_j)^2 = (\alpha_j^{(n)})^2, \quad E(\alpha_j^{(n)}\xi_j)^3 = 0,$$

further we have

$$\sum_{j=1}^{n}\left|Ee^{it\alpha_j^{(n)}\eta_j}-Ee^{it\alpha_j^{(n)}\xi_j}\right|\le\sum_{j=1}^{n}E\left|e^{it\alpha_j^{(n)}\eta_j}-1-it\alpha_j^{(n)}\eta_j-\frac{(it)^2}{2!}(\alpha_j^{(n)}\eta_j)^2\right|+$$

$$+\sum_{j=1}^{n}E\left|e^{it\alpha_j^{(n)}\xi_j}-1-it\alpha_j^{(n)}\xi_j-\frac{(it)^2}{2!}(\alpha_j^{(n)}\xi_j)^2-\frac{(it)^3}{3!}(\alpha_j^{(n)}\xi_j)^3\right|.$$

With application of the estimation (6) for any $\varepsilon>0$ we can write

$$\sum_{j=1}^{n}E\left|e^{it\alpha_j^{(n)}\eta_j}-1-it\alpha_j^{(n)}\eta_j-\frac{(it)^2}{2!}(\alpha_j^{(n)}\eta_j)^2\right|\le$$

$$\le\frac{|t|^3}{3!}\sum_{j:|\alpha_j^{(n)}\eta_j|\le\varepsilon}E\left|\alpha_j^{(n)}\eta_j\right|^3+\frac{2|t|^2}{2!}\sum_{j:|\alpha_j^{(n)}\eta_j|>\varepsilon}E\left|\alpha_j^{(n)}\eta_j\right|^2\le$$

$$\le\frac{|t|^3}{6}\varepsilon+t^2\sum_{j=1}^{n}E\left(\left(\alpha_j^{(n)}\eta_j\right)^2I\left(\left|\alpha_j^{(n)}\eta_j\right|>\varepsilon\right)\right).$$

Since $E\xi_j^4=3$ and $\left(\alpha_j^{(n)}\right)^2=E\left(\alpha_j^{(n)}\eta_j\right)^2$, we also have

$$\sum_{j=1}^{n}E\left|e^{it\alpha_j^{(n)}\xi_j}-1-it\alpha_j^{(n)}\xi_j-\frac{(it)^2}{2!}(\alpha_j^{(n)}\xi_j)^2-\frac{(it)^3}{3!}(\alpha_j^{(n)}\xi_j)^3\right|\le$$

$$\le\frac{t^4}{4!}\sum_{j=1}^{n}E\left|\alpha_j^{(n)}\xi_j\right|^4=\frac{t^4}{4!}\sum_{j=1}^{n}3\left(\alpha_j^{(n)}\right)^4\le\frac{t^4}{8}\max_{1\le j\le n}\left(\alpha_j^{(n)}\right)^2\sum_{j=1}^{n}\left(\alpha_j^{(n)}\right)^2=$$

$$=\frac{t^4}{8}\max_{1\le j\le n}E\left(\left(\alpha_j^{(n)}\eta_j\right)^2\right)\le\frac{t^4}{8}\max_{1\le j\le n}E\left(\left(\alpha_j^{(n)}\eta_j\right)^2I\left(\left|\alpha_j^{(n)}\eta_j\right|\le\varepsilon\right)\right)+$$

$$+\frac{t^4}{8}\max_{1\le j\le n}E\left(\left(\alpha_j^{(n)}\eta_j\right)^2I\left(\left|\alpha_j^{(n)}\eta_j\right|>\varepsilon\right)\right)\le$$

$$\le\frac{t^4}{8}\varepsilon^2+\frac{t^4}{8}\sum_{j=1}^{n}E\left(\left(\alpha_j^{(n)}\eta_j\right)^2I\left(\left|\alpha_j^{(n)}\eta_j\right|>\varepsilon\right)\right).$$

Finally we obtain

$$\left| Ee^{it\sum_{j=1}^{n}\alpha_j^{(n)}\eta_j} - e^{-t^2/2} \right| \leq$$

$$\leq \frac{|t|^3}{6}\varepsilon + \frac{t^4}{8}\varepsilon^2 + \left(t^2 + \frac{t^4}{8}\right)\sum_{j=1}^{n} E\left(\left(\alpha_j^{(n)}\eta_j\right)^2 I\left(\left|\alpha_j^{(n)}\eta_j\right| > \varepsilon\right)\right),$$

which completes the proof. □

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