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Nonlinear Diffusion Equation Involving a Generalized Laplacian Operator

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Abstract. This investigation focuses on identifying weak solutions for a nonlinear diffusion equation that includes $p(u)$-Laplacian-like operators. By imposing specific conditions on the initial value, we derive our result using a time-discrete approach in conjunction with a singular perturbation technique and Schauder's fixed-point theorem.

Key Words: $p(b(u))$-Laplacian-Like Operator, Weak Solutions, Parabolic Problem, Gelfand Triple, Boundary Value Problems
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Introduction

This article is concerned with the existence of weak solutions for the following local $p(u)$-Laplacian problem:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u + \frac{|\nabla u|^{2p(b(u))-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(b(u))}}}\right) + g(u) = f & \text{in } Q = \Omega\times]0,T[, \\ u = 0 & \text{on } \Gamma = \partial\Omega\times]0,T[, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $N \geq 2$; f, g are given functions, and $p : \mathbb{R} \to [1, +\infty[$ is a Lipschitz continuous and bounded function.

Petroleum extraction, flow through porous media, reaction-diffusion issues, non-Newtonian fluids, and other physical phenomena can all give rise to nonlinear boundary value problems with nonlinearities and nonstandard $p(x)$-growth conditions. Consequently, there has been a lot of interest in the study of these issues and their generalizations in recent years. The main focus of this research is when the function p is composed with another function that depends on the unknown solution u (see, for example, [1, 7, 13]).

In this setting, the problem becomes nonlocal and especially interesting. Indeed, one major incentive for studying nonlocal problems is the fact that measurements in the physical world are typically taken using local averages rather than punctual data.

In [5], an intriguing nonlocal model is proposed with $p = p(|\nabla u|)$, which enables a reduction of noise in the restored image. Likewise, in [14], the author provided multiple numerical illustrations indicating that taking into account exponents $p = p(u)$ maintains edges and lowers noise in the recovered images u. Additionally, some models of electrorheological fluids have been studied using parabolic equations involving the $p(u)$-Laplacian (see, for instance, [4, 6, 9, 10, 12]). Only a very small number of papers have been published regarding the parabolic type $p(b(u))$-Laplacian equations (see, for example, [2, 3, 15]).

Our motivation came from the works of Chipot et al. [7] and Zhang et al. [15], but unlike the mentioned articles, we work with a generalized capillarity operator and with a monotone nonlinearity, extending and complementing their results.

We developed this work in the following way. The second section presents some preliminary results. In the third section, we implement an approximation strategy utilizing singular perturbation techniques, the theory of monotone operators, and Schauders fixed-point theorem, which allows us to obtain our result of existence of weak solutions to problem (1).

1 Preliminaries

From now on, $p : \mathbb{R} \to [1, +\infty[$ is a function that is Lipschitz continuous and bounded with

$$1 < \alpha < p(s) \leq \beta < \infty \quad \text{for all } s \in \mathbb{R}.$$

Let $b : W_0^{1,\alpha}(\Omega) \to \mathbb{R}$ be a continuous and bounded function. Since $p(b(u)) \in \mathbb{R}$ is a real number for each fixed u, the Sobolev spaces involved in this work are the standard ones. For a *fixed* function $w \in L^2(\Omega)$, we define the standard Sobolev space

$$W_0^{1,p(b(w))}(\Omega) = W_0^{1,r}(\Omega)\big|_{r=p(b(w))},$$

which is a well-defined, reflexive, separable Banach space (since $p(b(w)) \in (\alpha, \beta) \subset (1, \infty)$), endowed with the norm $\|u\|_{W_0^{1,p(b(w))}} = \|\nabla u\|_{L^{p(b(w))}(\Omega)}$. We stress that the exponent $p(b(w))$ is a fixed real number determined by the *given* function w, not by the element u being considered.

For a given reference function $w \in L^2(Q)$, we define the functional space

$$X(Q,w) := \Big\{ u \in L^\infty(0,T;L^2(\Omega)) \cap L^\alpha(0,T;W_0^{1,\alpha}(\Omega)) : \\ \iint_Q |\nabla u|^{p(b(w))} dxdt < \infty \Big\}$$

equipped with the norm

$$\|u\|_{X(Q,w)} := \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|_{L^\alpha(Q)} + \left(\iint_Q |\nabla u|^{p(b(w))} dxdt \right)^{1/\alpha}.$$

The final solution space $X(Q)$ is defined as the set

$$X(Q) := \Big\{ u \in L^\infty(0,T;L^2(\Omega)) \cap L^\alpha(0,T;W_0^{1,\alpha}(\Omega)) : \\ \iint_Q |\nabla u|^{p(b(u))} dxdt < \infty \Big\}$$

with the associated norm

$$\|u\|_{X(Q)} := \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|_{L^\alpha(Q)} + \left(\iint_Q |\nabla u|^{p(b(u))} dxdt \right)^{1/\alpha}$$

and its dual space $X'(Q)$. Since $\alpha > 1$ and $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ compactly (by the assumption $\alpha > 2N/(N+2)$), the triple

$$(L^\alpha(0,T;W_0^{1,\alpha}(\Omega)),\ L^{\alpha'}(0,T;W^{-1,\alpha'}(\Omega)),\ L^2(Q))$$

forms a Gelfand evolution triple with the required density and continuity properties, so that the embedding

$$\{u \in L^\alpha(0,T;W_0^{1,\alpha}(\Omega)) : u_t \in L^{\alpha'}(0,T;W^{-1,\alpha'}(\Omega))\} \hookrightarrow C([0,T];L^2(\Omega))$$

holds continuously (see [8]). It is obvious that

$$X(Q) \hookrightarrow (L^\alpha(0,T;W_0^{1,\alpha}(\Omega)).$$

Lemma 1 (Chipot M. [7]) *Assume that*

(i) $1 < \alpha \le q_\nu(x) \le \beta < +\infty$ *for all* $\nu \in \mathbb{N}$ *and for some constants* $\alpha, \beta > 0$,

(ii) $q_\nu \to q$ *a.e. in* Ω *as* $\nu \to \infty$,

(iii) $\nabla u_\nu \rightharpoonup \nabla u$ *in* $(L^1(\Omega))^n$ *as* $\nu \to \infty$,

(iv) $\big\| |\nabla u_\nu|^{q_\nu(x)} \big\|_{L^1(\Omega)} \le C$ *for some positive constant* C *independent of* ν.

Then $\nabla u \in \left(L^{q(x)}(\Omega)\right)^N$ *and*

$$\int_\Omega |\nabla u|^{q(x)}\, dx \le \liminf_{\nu\to\infty} \int_\Omega |\nabla u_\nu|^{q_\nu(x)}\, dx.$$

Definition 1 *A function* $u \in X(Q) \cap C\left([0,T], L^2(\Omega)\right)$ *is a weak solution of problem* (1) *if it satisfies*

$$\begin{aligned} -\int_\Omega u_0(x)\phi(x,0)dx + \int_Q \Big[-u\phi_t + \Big(|\nabla u|^{p(b(u))-2}\nabla u \\ + \frac{|\nabla u|^{2p(b(u))-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(b(u))}}}\Big)\cdot \nabla\phi + g(u)\phi\Big] dx\, dt = \int_Q f\phi\, dx\, dt \end{aligned} \tag{2}$$

for all $\phi \in C^1(\overline{Q})$ with $\phi(.,T) = 0$.

2 Solutions to nonlinear diffusion equation

The main result of this paper is stated in the following theorem.

Theorem 1 *Consider the assumptions given above for the functions* p *and* b, *together with* $\alpha > 2N/(N+2)$. *If* $u_0 \in L^2(\Omega)$,

(A_0) $g : \mathbb{R} \to \mathbb{R}$ *is a nondecreasing continuous function, with* $g(0) = 0$, $|g(s)| \le C|s|$,

and

(F_0) $f \in L^{\alpha'}(Q)$,

then there exists a weak solution $u \in X(Q)\cap C([0,T]; L^2(\Omega))$ *for problem* (1).

Proof. We will use the Rothe method to investigate problem (1). Let N_0 be a positive integer, $t_k = kh$ and $h := T/N_0$. The key idea, which avoids self-reference, is to *freeze the exponent at the previous time level*: for each $k \ge 1$, we look for

$$u_k \in W_0^{1,p(b(u_{k-1}))}(\Omega)$$

satisfying the elliptic problem

$$\begin{cases} \dfrac{u_k - u_{k-1}}{h} + L_{k-1}(u_k) = \hat{f}((k-1)h), \quad x \in \Omega, \\ u_k\big|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where

$$L_{k-1}(v) := -\operatorname{div}\Big(|\nabla v|^{p(b(u_{k-1}))-2}\nabla v + \frac{|\nabla v|^{2p(b(u_{k-1}))-2}\nabla v}{\sqrt{1+|\nabla v|^{2p(b(u_{k-1}))}}}\Big) + g(v),$$

and

$$\hat{f}(x,t) = h^{-1}\int_t^{t+h} f(x,\tau)\,\mathrm{d}\tau.$$

Notice that $\hat{f}(.) \in L^{\alpha'}(\Omega)$. The next two steps will allow us to conclude the theorem.

Step 1. *Calculation of the solution u_k of* (3). We detail the case $k=1$ (the inductive step is identical). Fix $w \in L^2(\Omega)$. Consider the auxiliary perturbed problem: find $u \in W_0^{1,\beta}(\Omega)$ such that for all $z \in W_0^{1,\beta}(\Omega)$,

$$\begin{aligned}&\int_\Omega \frac{u-u_0}{h} z\ \mathrm{d}x + \int_\Omega \Big(|\nabla u|^{p(b(w))-2}\nabla u + \frac{|\nabla u|^{2p(b(w))-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(b(w))}}}\Big)\cdot \nabla z\ \mathrm{d}x\\ &+\int_\Omega g(u) z\ \mathrm{d}x + \varepsilon\int_\Omega \Big(|\nabla u|^{\beta-2}\nabla u + \frac{|\nabla u|^{2\beta-2}\nabla u}{\sqrt{1+|\nabla u|^{2\beta}}}\Big)\cdot\nabla z\ \mathrm{d}x = \int_\Omega \hat{f}(0)\, z\ \mathrm{d}x,\end{aligned} \tag{4}$$

where $\varepsilon > 0$ and $p(b(w)) \in [\alpha,\beta]$ is a *fixed constant* (since w is fixed). Next, we will show that equation (4) possesses a weak solution $u_1(.)$ in $W_1 = W_0^{1,p(b(u_1))}(\Omega) \cap L^2(\Omega)$. For this, we employ a singular perturbation and the Schauder fixed-point theorem.

We examine the following auxiliary problem:

$$\begin{cases} \dfrac{u-u_0}{h} - \operatorname{div}\left(|\nabla u|^{p(b(w))-2}\nabla u + \dfrac{|\nabla u|^{2p(b(w))-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(b(w))}}}\right) + g(u) \\ -\epsilon \operatorname{div}\left(|\nabla u|^{\beta-2}\nabla u + \dfrac{|\nabla u|^{2\beta-2}\nabla u}{\sqrt{1+|\nabla u|^{2\beta}}}\right) = \hat{f}(0),\ \epsilon > 0, \end{cases} \tag{5}$$

where $2N/(N+2) < \alpha < p(b(w)) \leq \beta < \infty$ for a.e. $x \in \Omega$.

Let us build the operator $L_\epsilon : W_0^{1,\beta}(\Omega) \to W^{-1,\beta'}(\Omega)$ given by

$$\langle L_\epsilon u, \varphi\rangle = L(u) - \epsilon \operatorname{div}\left(|\nabla u|^{\beta-2}\nabla u + \frac{|\nabla u|^{2\beta-2}\nabla u}{\sqrt{1+|\nabla u|^{2\beta}}}\right).$$

Noting that the sum of two (nonlinear) bounded, continuous, and coercive operators is bounded, continuous, and coercive, it is simple to verify, using Proposition 3.1 in [11], that

i) L_ϵ is bounded, continuous and strictly monotone,

ii) L_ϵ is coercive.

It follows from i) and ii), by applying Browder-Minty theorem, that L_ϵ is a surjection. Hence, we obtain a unique solution $u_w \in W_0^{1,\beta}(\Omega)$ of (5), which satisfies

$$\int_\Omega \frac{u_w - u_0}{h} z\,dx + \int_\Omega \left(|\nabla u_w|^{p(b(w))-2}\nabla u_w + \frac{|\nabla u_w|^{2p(b(w))-2}\nabla u_w}{\sqrt{1+|\nabla u_w|^{2p(b(w))}}} \right)\cdot \nabla z\,dx$$
$$+ \int_\Omega g(u_w) z\,dx + \epsilon \int_\Omega \left(|\nabla u_w|^{\beta-2}\nabla u_w + \frac{|\nabla u_w|^{2\beta-2}\nabla u_w}{\sqrt{1+|\nabla u_w|^{\beta}}} \right)\cdot \nabla z\,dx$$
$$= \int_\Omega \hat{f}(0) z\,dx \tag{6}$$

for all $z \in W_0^{1,\beta}(\Omega)$. Substituting $z = u_w$ in (6), we obtain

$$\frac{1}{2h}\int_\Omega u_w^2\,dx + \int_\Omega \left[(|\nabla u_w|^{p(b(w))} + \frac{|\nabla u_w|^{2p(b(w))}}{\sqrt{1+|\nabla u|^{2p(b(w))}}}) + g(u_w)u_w \right] dx$$
$$+ \epsilon \int_\Omega (|\nabla u_w|^{\beta} + \frac{|\nabla u_w|^{2\beta}}{\sqrt{1+|\nabla u_w|^{2\beta}}})\,dx \le \frac{1}{2h}\int_\Omega u_0^2\,dx + C\|\nabla u_w\|_{L^\beta(\Omega)}$$

for some $C > 0$ (with C depending on $\alpha, \beta, \Omega, \|\hat{f}(0)\|_{L^{\alpha'}(\Omega)}$).

The Hölder inequality implies that

$$\begin{gathered} \|u_w\|_{L^2(\Omega)} \le C_0 \\ \||\nabla u_w|\|_{L^\beta(\Omega)} \le C_0, \\ \int_\Omega \frac{|\nabla u_w|^{2\beta}}{\sqrt{1+|\nabla u_w|^{2\beta}}}\,dx \le C_0 \end{gathered} \tag{7}$$

for a constant $C_0 = C_0(\alpha, \beta, \Omega, h, u_0, \hat{f}(0))$ independent of both ϵ and w. Consequently, the ball $B := \{v \in L^2(\Omega) : \|v\|_{L^2(\Omega)} \le C_0\}$ has radius C_0 independent of ϵ.

Moreover, using the Hölder inequality together with the uniform L^β-bound on $|\nabla u_w|$ and the fact that $\alpha \le p(b(w)) \le \beta$, we also obtain

$$\|\nabla u_w\|_{L^\alpha(\Omega)} \le C\left(\int_\Omega |\nabla u_w|^{p(b(w))}\,dx + 1 \right)^{1/\alpha} \le C, \tag{8}$$

with C independent of w and ε. Since $\alpha > 2N/(N+2)$ implies the compact embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$, the uniform $W^{1,\alpha}$-bound (8) ensures that $T : B \to B$ such that $T(w) = u_w$, where u_w is weak solution of (5), maps bounded sets into precompact sets in $L^2(\Omega)$. Combined with the continuity of T established below, Schauder's fixed-point theorem enables us to locate a point $u_\varepsilon \in B$ with $T_\varepsilon(u_\varepsilon) = u_\varepsilon$, i.e., u_ε solves (6) with $w = u_\varepsilon$. Note that the exponent is now $p(b(u_\varepsilon))$, determined by u_ε itself.

We take a sequence (w_ν) in $L^2(\Omega)$ such that

$$w_\nu \to w \text{ strongly in } L^2(\Omega) \text{ as } \nu \to \infty. \tag{9}$$

Set $u_\nu := T(w_\nu)$ for any positive integer ν. From (7) we have

$$\|\nabla u_\nu\|_{L^\beta(\Omega)} \le C, \qquad \text{for some } C>0 \text{ (without } \nu\text{-dependence).}$$

Then, we get a subsequence u_ν (still denoted by u_ν) such that

$$\begin{aligned} u_\nu &\rightharpoonup u \quad \text{in } W_0^{1,\beta}(\Omega), \text{as } \nu\to\infty,\\ u_\nu &\to u \quad \text{strongly in } L^2(\Omega) \text{ as } \nu\to\infty. \end{aligned} \tag{10}$$

Now, making (u_ν, w_ν) instead (u_w, w) in (6), we have

$$\begin{cases} \displaystyle\int_\Omega \frac{u_\nu - u_0}{h} z\,dx + \int_\Omega \big[(|\nabla u_\nu|^{p(b(w_\nu))-2}\nabla u_\nu + \frac{|\nabla u_\nu|^{2p(b(w_\nu))-2}\nabla u_\nu}{\sqrt{1+|\nabla u|^{2p(b(w_\nu))}}}).\nabla z \\ \displaystyle +g(u_\nu)z\big]\,dx + \epsilon\int_\Omega (|\nabla u_\nu|^{\beta-2}\nabla u_\nu + \frac{|\nabla u_\nu|^{2\beta-2}\nabla u_\nu}{\sqrt{1+|\nabla u_\nu|^{2\beta}}}).\nabla z\,dx = \int_\Omega \hat{f}(0) z\,dx, \end{cases} \tag{11}$$

for all $z \in W_0^{1,\beta}(\Omega)$. Then, using the monotonicity of the capillarity operator and the function g, we get

$$\begin{aligned} &\int_\Omega \big[(|\nabla u_\nu|^{p(b(w_\nu))-2}\nabla u_\nu + \frac{|\nabla u_\nu|^{2p(b(w_\nu))-2}\nabla u_\nu}{\sqrt{1+|\nabla u|^{2p(b(w_\nu))}}}) \\ &\quad + \epsilon(|\nabla u_\nu|^{\beta-2}\nabla u_\nu + \frac{|\nabla u_\nu|^{2\beta-2}\nabla u_\nu}{\sqrt{1+|\nabla u_\nu|^{2\beta}}})\big]\cdot\nabla(u_\nu - z)\,dx \\ &- \int_\Omega \big[(|\nabla z|^{p(b(w_\nu))-2}\nabla z + \frac{|\nabla v|^{2p(b(w_\nu))-2}\nabla z}{\sqrt{1+|\nabla z|^{2p(b(w_\nu))}}}) \\ &\quad + \epsilon(|\nabla z|^{\beta-2}\nabla z + \frac{|\nabla z|^{2\beta-2}\nabla z}{\sqrt{1+|\nabla z|^{2\beta}}})\big]\cdot\nabla(u_\nu - z)\,dx \\ &+ \int_\Omega (g(u_\nu)-g(z))(u_\nu - z)\,dx + \int_\Omega \frac{(u_\nu - z)^2}{h}\,dx \ge 0. \end{aligned} \tag{12}$$

Taking $z = u_\nu - z$ as the test function in (11), using (12), leads us to

$$\begin{aligned} &\int_\Omega \frac{(u_0 - z)}{h}(u_\nu - z)\,dx + \int_\Omega \hat{f}(0)(u_\nu - z)\,dx \\ &- \int_\Omega \big[(|\nabla z|^{p(b(w_\nu))-2}\nabla z + \frac{|\nabla z|^{2p(b(w_\nu))-2}\nabla z}{\sqrt{1+|\nabla z|^{2p(b(w_\nu))}}}) + \epsilon(|\nabla z|^{\beta-2}\nabla z \\ &+ \frac{|\nabla z|^{2\beta-2}\nabla z}{\sqrt{1+|\nabla z|^{2\beta}}})\big].\nabla(u_\nu - z)\,dx - \int_\Omega g(z)(u_\nu - z) \ge 0 \end{aligned} \tag{13}$$

for all $z \in W_0^{1,\beta}(\Omega)$. From (9), we get a subsequence (still denoted with the same sequence) such that

$$w_\nu \to w \quad \text{almost everywhere on } \Omega \quad \text{as } \nu\to\infty.$$

By the monotone convergence theorem and the Lipschitz-continuity of p, we arrived at

$$|\nabla z|^{p(b(w_\nu))-2}\nabla z+\frac{|\nabla z|^{2p(b(w_\nu))-2}\nabla z}{\sqrt{1+|\nabla z|^{2p(b(w_\nu))}}}\to |\nabla z|^{p(b(w))-2}\nabla z+\frac{|\nabla z|^{2p(b(w))-2}\nabla z}{\sqrt{1+|\nabla z|^{2p(b(w))}}} \tag{14}$$

strongly in $(L^{\beta'}(\Omega))^N$ for all $z\in W_0^{1,\beta}(\Omega)$.

Using (10), (13) and (14), we get

$$\begin{aligned}
&\int_\Omega \frac{(u_0-z)}{h}(u-z)\,dx+\int_\Omega \hat{f}(0)(u-z)\,dx\\
&-\int_\Omega \big[(|\nabla z|^{p(b(w))-2}\nabla z+\frac{|\nabla z|^{2p(b(w))-2}\nabla z}{\sqrt{1+|\nabla z|^{2p(b(w))}}})+\epsilon(|\nabla z|^{\beta-2}\nabla z\\
&+\frac{|\nabla z|^{2\beta-2}\nabla z}{\sqrt{1+|\nabla z|^{2\beta}}})\big].\nabla(u-z)\,dx-\int_\Omega g(z)(u-z)\,dx\ge 0
\end{aligned} \tag{15}$$

for all $z\in W_0^{1,\beta}(\Omega)$. Making $z=u\mp\theta y$ where $y\in W_0^{1,\beta}(\Omega)$ and $\theta>0$, we have from (15) that

$$\begin{aligned}
&\pm\Big\{\int_\Omega \frac{u_0-(u\mp\theta y)}{h}y\,dx+\int_\Omega \hat{f}(0)y\,dx\\
&-\int_\Omega \big[(|\nabla(u\mp\theta y)|^{p(b(w))-2}\nabla(u\mp\theta y)+\frac{|\nabla(u\mp\theta y)|^{2p(b(w))-2}\nabla(u\mp\theta y)}{\sqrt{1+|\nabla(u\mp\theta y)|^{2p(b(w))}}})\\
&+\epsilon(|\nabla(u\mp\theta y)|^{\beta-2}\nabla(u\mp\theta y)+\frac{|\nabla(u\mp\theta y)|^{2\beta-2}\nabla(u\mp\theta y)}{\sqrt{1+|\nabla u\mp\theta y|^{2\beta}}})\big].\nabla y\,dx\\
&-\int_\Omega g(u\mp\theta y)y\,dx\Big\}\ge 0.
\end{aligned}$$

Next, letting $\theta\to 0$ and using the hemicontinuity of the operators, we obtain

$$\begin{aligned}
&\int_\Omega \frac{u-u_0}{h}y\,dx+\int_\Omega \big[(|\nabla u|^{p(b(w))-2}\nabla u+\frac{|\nabla u|^{2p(b(w))-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(b(w))}}})\\
&+\epsilon(|\nabla u|^{\beta-2}\nabla u+\tfrac{|\nabla u|^{2\beta-2}\nabla u}{\sqrt{1+|\nabla u|^{2\beta}}})\big].\nabla y\,dx+\int_\Omega g(u)y\,dx=\int_\Omega \hat{f}(0)y\,dx
\end{aligned}$$

for all $y\in W_0^{1,\beta}(\Omega)$. Hence, $u=u_w$. By the uniqueness of the limit, taking into account (15), it follows that

$$u_\nu\to u_w\quad\text{in } L^2(\Omega),$$

showing the continuity of T. Therefore, (5) possesses a weak solution.

Thus, for every $\epsilon > 0$, we get $u_\epsilon \in W_0^{1,\beta}(\Omega)$ verifying

$$\int_\Omega \frac{u_\epsilon - u_0}{h} z\,dx + \int_\Omega \big[(|\nabla u_\epsilon|^{p(b(u_\epsilon))-2}\nabla u_\epsilon + \frac{|\nabla u_\epsilon|^{2p(b(u_\epsilon))-2}\nabla u_\epsilon}{\sqrt{1+|\nabla u_\epsilon|^{2p(b(u_\epsilon))}}})$$
$$+ \epsilon(|\nabla u_\epsilon|^{\beta-2}\nabla u_\epsilon + \frac{|\nabla u_\epsilon|^{2\beta-2}\nabla u_\epsilon}{\sqrt{1+|\nabla u_\epsilon|^{2\beta}}})\big] \cdot \nabla z\,dx + \int_\Omega g(u_\epsilon) z\,dx = \int_\Omega \hat{f}(0) z\,dx$$

for all $z \in W_0^{1,\beta}(\Omega)$.
Step 2. *Passage to the limit $\varepsilon \to 0$.* The fixed point u_ε satisfies (5) with $w = u_\varepsilon$. From the uniform bound (7) and (8) (which hold independently of ε), passing to a subsequence $\varepsilon_\nu \to 0$,

$$\begin{aligned} u_{\varepsilon_\nu} &\rightharpoonup u_1 \quad \text{in } W_0^{1,\alpha}(\Omega),\\ u_{\varepsilon_\nu} &\to u_1 \quad \text{in } L^2(\Omega),\\ u_{\varepsilon_\nu} &\to u_1 \quad \text{a.e. in } \Omega. \end{aligned}$$

By Lipschitz continuity of p and b, $p(b(u_{\varepsilon_\nu})) \to p(b(u_1))$ a.e., Lemma 1 gives $\nabla u_1 \in (L^{p(b(u_1))}(\Omega))^N$, i.e., $u_1 \in W_0^{1,p(b(u_1))}(\Omega)$.

Remark 1 *The iterate u_1 belongs to two spaces for logically distinct reasons:*

- $u_1 \in W_0^{1,p(b(u_0))}(\Omega)$*: this holds* by construction, *since u_1 was found as the solution of* (3) *with $k = 1$, whose exponent is the* known *datum $p(b(u_0))$.*
- $u_1 \in W_0^{1,p(b(u_1))}(\Omega)$*: this is a* consequence *of Lemma 1 applied to the limit $\varepsilon \to 0$. The fixed-point-plus-limit procedure forces $p(b(u_{\varepsilon_\nu})) \to p(b(u_1))$ a.e., and the lemma transfers the uniform gradient bound to the limit with the* updated *exponent $p(b(u_1))$.*

Identification of the nonlinear limit. Denote

$$A(v,q) := |q|^{p(b(v))-2}q + \frac{|q|^{2p(b(v))-2}q}{\sqrt{1+|q|^{2p(b(v))}}}.$$

We wish to identify the weak limit:

$$A(u_{\varepsilon_\nu}, \nabla u_{\varepsilon_\nu}) \rightharpoonup \xi \quad \text{in } (L^{\alpha'}(\Omega))^N, \qquad \xi = A(u_1, \nabla u_1).$$

Let $z \in W_0^{1,\beta}(\Omega)$ be arbitrary. Again, the monotonicity of the capillarity operator gives

$$\int_\Omega \big[A(u_{\varepsilon_\nu}, \nabla u_{\varepsilon_\nu}) - A(u_{\varepsilon_\nu}, \nabla z)\big] \cdot (\nabla u_{\varepsilon_\nu} - \nabla z)\ \mathrm{d}x \geq 0. \tag{16}$$

From the variational formulation of u_{ε_ν} we have

$$\begin{aligned}&\int_\Omega A(u_{\varepsilon_\nu}, \nabla u_{\varepsilon_\nu}) \cdot \nabla(u_{\varepsilon_\nu} - z)\ \mathrm{d}x \\ &\leq \int_\Omega \frac{u_0 - z}{h}(u_{\varepsilon_\nu} - z)\ \mathrm{d}x + \int_\Omega \hat{f}(0)(u_{\varepsilon_\nu} - z)\ \mathrm{d}x - \int_\Omega g(u_{\varepsilon_\nu})(u_{\varepsilon_\nu} - z)\ \mathrm{d}x.\end{aligned} \tag{17}$$

Combining (16) and (17),

$$\begin{aligned}&\int_\Omega A(u_{\varepsilon_\nu}, \nabla z) \cdot \nabla(u_{\varepsilon_\nu} - z)\ \mathrm{d}x \\ &\leq \int_\Omega \frac{u_0 - z}{h}(u_{\varepsilon_\nu} - z)\ \mathrm{d}x + \int_\Omega \hat{f}(0)(u_{\varepsilon_\nu} - z)\ \mathrm{d}x - \int_\Omega g(u_{\varepsilon_\nu})(u_{\varepsilon_\nu} - z)\ \mathrm{d}x.\end{aligned} \tag{18}$$

Since $u_{\varepsilon_\nu} \to u_1$ in $L^2(\Omega)$ and $p(b(u_{\varepsilon_\nu})) \to p(b(u_1))$ a.e., the Lipschitz continuity of p and dominated convergence imply

$$A(u_{\varepsilon_\nu}, \nabla z) \to A(u_1, \nabla z) \quad \text{strongly in } (L^{\beta'}(\Omega))^N.$$

Passing to the limit $\nu \to \infty$ in (18), we obtain

$$\begin{aligned}\int_\Omega A(u_1, \nabla z) \cdot \nabla(u_1 - z)\ \mathrm{d}x \leq \int_\Omega \frac{u_0 - z}{h}(u_1 - z)\ \mathrm{d}x + \int_\Omega \hat{f}(0)(u_1 - z)\ \mathrm{d}x \\ - \int_\Omega g(u_1)(u_1 - z)\ \mathrm{d}x.\end{aligned}$$

Setting $z = u_1 \mp \theta y$ for arbitrary $y \in W_0^{1,\beta}(\Omega)$ and $\theta > 0$, then dividing by θ, letting $\theta \to 0$ and, again using the hemicontinuity, we get

$$\int_\Omega \frac{u_1 - u_0}{h} y\ \mathrm{d}x + \int_\Omega A(u_1, \nabla u_1) \cdot \nabla y\ \mathrm{d}x + \int_\Omega g(u_1) y\ \mathrm{d}x = \int_\Omega \hat{f}(0)\, y\ \mathrm{d}x$$

for all $y \in W_0^{1,\beta}(\Omega)$.

Hence u_1 is a weak solution of (3) for $k = 1$, and the identification $\xi = A(u_1, \nabla u_1)$ is established.

Step 3. *Inductive construction and a priori estimates.* Repeating Steps 1–2 for $k = 2, \ldots, N_0$, and noting that at each step the exponent is frozen at u_{k-1} (already known), yields weak solutions $u_1, \ldots, u_{N_0}$, that is,

$$\int_\Omega \frac{u_k - u_0}{h} y\ \mathrm{d}x + \int_\Omega A(u_k, \nabla u_k) \cdot \nabla y\ \mathrm{d}x + \int_\Omega g(u_k) y\ \mathrm{d}x = \int_\Omega \hat{f}((k-1)h)\, y\ \mathrm{d}x$$

for all $y \in W_0^{1,\beta}(\Omega)$, with

$$u_k \in W_0^{1,p(b(u_{k-1}))}(\Omega), \qquad k = 1, \ldots, N_0.$$

Let us build the function

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & 0 < t \le h, \\ \vdots & \vdots \\ u_j(x), & (j-1)h < t \le jh, \\ \vdots & \vdots \\ u_{N_0}(x), & (N_0 - 1)h < t \le N_0 h. \end{cases}$$

Let also $u_{h,-}(x,t) := u_{k-1}(x)$ for $(k-1)h < t \le kh$, so that the Rothe equation reads

$$\int_\Omega \frac{u_h - u_{h,-}}{h} y \,\mathrm{d}x + \int_\Omega A(u_{h,-}, \nabla u_h) \cdot \nabla y \,\mathrm{d}x + \int_\Omega g(u_h) y \,\mathrm{d}x = \int_\Omega \hat{f}\, y \,\mathrm{d}x, \tag{19}$$

$y \in W_0^{1,\beta}(\Omega)$. Note that the operator in (19) uses the lagged function $u_{h,-}$ in the exponent and u_h in the gradient, reflecting the structure $u_k \in W_0^{1,p(b(u_{k-1}))}(\Omega)$.

Testing (19) with $y = u_h$, summing over k, and using $g(u_k)u_k \ge 0$, we obtain

$$\begin{aligned} &\|u_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \int_0^T \int_\Omega \Big(|\nabla u_h|^{p(b(u_{h,-}))} + \frac{|\nabla u_h|^{2p(b(u_{h,-}))}}{\sqrt{1 + |\nabla u_h|^{2p(b(u_{h,-}))}}} \Big) \,\mathrm{d}x \,\mathrm{d}t \\ &\le \|u_0\|^2_{L^2(\Omega)} + CT, \end{aligned} \tag{20}$$

whence, in particular,

$$\|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u_h\|_{L^\alpha(Q)} \le C \quad \text{uniformly in } h. \tag{21}$$

Step 4. *Passage to the limit $h \to 0$ and identification of the limit χ.* Set $q_h := p(b(u_h))$ and $q := p(b(u))$. From (21), passing to a subsequence (not relabeled), we get

$$\begin{aligned} u_h &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;L^2(\Omega)), && (22) \\ u_h &\rightharpoonup u && \text{in } L^\alpha\big(0,T;W_0^{1,\alpha}(\Omega)\big), && (23) \\ u_h &\to u && \text{in } L^2(Q) \text{ and a.e. in } Q, && (24) \\ u_{h,-} &\to u && \text{in } L^2(Q) \text{ and a.e. in } Q, && (25) \\ q_h &\to q && \text{a.e. in } Q, && (26) \\ A(u_h, \nabla u_h) &\rightharpoonup \chi && \text{in } (L^{\alpha'}(Q))^N, && (27) \\ g(u_h) &\rightharpoonup g(u) && \text{in } L^2(Q). && (28) \end{aligned}$$

Convergences (22)–(23) follow from the uniform bound (20) and reflexivity of $L^{\alpha'}(Q)$. Convergence (24) follows from the compact embedding

$L^\alpha(0,T;W_0^{1,\alpha}(\Omega)) \overset{c}{\hookrightarrow} L^2(Q)$. Convergence (25) follows because $\|u_{h,-} - u_h\|_{L^2(Q)} \to 0$ as $h \to 0$ (standard Rothe estimate). Convergence (26) follows from Lipschitz continuity of $p \circ b$ and (24).

For (28), $|g(u_h)| \le C|u_h|$ gives boundedness in $L^2(Q)$, and a.e. convergence together with Lions' lemma yields the weak limit.

From (20) and (26), applying Lemma 1 with $q_\nu = p(b(u_{h,-}))$, $u_\nu = u_h$, $q = p(b(u))$, we get

$$\iint_Q |\nabla u|^{p(b(u))} \,\mathrm{d}x\,\mathrm{d}t \;\le\; \liminf_{h\to 0} \iint_Q |\nabla u_h|^{p(b(u_{h,-}))} \,\mathrm{d}x\,\mathrm{d}t \;<\; \infty.$$

Therefore, $u \in X(Q)$.

On the other hand, for fixed h and each level k, the strict monotonicity of A in its second argument gives, for every $v_k \in W_0^{1,\beta}(\Omega)$,

$$\int_\Omega \big[A(u_k,\nabla u_k) - A(u_k,\nabla v_k)\big]\cdot(\nabla u_k - \nabla v_k)\ \mathrm{d}x \ge 0.$$

Let $v \in L^\beta(0,T;W_0^{1,\beta}(\Omega))$ and set $v_k := v(\cdot,kh)$. Expanding and summing over $k = 1,\dots,N_0$, we get

$$\int_Q A(u_h,\nabla u_h)\cdot(\nabla u_h - \nabla v)\ \mathrm{d}x\ \mathrm{d}t \;\ge\; \int_Q A(u_h,\nabla v)\cdot(\nabla u_h - \nabla v)\ \mathrm{d}x\ \mathrm{d}t. \tag{29}$$

We choose $z = u_h - v$ as a test functions in the variational formulation of (3), and obtain

$$\begin{aligned}\int_Q A(u_h,\nabla u_h)\cdot(\nabla u_h - \nabla v)\ \mathrm{d}x\ \mathrm{d}t &\le \int_Q \frac{u_{h,-} - u_h}{h}(u_h - v)\ \mathrm{d}x\ \mathrm{d}t \\ &+ \int_Q \hat{f}\,(u_h - v)\ \mathrm{d}x\ \mathrm{d}t - \int_Q g(u_h)(u_h - v)\ \mathrm{d}x\ \mathrm{d}t =: R_h.\end{aligned} \tag{30}$$

Combining (29) and (30),

$$\int_Q A(u_h,\nabla v)\cdot(\nabla u_h - \nabla v)\ \mathrm{d}x\ \mathrm{d}t \;\le\; R_h. \tag{31}$$

To pass to the limit in the term $A(u_h,\nabla v)$ when $h \to 0$, we have by (26), that $q_h \to q$ for a.e. (x,t),

$$A(u_h,\nabla v)(x,t) \;\longrightarrow\; A(u,\nabla v)(x,t) \quad \text{a.e. in } Q.$$

and, since $\alpha \le q_h \le \beta$ uniformly, the pointwise bound

$$|A(u_h,\nabla v)| \;\le\; C\big(|\nabla v|^{\alpha-1} + |\nabla v|^{\beta-1} + |\nabla v|^{2\alpha-1} + |\nabla v|^{2\beta-1} + 1\big) \;\in\; L^{\alpha'}(Q),$$

provides a *fixed* dominating function in $L^{\alpha'}(Q)$ (independent of h). The dominated convergence theorem therefore applies, giving

$$A(u_h, \nabla v) \;\longrightarrow\; A(u, \nabla v) \quad \text{strongly in } (L^{\alpha'}(Q))^N. \tag{32}$$

By standard Rothe estimates, the discrete time-derivative term satisfies

$$\int_Q \frac{u_{h,-} - u_h}{h}(u_h - v)\ \mathrm{d}x\ \mathrm{d}t \;\longrightarrow\; -\int_Q u_t(u-v)\ \mathrm{d}x\ \mathrm{d}t.$$

The remaining terms in R_h converge, and using (24) and (28), we get

$$R_h \longrightarrow -\int_Q u_t(u-v)\ \mathrm{d}x\ \mathrm{d}t + \int_Q f(u-v)\ \mathrm{d}x\ \mathrm{d}t - \int_Q g(u)(u-v)\ \mathrm{d}x\ \mathrm{d}t =: R.$$

From (32) and the weak convergence $\nabla u_h \to \nabla u$ in $(L^{\alpha}(Q))^N$,

$$\int_Q A(u_h, \nabla v)\cdot(\nabla u_h - \nabla v)\ \mathrm{d}x\ \mathrm{d}t \;\longrightarrow\; \int_Q A(u, \nabla v)\cdot(\nabla u - \nabla v)\ \mathrm{d}x\ \mathrm{d}t.$$

Hence, passing $h \to 0$ in (31),

$$\int_Q A(u, \nabla v)\cdot(\nabla u - \nabla v)\ \mathrm{d}x\ \mathrm{d}t \;\le\; R, \qquad \forall\, v \in L^{\beta}\big(0,T; W_0^{1,\beta}(\Omega)\big). \tag{33}$$

Next, we will obtain the identification $\chi = A(u, \nabla u)$. By passing $h \to 0$ in the summed Rothe identities using (22)–(28), we obtain

$$\int_Q \chi\cdot\nabla\phi\ \mathrm{d}x\ \mathrm{d}t = \int_Q \big[f\phi - u_t\phi - g(u)\phi\big]\ \mathrm{d}x\ \mathrm{d}t, \qquad \phi \in C_0^1(Q). \tag{34}$$

Here, we observe that, since $\chi \in (L^{\alpha'}(Q))^N$ and $u_t \in L^{\alpha}(0,T; W_0^{1,\alpha}(\Omega)'$, both sides of (34) extend by continuity to all $\phi \in L^{\alpha}(0,T; W_0^{1,\alpha}(\Omega)$. Taking $\phi = u - v$ in (34) shows that $R = \int_Q \chi\cdot(\nabla u - \nabla v)\ \mathrm{d}x\ \mathrm{d}t$, and thus, inequality (33) becomes

$$\int_Q \big[A(u, \nabla v) - \chi\big]\cdot(\nabla u - \nabla v)\ \mathrm{d}x\ \mathrm{d}t \;\le\; 0, \qquad v \in L^{\beta}\big(0,T; W_0^{1,\beta}(\Omega)\big).$$

Now substitute $v = u - \theta\phi$ for arbitrary $\phi \in L^{\beta}(0,T; W_0^{1,\beta}(\Omega))$ and $\theta > 0$:

$$\int_Q \big[A(u, \nabla u - \theta\nabla\phi) - \chi\big]\cdot\theta\nabla\phi\ \mathrm{d}x\ \mathrm{d}t \;\le\; 0. \tag{35}$$

Divide (35) by $\theta > 0$ and let $\theta \to 0$. Since $\nabla v = \nabla u - \theta\nabla\phi$ is now a *fixed* vector as $\theta \to 0$, and $q = p(b(u))$ does not depend on θ, the dominated convergence theorem, applied to $A(u, \nabla u - \theta\nabla\phi)$ with $\theta \in [0,1]$, gives

$$\int_Q \big[A(u, \nabla u) - \chi\big]\cdot\nabla\phi\ \mathrm{d}x\ \mathrm{d}t \;\le\; 0.$$

Now, replacing ϕ by $-\phi$, we arrive at

$$\chi = A(u, \nabla u) \quad \text{a.e. in } Q.$$

Step 5. *Verification of the weak formulation and regularity.* Taking the limit $h \to 0$ in the Rothe identity (integrated against test functions $\varphi \in C^1(Q)$ with $\varphi(\cdot, T) = 0$) and using the convergences of Steps 3–4, we obtain precisely (2). The regularity $\nabla u \in (L^{p(b(u))}(Q))^N$ follows from Lemma 1 applied to the sequences u_h.

Finally, the time-derivative satisfies $u_t \in L^{\alpha'}(0, T; W^{-1,\alpha'}(\Omega))$ (in the sense of distributions, which follows from the weak formulation by a standard duality argument). Thus, we conclude

$$u \in C\big([0, T]; L^2(\Omega)\big),$$

and the initial condition $u(\cdot, 0) = u_0$ is attained in $L^2(\Omega)$. Theorem 1 is verified. $\square$

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