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Uniqueness Theorems for Multiple Series by Vilenkin and Generalized Haar Systems

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Abstract. In this paper we discuss the uniqueness property of a summation method for multiple series with respect to Vilenkin and generalized Haar systems. It is proved that if the multiple series with respect to these systems is a.e. summable by that method to an integrable function on $[0,1)^d$ and satisfies an extra condition, then it is the Fourier series of this function.

Key Words: Uniqueness Theorem, Summation Methods, Vilenkin System, Haar System, Fourier Series
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Introduction

First we recall the definitions of Vilenkin and generalized Haar systems (see [1]). Let $\{p_k\}$, $p_k \geq 2$, $k \in \mathbb{N}$, be a sequence of natural numbers and $m_0 = 1$, $m_k = m_{k-1}p_k$, $k \in \mathbb{N}$. Then every nonnegative integer n is uniquely represented by the series

$$n = \sum_{k=1}^{\infty} n_k m_{k-1}, \quad \text{where} \quad n_k \in \{0, 1, \ldots, p_k - 1\},\ k \in \mathbb{N}.$$

Every point $x \in [0,1)$ can be represented in the following form:

$$x = \sum_{k=1}^{\infty} \frac{x_k}{m_k}, \quad \text{where} \quad x_k \in \{0, 1, \ldots, p_k - 1\},\ k \in \mathbb{N},$$

where we assume that for infinitely many $k \in \mathbb{N}$, $x_k \neq p_k - 1$.

Let n be a natural number of the form $n = m_k + r(p_{k+1} - 1) + s - 1$, where $0 \leq r \leq m_k - 1$, $1 \leq s \leq p_{k+1} - 1$. Denote

$$\chi_0(x) \equiv 1,$$

$$\chi_n(x) := \chi_{r,s}^{(k)}(x) := \begin{cases} \sqrt{m_k}\exp\left(2\pi i \frac{x_{k+1}}{p_{k+1}} s\right), & \text{if } x \in \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right), \\ 0 & \text{if } x \notin \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right), \end{cases} \tag{1}$$

The system $\{\chi_n(x)\}_{n=0}^{\infty}$ is called the generalized Haar system generated by the sequence $\{p_k\}$, $k \in \mathbb{N}$. When $p_k = 2$ for all $k \in \mathbb{N}$, the generalized Haar system coincides with the classical Haar system.

The functions

$$R_k(x) := \exp\left(\frac{2\pi i x_k}{p_k}\right), \quad k \in \mathbb{N},$$

are called generalized Rademacher functions.

The Vilenkin system $\{\Psi_n(x)\}_{n=0}^{\infty}$ is defined as follows:

$$\Psi_0(x) \equiv 1 \quad \text{and} \quad \Psi_n(x) := \prod_{k=1}^{\infty} R_k^{n_k}(x) = \exp\left(2\pi i \sum_{k=1}^{\infty} \frac{n_k x_k}{p_k}\right).$$

Note that in the case $p_k = 2$, $k \in \mathbb{N}$, Vilenkin system coincides with the Walsh system. These systems were defined in 1947 by N. Vilenkin [1] and since then are investigated by many mathematicians. When $\sup\{p_k\} = +\infty$, Vilenkin and generalized Haar systems are essentially different from the Walsh and Haar systems, correspondingly.

In the sequel, writing $\{f_n\}_{n=0}^{\infty}$ we mean either the generalized Haar system or Vilenkin system.

In [2] and [3] a new method of summation is defined for Vilenkin and generalized Haar systems. We denote

$$\mathcal{J}_k := \left\{ \left[\frac{j}{m_k}, \frac{j+1}{m_k}\right) : \quad j = 0, 1, \ldots, m_k - 1 \right\}, \qquad k = 0, 1, 2, \ldots. \tag{2}$$

For the interval $J \in \mathcal{J}_k$, $k \in \mathbb{N}$, we denote by $\tilde{J}$ the interval from $\mathcal{J}_{k-1}$, which contains J. The intervals $(J)_l$, $(l \in \mathbb{Z})$ are defined in the following way:

1. $(J)_0 = J, \quad (J)_l \in \mathcal{J}_k, \quad (J)_l \subset \tilde{J},$
2. the right endpoint of $(J)_l$ coincides with the left endpoint of $(J)_{l+1}$, with the convention that the endpoints of $\tilde{J}$ are identified, i.e. if the right endpoint of $(J)_l$ is $\dfrac{j}{m_{k-1}}$, then the left endpoint of $(J)_{l+1}$ is $\dfrac{j-1}{m_{k-1}}$.

For each interval $J \in \mathcal{J}_k$ and natural number $q \leq \dfrac{p_k}{2}$ we denote

$$(J)^q := \bigcup_{l=-q}^{q} (J)_l, \tag{3}$$

$$\varphi_J^{(q)}(t) := \begin{cases} \dfrac{m_k}{q}\left(1 - \dfrac{|l|}{q}\right), & \text{if} \quad t \in (J)_l \quad |l| < q, \\ 0, & \text{if} \quad t \notin (J)^{q-1}. \end{cases} \tag{4}$$

It is clear that $(J)^0 = (J)_0 = J$ and

$$\varphi_J^{(1)}(t) = m_k \mathbb{I}_J(t), \quad \int_0^1 \varphi_J^{(q)}(t)dt = \int_{(J)^{q-1}} \varphi_J^{(q)}(t)dt = 1 \quad \text{for any } q \le \frac{p_k}{2}, \tag{5}$$

where by $\mathbb{I}_J$ we denote the characteristic function of J.

For each natural number k and for each $x \in [0,1)$ denote by $I_{k,x}$ the interval from $\mathcal{J}_k$, which contains the point x. Sometimes, when $J = I_{k,x}$, instead of $\varphi_J^{(q)}$ we use the notation $\varphi_{k,x}^{(q)}$, i.e. $\varphi_{k,x}^{(q)}(t) := \varphi_{I_{k,x}}^{(q)}(t)$.

Taking into account the definition of the system $\{f_n(x)\}$, it is clear that for each $\varphi_{k,x}^{(q)}$ we have

$$(f_n, \varphi_{k,x}^{(q)}) := \int_0^1 f_n(t)\varphi_{k,x}^{(q)}(t)dt = 0, \quad \text{when} \quad n \ge m_k.$$

Therefore, for each series

$$\sum_{n=0}^{\infty} a_n f_n(x), \tag{6}$$

for any $x \in [0,1)$ and any pair of natural numbers k, q, $2q < p_k$, the sums

$$\sigma_{k,q}(x) := \sum_{n=0}^{\infty} a_n (f_n, \varphi_{k,x}^{(q)})$$

are well-defined. Let $S_m(x)$ be the partial sum of the series (6), i.e.

$$S_m(x) := \sum_{n=0}^{m} a_n f_n(x).$$

It is obvious that for any $k \in \mathbb{N}$ and $q < p_k/2$

$$\sigma_{k,1}(x) = S_{m_k-1}(x) \quad \text{and} \quad \sigma_{k,q}(x) = \int_0^1 S_{m_r-1}(t)\varphi_{k,x}^{(q)}(t)dt \quad \forall r \ge k. \tag{7}$$

Denote

$$S^*(x) := \sup_m |S_m(x)|, \qquad \text{and} \qquad \sigma^*(x) := \sup_{k,q} |\sigma_{k,q}(x)|. \tag{8}$$

We will denote by $\text{mes}_d(E)$ the Lebesgue measure of E in $\mathbb{R}^d$. In the case when $d = 1$ we will write $\text{mes}(E)$ instead of $\text{mes}_1(E)$.

The following two theorems were announced in [3] and proved in [4] and [5].

Theorem 1 *There exists a constant $C > 0$, independent both of coefficients $\{a_n\}$ and the sequence $\{p_k\}$, such that*

$$\sigma^*(x) < C \cdot S^*(x) \qquad \text{for any} \quad x \in [0,1).$$

In addition, if the series (6) *converges to $S(x)$ at a point x, then*

$$\lim_{k\to\infty} \sigma_{k,q}(x) = S(x).$$

Theorem 2 *If the sums $\sigma_{k,q}(x)$ converge in measure to a function $f(x)$, as $k \to \infty$, and for some increasing sequence $\lambda_\nu \to \infty$ the condition*

$$\lim_{\nu\to\infty} \lambda_\nu \cdot \operatorname{mes}\{x \in [0,1) : \ \sigma^*(x) > \lambda_\nu\} = 0 \tag{9}$$

holds, then for every natural number n,

$$a_n = \lim_{\nu\to\infty} \int_0^1 [f(x)]_{\lambda_\nu} \overline{f_n(x)} dx,$$

where

$$[g(x)]_\lambda := \begin{cases} g(x), & \text{if } \ |g(x)| \le \lambda, \\ 0, & \text{if } \ |g(x)| > \lambda. \end{cases}$$

The next theorem is proved in [2], Theorems 2 and 3.

Theorem 3 *If the series* (6) *is the Fourier series of an integrable function f, then*

$$\lim_{k\to\infty} \sigma_{k,q}(x) = f(x) \quad \text{a.e. on} \ \ [0,1)$$

and

$$\lim_{\lambda\to\infty} \lambda \cdot \operatorname{mes}\{x \in [0,1) : \ \sigma^*(x) > \lambda\} = 0.$$

Uniqueness theorems for classical series by Vilenkin and generalized Haar systems, generated by a bounded sequence $\{p_k\}$, when the majorant of partial sums satisfies some condition, were considered in [6] and [7]. A similar theorem for the multiple Haar system was proved in [8]. For the trigonometric system it is proved in [9] that if a multiple trigonometric series is summable almost everywhere by the Riemann's method to an integrable function f and satisfies an additional condition similar to (9), then it is the Fourier series of f.

In this paper we consider analogous questions for multiple series by Vilenkin and generalized Haar systems generated by an arbitrary sequence $\{p_n\}$.

Let $\mathbb{N}_0$ be the set of non-negative integers. For any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in [0,1)^d$, $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d$, $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$ and $\mathbf{q} = (q_1, q_2, \ldots, q_d) \in \mathbb{N}^d$, $(2q_i < p_{k_i}, \ \ i = 1, 2, \ldots, d)$, denote

$$f_{\mathbf{n}}(\mathbf{x}) := f_{n_1}(x_1) f_{n_2}(x_2) \cdots f_{n_d}(x_d),$$

$$(f_{\mathbf{n}}, \varphi^{(\mathbf{q})}_{\mathbf{k},\mathbf{x}}) := \prod_{i=1}^{d} (f_{n_i}, \varphi^{(q_i)}_{k_i,x_i}) = \prod_{i=1}^{d} \int_0^1 f_{n_i}(t_i)\varphi^{(q_i)}_{k_i,x_i}(t_i)dt_i$$

and consider the following series

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} f_{n_1}(x_1) f_{n_2}(x) \cdots f_{n_d}(x_d). \tag{10}$$

We denote

$$\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{x}) := \sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} (f_{\mathbf{n}}, \varphi^{(\mathbf{q})}_{\mathbf{k},\mathbf{x}}).$$

In this paper, writing $\mathbf{k} \to \infty$ we mean $\min\{k_i\} \to \infty$.

The following theorem holds:

Theorem 4 *If the sums $\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{x})$ of (10) converge a.e. on the cube $[0,1)^d$ to an integrable function $f(\mathbf{x})$ as $\mathbf{k} \to \infty$, and*

$$\lim_{\lambda\to\infty} \lambda \cdot \mathrm{mes}_d \Big\{ \mathbf{x} \in [0,1)^d : \sup_{\mathbf{k},\mathbf{q}} |\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{x})| > \lambda \Big\} = 0,$$

then the series (10) is the Fourier series of f with respect to the system $\{f_{\mathbf{n}}\}$, i.e.

$$a_{\mathbf{n}} = \int_{[0,1)^d} f(\mathbf{x}) \overline{f_{\mathbf{n}}(\mathbf{x})} d\mathbf{x}.$$

Theorem 4 follows from the more general theorem 5. Before formulating it, we introduce some notations. Suppose that

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}}(\mathbf{r},\mathbf{s}) f_{\mathbf{n}}(\mathbf{x})$$

is a multiple series with respect to a system $\{f_n\}$, where coefficients $a_{\mathbf{n}}(\mathbf{r},\mathbf{s})$ depend on $\mathbf{r} \in \mathbf{R} \subset \mathbb{N}^m$ and $\mathbf{s} \in \mathbf{S} \subset \mathbb{R}^l$. We denote

$$\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x}) := \sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}}(\mathbf{r},\mathbf{s}) (f_{\mathbf{n}}, \varphi^{(\mathbf{q})}_{\mathbf{k},\mathbf{x}}) \quad \text{and} \quad \sigma^*(\mathbf{x}) := \sup_{\mathbf{k},\mathbf{q},\mathbf{r},\mathbf{s}} |\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x})|.$$

For any positive number λ we set

$$E_\lambda := \{\mathbf{x} \in [0,1)^d : \sigma^*(\mathbf{x}) > \lambda\}.$$

Theorem 5 *If the sums $\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x})$ converge a.e. on the cube $[0,1)^d$ to an integrable function $f(\mathbf{x})$ as $\mathbf{k} \to \infty$ and $\mathbf{r} \to \infty$, (i.e. $\min\{k_1, k_2, \dots, k_d, r_1, \dots, r_m\} \to \infty$), and*

$$\lim_{\lambda\to\infty} \lambda \cdot \mathrm{mes}_d(E_\lambda) = 0,$$

then for any $\mathbf{n} \in \mathbb{N}_0^d$,

$$\lim_{\mathbf{r}\to\infty} a_{\mathbf{n}}(\mathbf{r},\mathbf{s}) = \int_{[0,1)^d} f(\mathbf{x}) \overline{f_{\mathbf{n}}(\mathbf{x})} d\mathbf{x}.$$

1 Auxiliary Results

The following lemma was proved in [5], Lemma 3.

Lemma 1 *Let $I \in \mathcal{J}_s$ and $E \subset I$. If*

$$\operatorname{mes}(E) < \frac{1}{20p_{s+1}}\operatorname{mes}(I),$$

then for any $\nu > s$ the characteristic function of I can be represented in the form

$$\mathbb{I}_I(t) = \sum_{k=s+1}^{\nu}\sum_{\Delta\in\Omega_k^1}\alpha_\Delta\varphi_\Delta^{(q_\Delta)}(t) + \sum_{k=s+1}^{\nu-1}\sum_{\Delta\in\Omega_k^2}\beta_\Delta\varphi_\Delta^{(1)}(t) + \sum_{\Delta\in\Omega_\nu^3}\gamma_\Delta\varphi_\Delta^{(1)}(t), \quad (11)$$

where $\Omega_k^i \subset \mathcal{J}_k,\ i = 1,2,3,$ with the following properties:

$$\alpha_\Delta \geq 0, \quad \beta_\Delta \geq 0, \quad \gamma_\Delta \geq 0, \quad (12)$$

$$(\operatorname{supp}(\varphi_\Delta^{(q_\Delta)})\cap E) > \frac{1}{6}\operatorname{mes}(\operatorname{supp}(\varphi_\Delta^{(q_\Delta)})), \quad \text{if } \Delta\in\Omega_k^1,\ k = s+1,\dots,\nu, \quad (13)$$

$$\operatorname{mes}(\operatorname{supp}(\varphi_\Delta^{(1)})\cap E) > \frac{1}{20}\operatorname{mes}(\operatorname{supp}(\varphi_\Delta^{(1)})), \quad \text{if } \Delta\in\Omega_k^2,\ k = s+1,\dots,\nu-1 \quad (14)$$

and if $\Delta \in \Omega_k^i$ for some k and i, then

$$\operatorname{mes}(\Delta\cap E) \leq \frac{1}{2}\operatorname{mes}(\Delta).$$

The next lemma was proved in [9], Lemma 1.

Lemma 2 *Let d be a natural number larger than 1 and let $\varphi(\mathbf{x})$ be a non-negative function defined on some d-dimensional cube $[a,b]^d$. If*

$$\liminf_{\lambda\to+\infty}\lambda\cdot\operatorname{mes}_d\{\mathbf{x} = (x_1,x_2,\dots,x_d)\in[0,1]^d :\ \varphi(\mathbf{x}) > \lambda\} = 0,$$

then

$$\operatorname{mes}_{d-1}\Big\{(x_2,\dots,x_d)\in[0,1]^{d-1} : \liminf_{\lambda\to+\infty}\lambda\cdot\operatorname{mes}\{x_1\in[0,1] :\ \varphi(\mathbf{x}) > \lambda\} = 0\Big\} = (b-a)^{d-1}.$$

Lemma 3 *Suppose that the coefficients of the series* (6) *depend on some parameter $\mathbf{r}\in\mathbf{R}$ (i.e. $a_n = a_n(\mathbf{r})$),*

$$\sigma_{k,q}(\mathbf{r},x) := \sum_{n=0}^{\infty}a_n(\mathbf{r})(f_n,\varphi_{k,x}^{(q)}) \qquad \text{and} \qquad \sigma^*(x) := \sup_{k,q,\mathbf{r}}|\sigma_{k,q}(\mathbf{r},x)|.$$

If for some positive number λ and some $k \in \mathbb{N}$ the inequality

$$\operatorname{mes}\{x \in [0,1): \ \sigma^*(x) > \lambda\} < \frac{1}{m_k} \tag{15}$$

holds, then for any $n < m_k$ and $\mathbf{r} \in \mathbf{R}$ the coefficients $a_n(\mathbf{r})$ satisfy

$$|a_n(\mathbf{r})| \leq \lambda.$$

Proof. Suppose for some $\lambda > 0$ and $k \in \mathbb{N}$ the inequality (15) holds. Note that for any fixed $\mathbf{r}$ the function $\sigma_{k,1}(\mathbf{r}, x) = S_{m_k-1}(\mathbf{r}, x)$ (see (7)) is constant on each interval $J \in \mathcal{J}_k$ with length $\dfrac{1}{m_k}$. Therefore, according to the inequality (15) and

$$\operatorname{mes}\{x \in [0,1): \ \sigma_{k,1}(\mathbf{r}, x) > \lambda\} < \operatorname{mes}\{x \in [0,1): \ \sigma^*(x) > \lambda\} < \frac{1}{m_k},$$

we obtain that

$$|\sigma_{k,1}(\mathbf{r}, x)| = \left| \sum_{n=0}^{m_k-1} a_n(\mathbf{r}) f_n(x) \right| \leq \lambda \quad \text{for all } x \in [0,1) \quad \text{and} \quad \mathbf{r} \in \mathbf{R}.$$

Since $f_n(x)$ is an orthonormal system on $[0,1)$, then in view of the last inequality, we obtain that for any $n < m_k$ and $\mathbf{r} \in \mathbf{R}$ the following inequality holds:

$$|a_n(\mathbf{r})| \leq \lambda.$$

□

In [2], for any integrable function f on $[0,1)$, the following function is defined:

$$\mathcal{M}^*(f, x) := \sup_{\substack{q, J: x \in J \in \mathcal{J}_k \\ 0 \leq q \leq \frac{p_k}{2}}} \frac{1}{\operatorname{mes}(J)^q} \int_{(J)^q} |f(t)| dt,$$

and it is proved that

$$\operatorname{mes}\{x : \ \mathcal{M}^*(f, x) > \lambda\} \leq \frac{3}{\lambda} \|f\|_1, \tag{16}$$

$$\lim_{\lambda \to \infty} \lambda \cdot \operatorname{mes}\{x : \ \mathcal{M}^*(f, x) > \lambda\} = 0. \tag{17}$$

2 Proof of the main results

We prove Theorem 5 by mathematical induction. The induction is carried out over the dimension d.

First let us separately state and prove the theorem in the case $d = 1$.

Suppose that

$$\sum_{n=0}^{\infty} a_n(\mathbf{r},\mathbf{s}) f_n(x)$$

is a series with respect to the system $\{f_n\}$, where coefficients $a_n(\mathbf{r},\mathbf{s})$ depend on $\mathbf{r} \in \mathbf{R} \subset \mathbb{N}^m$ and $\mathbf{s} \in \mathbf{S} \subset \mathbb{R}^l$. Denote

$$\sigma_{k,q}(\mathbf{r},\mathbf{s},x) := \sum_{n=0}^{\infty} a_n(\mathbf{r},\mathbf{s})(f_n, \varphi_{k,x}^{(q)}) \qquad \text{and} \qquad \sigma^*(x) := \sup_{k,q,\mathbf{r},\mathbf{s}} |\sigma_{k,q}(\mathbf{r},\mathbf{s},x)|.$$

Theorem 6 *If the sums $\sigma_{k,q}(\mathbf{r},\mathbf{s},x)$ converge a.e. on the interval $[0,1)$ to an integrable function $f(x)$ as $k \to \infty$ and $\mathbf{r} \to \infty$, (i.e. $\min\{k, r_1, \dots, r_m\} \to \infty$), and*

$$\lim_{\lambda\to\infty} \lambda \cdot \mathrm{mes}\{x \in [0,1) : \ \sigma^*(x) > \lambda\} = 0, \tag{18}$$

then for any $n \in \mathbb{N}_0$,

$$\lim_{\mathbf{r}\to\infty} a_n(\mathbf{r},\mathbf{s}) = \int_0^1 f(x)\overline{f_n(x)}dx.$$

Proof. First consider the case when $f(x) = 0$.

Suppose the sums $\sigma_{k,q}(\mathbf{r},\mathbf{s},x)$ converge a.e. to $f(x) = 0$ and the condition (18) holds.

Let n be a non-negative integer and i be the smallest natural number for which $n < m_i$. It is clear that $f_n(x)$ is constant on each interval $I_u \in \mathcal{J}_i$. Denote by $f_n(I_u)$ the value of f_n on the interval I_u. Notice that

$$a_n(\mathbf{r},\mathbf{s}) = \int_0^1 S_{m_i-1}(\mathbf{r},\mathbf{s},t)\overline{f_n(t)}dt = \sum_{u=1}^{m_i} \overline{f_n(I_u)} \int_{I_u} S_{m_i-1}(\mathbf{r},\mathbf{s},t)dt,$$

where

$$S_N(\mathbf{r},\mathbf{s},x) := \sum_{n=0}^{N} a_n(\mathbf{r},\mathbf{s}) f_n(x).$$

Therefore, in order to prove theorem 6 in the case $f(x) = 0$, it is sufficient to prove that for any natural number i and for each $I \in \mathcal{J}_i$, we have

$$\lim_{\mathbf{r}\to\infty} \int_I S_{m_i-1}(\mathbf{r},\mathbf{s},t)dt = 0. \tag{19}$$

Let $I \in \mathcal{J}_i$ for some $i \in \mathbb{N}$ and $\varepsilon \in \left(0, \frac{1}{20p_{i+1}}\right)$. Denote

$$E_\lambda := \{x \in I : \ \sigma^*(x) > \lambda\}. \tag{20}$$

In view of (18), one can choose a number $\lambda > 1$ such that

$$\lambda \cdot \text{mes}(E_\lambda) < \varepsilon \cdot \text{mes}(I). \tag{21}$$

Since $\text{mes}(E_\lambda) < \dfrac{1}{20p_{i+1}} \cdot \text{mes}(I)$, then, applying Lemma 1 for any $\nu > i$, we obtain the following representation for the characteristic function of I

$$\mathbb{I}_I(t) = \sum_{k=i+1}^{\nu} \sum_{\Delta \in \Omega_k^1} \alpha_\Delta \varphi_\Delta^{(q_\Delta)}(t) + \sum_{k=i+1}^{\nu-1} \sum_{\Delta \in \Omega_k^2} \beta_\Delta \varphi_\Delta^{(1)}(t) + \sum_{\Delta \in \Omega_\nu^3} \gamma_\Delta \varphi_\Delta^{(1)}(t), \tag{22}$$

where $\Omega_k^i \subset \mathcal{J}_k, \ i = 1, 2, 3$, with the following properties:

$$\alpha_\Delta \geq 0, \quad \beta_\Delta \geq 0, \quad \gamma_\Delta \geq 0, \tag{23}$$

$$\text{mes}(\text{supp}(\varphi_\Delta^{(q_\Delta)}) \cap E_\lambda) > \frac{1}{6}\text{mes}(\text{supp}(\varphi_\Delta^{(q_\Delta)})), \quad \text{if} \ \ \Delta \in \Omega_k^1, \ k = i+1, \dots, \nu, \tag{24}$$

$$\text{mes}(\text{supp}(\varphi_\Delta^{(1)}) \cap E_\lambda) > \frac{1}{20}\text{mes}(\text{supp}(\varphi_\Delta^{(1)})), \quad \text{if} \ \ \Delta \in \Omega_k^2, \ k = i+1, \dots, \nu-1 \tag{25}$$

and if $\Delta \in \Omega_k^i$ for some k and i, then

$$\text{mes}(\Delta \cap E_\lambda) \leq \frac{1}{2}\text{mes}(\Delta). \tag{26}$$

According to (5), (22) and (23) we get that

$$\sum_{\Delta \in \Omega_\nu^3} \gamma_\Delta \leq \text{mes}(I). \tag{27}$$

For any $\nu > i$, we denote

$$F_\nu := \left(\bigcup_{k=i+1}^{\nu} \bigcup_{\Delta \in \Omega_k^1} \text{supp}\left(\varphi_\Delta^{(q_\Delta)}\right) \right) \bigcup \left(\bigcup_{k=i+1}^{\nu-1} \bigcup_{\Delta \in \Omega_k^2} \text{supp}\left(\varphi_\Delta^{(1)}\right) \right). \tag{28}$$

In view of (24) and (25) we have that

$$\mathcal{M}^*(\mathbb{I}_{E_\lambda}, x) > \frac{1}{20} \quad \text{for any } x \in F_\nu,$$

which means (see also (16)) that

$$\text{mes}(F_\nu) \leq 60\text{mes}(E_\lambda). \tag{29}$$

Combining the last inequality with (5), (22) and (23), we obtain that

$$\sum_{k=i+1}^{\nu}\sum_{\Delta\in\Omega_k^1}\alpha_\Delta+\sum_{k=i+1}^{\nu-1}\sum_{\Delta\in\Omega_k^2}\beta_\Delta+\sum_{\Delta\in\Omega_\nu^3,\Delta\subset F_\nu}\gamma_\Delta=\sum_{k=i+1}^{\nu}\sum_{\Delta\in\Omega_k^1}\alpha_\Delta\int_I\varphi_\Delta^{(q_\Delta)}(t)dt+$$
$$+\sum_{k=i+1}^{\nu-1}\sum_{\Delta\in\Omega_k^2}\beta_\Delta\int_I\varphi_\Delta^{(1)}(t)dt+\sum_{\Delta\in\Omega_\nu^3,\Delta\subset F_\nu}\gamma_\Delta\int_I\varphi_\Delta^{(1)}(t)dt\le$$
$$\int_{F_\nu}1dt\le 60\mathrm{mes}(E_\lambda). \quad (30)$$

According to (22), (7) and the definition of functions f_n, we get that for any $\nu>i$,

$$\int_I S_{m_i-1}(\mathbf{r},\mathbf{s},t)dt=\int_I S_{m_\nu-1}(\mathbf{r},\mathbf{s},t)dt=$$
$$\sum_{k=i+1}^{\nu}\sum_{\Delta\in\Omega_k^1}\alpha_\Delta\int_I S_{m_k-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(q_\Delta)}(t)dt+$$
$$+\sum_{k=i+1}^{\nu-1}\sum_{\Delta\in\Omega_k^2}\beta_\Delta\int_I S_{m_k-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(1)}(t)dt+$$
$$+\sum_{\Delta\in\Omega_\nu^3,\Delta\subset F_\nu}\gamma_\Delta\int_I S_{m_\nu-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(1)}(t)dt+$$
$$+\sum_{\Delta\in\Omega_\nu^3,\Delta\subset I\setminus F_\nu}\gamma_\Delta\int_I S_{m_\nu-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(1)}(t)dt=$$
$$:A_{\nu,1}+A_{\nu,2}+A_{\nu,3}+A_{\nu,4}. \quad (31)$$

Since for any fixed (admissible) $k>i,q,\mathbf{r},\mathbf{s}$ the function $\sigma_{k,q}(\mathbf{r},\mathbf{s},x)$ is constant on the interval $\Delta\in\mathcal{J}_k$ and (see (5) and (7))

$$\sigma_{k,q}(\mathbf{r},\mathbf{s},x)=\int_0^1 S_{m_k-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(q)}(t)dt=\int_I S_{m_k-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(q)}(t)dt,\quad x\in\Delta,$$

then according to (26) and (20), we obtain that for any natural $k>i$

$$\left|\int_I S_{m_k-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(q_\Delta)}(t)dt\right|\le\lambda,\quad \text{if}\quad \Delta\in\Omega_k^1$$
$$\left|\int_I S_{m_k-1}(\mathbf{r},\mathbf{s},t)\varphi_\Delta^{(1)}(t)dt\right|\le\lambda,\quad \text{if}\quad \Delta\in\Omega_k^2 \text{ or } \Delta\in\Omega_k^3. \quad (32)$$

Therefore, using (21), (30) and (31) we obtain

$$|A_{\nu,1}|+|A_{\nu,2}|+|A_{\nu,3}|\le 60\lambda\cdot\mathrm{mes}(E_\lambda)\le 60\varepsilon\cdot\mathrm{mes}(I). \quad (33)$$

Denote

$$B_\nu := B_\nu(\mathbf{r},\mathbf{s},q) := \{x \in I \setminus F_\nu : \ |\sigma_{\nu,q}(\mathbf{r},\mathbf{s},x)| > \varepsilon\}, \quad B'_\nu := I \setminus (F_\nu \cup B_\nu).$$

Since $\sigma_{\nu,q}(\mathbf{r},\mathbf{s},x)$ converges a.e. to $f(x)=0$, then there exists N_0 such that

$$\operatorname{mes}(B_\nu) < \operatorname{mes}(E_\lambda) \quad \text{for any} \quad \nu, \ \mathbf{r} \ \text{ with } \min\{\nu, r_1, \dots, r_m\} > N_0. \qquad (34)$$

Note that both B_ν and B'_ν are unions of intervals of $\mathcal{J}_\nu$, because $\sigma_{\nu,q}(\mathbf{r},\mathbf{s},x)$ is constant on each interval $\Delta \in \mathcal{J}_\nu$. Hence, $A_{\nu,4}$ in (31) can be represented as follows

$$A_{\nu,4} = \sum_{\Delta\subset B_\nu} \gamma_\Delta \int_I S_{m_\nu-1}(\mathbf{r},\mathbf{s},t)\varphi^{(1)}_\Delta(t)dt + \sum_{\Delta\subset B'_\nu} \gamma_\Delta \int_I S_{m_\nu-1}(\mathbf{r},\mathbf{s},t)\varphi^{(1)}_\Delta(t)dt =: A'_{\nu,4} + A''_{\nu,4}.$$

According to (5), (32), (21) (22), (27), (28) and (34) we obtain that

$$|A'_{\nu,4}| \le \sum_{\Delta\subset B_\nu} \gamma_\Delta \cdot \lambda = \lambda \sum_{\Delta\subset B_\nu} \gamma_\Delta \int_I \varphi^{(1)}_\Delta(t)dt = \lambda \int_I \mathbb{I}_{B_\nu}(t)dt \le \lambda \cdot \operatorname{mes}(E_\lambda) \le \varepsilon \cdot \operatorname{mes}(I),$$

$$|A''_{\nu,4}| \le \varepsilon \sum_{\Delta\subset B'_\nu} \gamma_\Delta \le \varepsilon \cdot \operatorname{mes}(I),$$

as $\min\{\nu, r_1, \dots, r_m\} > N_0$. Therefore, $|A_{\nu,4}| \le 2\varepsilon \cdot \operatorname{mes}(I)$. The last inequality with (33) and (31) completes the proof of theorem 6 in the case when $f(x) = 0$.

Now suppose the sums $\sigma_{k,q}(\mathbf{r},\mathbf{s},x)$ converge a.e. to an integrable function f and the condition (18) holds. Let $\sum_{n=0}^{\infty} c_n f_n(x)$ be the Fourier series of f and $\tilde\sigma_{k,q}(x) := \sum_{n=0}^{\infty} c_n(f_n, \varphi^{(q)}_{k,x})$. By theorem 3, we have

$$\lim_{k\to\infty} \tilde\sigma_{k,q}(x) = f(x) \quad \text{a.e. on} \quad [0,1)$$

and

$$\lim_{\lambda\to\infty} \lambda \cdot \operatorname{mes}\{x \in [0,1) : \ \sup_{k,q} |\tilde\sigma_{k,q}(x)| > \lambda\} = 0.$$

Denote

$$c_n(\mathbf{r},\mathbf{s}) := a_n(\mathbf{r},\mathbf{s}) - c_n \quad \text{and} \quad \widehat\sigma_{k,q}(\mathbf{r},\mathbf{s},x) := \sum_{n=0}^{\infty} c_n(\mathbf{r},\mathbf{s})(f_n, \varphi^{(q)}_{k,x}).$$

It can be easily seen that

$$\widehat{\sigma}_{k,q}(\mathbf{r},\mathbf{s},x)=\sigma_{k,q}(\mathbf{r},\mathbf{s},x)-\tilde{\sigma}_{k,q}(x)\to 0 \quad \text{a.e. as } k\to\infty,\ \mathbf{r}\to\infty,$$

and

$$\lim_{\lambda\to\infty}\lambda\cdot\text{mes}\{x\in[0,1):\ \sup_{k,q}|\widehat{\sigma}_{k,q}(x)|>\lambda\}\leq \lim_{\lambda\to\infty}\lambda\cdot\text{mes}\left\{x\in[0,1):\ \sup_{k,q}|\sigma_{k,q}(x)|>\frac{\lambda}{2}\right\}+ \lim_{\lambda\to\infty}\lambda\cdot\text{mes}\left\{x\in[0,1):\ \sup_{k,q}|\tilde{\sigma}_{k,q}(x)|>\frac{\lambda}{2}\right\}=0.$$

Therefore, for any $n\in\mathbb{N}_0$,

$$\lim_{\mathbf{r}\to\infty}c_n(\mathbf{r},\mathbf{s})=\lim_{\mathbf{r}\to\infty}a_n(\mathbf{r},\mathbf{s})-c_n=0,$$

which proves the theorem 6.

□

Proof of Theorem 5. Theorem 5 has already been proved in the case $d=1$ (see the previous theorem). Suppose that its statement is true for dimension $d=\nu-1$ and let us prove it for $d=\nu$.

Suppose that

$$\sum_{\mathbf{n}\in\mathbb{N}_0^\nu}a_{\mathbf{n}}(\mathbf{r},\mathbf{s})f_{\mathbf{n}}(\mathbf{x})$$

is a ν-dimensional series with respect to the system $\{f_n\}$, where coefficients $a_{\mathbf{n}}(\mathbf{r},\mathbf{s})$ depend on $\mathbf{r}\in\mathbf{R}\subset\mathbb{N}^m$ and $\mathbf{s}\in\mathbf{S}\subset\mathbb{R}^l$. Also suppose that the sums

$$\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x})=\sum_{\mathbf{n}\in\mathbb{N}_0^\nu}a_{\mathbf{n}}(\mathbf{r},\mathbf{s})(f_{\mathbf{n}},\varphi_{\mathbf{k},\mathbf{x}}^{(\mathbf{q})})$$

converge almost everywhere on $[0,1)^\nu$ to an integrable function f as $\mathbf{k}\to\infty$ and $\mathbf{r}\to\infty$, and

$$\lim_{\lambda\to\infty}\lambda\cdot\text{mes}_\nu\{\mathbf{x}\in[0,1)^\nu:\quad \sigma^*(\mathbf{x})>\lambda\}=0. \tag{35}$$

For any $\mathbf{r}=(r_1,r_2,\dots,r_m)\in\mathbf{R}$, $\mathbf{k}=(k_1,k_2,\dots,k_\nu)\in\mathbb{N}^\nu$, $\mathbf{s}=(s_1,s_2,\dots,s_l)\in\mathbf{S}$, $\mathbf{q}=(q_1,q_2,\dots,q_\nu)$ and $\mathbf{x}=(x_1,x_2,\dots,x_\nu)\in[0,1)^\nu$ denote

$$\widehat{\mathbf{r}}:=(r_1,r_2,\dots,r_m,k_2,\dots,k_\nu),\quad \widehat{\mathbf{s}}:=(s_1,s_2,\dots,s_l,q_2,\dots q_\nu,x_2,\dots,x_\nu).$$

If for each $n_1\in\mathbb{N}_0$ we denote

$$A_{n_1}(\widehat{\mathbf{r}},\widehat{\mathbf{s}}):=\sum_{n_i\in\mathbb{N}_0,i=2,\dots\nu}a_{\mathbf{n}}(\mathbf{r},\mathbf{s})\prod_{i=2}^{\nu}\left(f_{n_i},\varphi_{k_i,x_i}^{(q_i)}\right),\quad (\mathbf{n}=(n_1,n_2,\dots,n_\nu)), \tag{36}$$

then $\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x})$ can be written as follows:

$$\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x}) = \sum_{n_1=0}^{\infty} A_{n_1}(\widehat{\mathbf{r}},\widehat{\mathbf{s}})\Big(f_{n_1},\varphi_{k_1,x_1}^{(q_1)}\Big) =: \tilde{\sigma}_{k_1,q_1}(\widehat{\mathbf{r}},\widehat{\mathbf{s}},x_1). \tag{37}$$

Also note that

$$\sigma^*(\mathbf{x}) = \sup_{\mathbf{k},\mathbf{q},\mathbf{r},\mathbf{s}} |\sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x})| = \sup_{k_1,q_1,\widehat{\mathbf{r}},\widehat{\mathbf{s}}} |\tilde{\sigma}_{k_1,q_1}(\widehat{\mathbf{r}},\widehat{\mathbf{s}},x_1)|. \tag{38}$$

According to (35) and Lemma 2, we get that for almost all $(x_2,\dots,x_\nu) \in [0,1)^{\nu-1}$,

$$\lim_{\lambda\to\infty} \lambda\cdot \mathrm{mes}\{x_1 \in [0,1) : \ \sigma^*(\mathbf{x}) > \lambda\} = 0, \quad \text{where } \mathbf{x} = (x_1,x_2,\dots,x_\nu). \tag{39}$$

Denote by H the set of all points $(x_2,\dots,x_\nu) \in [0,1)^{\nu-1}$ for which $f(\cdot,x_2,\dots,x_\nu) \in L^1[0,1)$,

$$\mathrm{mes}\{x_1 : \lim_{\substack{\mathbf{k}\to\infty\\ \mathbf{r}\to\infty}} \sigma_{\mathbf{k},\mathbf{q}}(\mathbf{r},\mathbf{s},\mathbf{x}) = f(\mathbf{x}) \ \ a.e.\} = 1$$

and (39) holds. It is clear that

$$\mathrm{mes}_{\nu-1}(H) = \mathrm{mes}_{\nu-1}[0,1)^{\nu-1} = 1.$$

Now, for any $(x_2,\dots,x_\nu) \in H$, applying Theorem 6 we obtain that

$$\lim_{\widehat{\mathbf{r}}\to\infty} A_{n_1}(\widehat{\mathbf{r}},\widehat{\mathbf{s}}) = \int_0^1 f(t,x_2,\dots,x_\nu)\overline{f_{n_1}(t)}dt \quad \text{for all } n_1 \in \mathbb{N}_0. \tag{40}$$

Let n_1 be a natural number and let i be the smallest number for which $n_1 < m_i$. Suppose that $\mathbf{k}' = (k_2,\dots,k_\nu)$, $\mathbf{q}' = (q_2,\dots,q_\nu)$ and

$$A^*_{n_1}(x_2,\dots,x_\nu) := \sup_{\mathbf{k}',\mathbf{q}',\mathbf{r},\mathbf{s}} |A_{n_1}(\widehat{\mathbf{r}},\widehat{\mathbf{s}})|$$

In view of Lemma 1, (37) and (38), we obtain, that if for some $(x_2,\dots,x_\nu)$ and $\lambda \in \mathbb{R}$ the inequality

$$\mathrm{mes}\{x_1 : \ \sigma^*(\mathbf{x}) > \lambda\} < \frac{1}{m_i}$$

holds, then

$$A^*_{n_1}(x_2,\dots,x_\nu) \le \lambda.$$

Therefore, if

$$A^*_{n_1}(x_2,\dots,x_\nu) > \lambda,$$

then

$$\operatorname{mes}\{x_1 : \ \sigma^*(\mathbf{x}) > \lambda\} \geq \frac{1}{m_i},$$

which means that

$$\operatorname{mes}_{\nu-1}\{(x_2, \dots, x_\nu) : \ A^*_{n_1}(x_2, \dots, x_\nu) > \lambda\} \leq \\ m_i \operatorname{mes}_\nu\{\mathbf{x} \in [0,1)^\nu : \ \sigma^*(\mathbf{x}) > \lambda\}.$$

Therefore, using also (35), we get that for any $n_1 \in \mathbb{N}_0$

$$\lim_{\lambda\to\infty} \lambda \cdot \operatorname{mes}_{\nu-1}\{(x_2, \dots, x_\nu) : \ A^*_{n_1}(x_2, \dots, x_\nu) > \lambda\} = 0. \tag{41}$$

Now according to (36), (40), (41) and the inductive assumption for $d = \nu - 1$, we obtain that

$$\lim_{\mathbf{r}\to\infty} a_{\mathbf{n}}(\mathbf{r}, \mathbf{s}) = \int_{[0,1)^\nu} f(\mathbf{x})\overline{f_{\mathbf{n}}(\mathbf{x})}d\mathbf{x}.$$

Theorem 5 is proved.

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