

On Weyl Multipliers of Almost Everywhere Unconditional Convergence of Series With Respect to General Haar and Franklin Systems

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Abstract. We obtain necessary and sufficient conditions for an increasing nonnegative sequence ω_n , $n \in \mathbb{N}$, to be a Weyl multiplier for a.e. unconditional convergence for general Haar or Franklin systems corresponding to an admissible sequence \mathcal{T} .

Key Words: General Haar System, General Franklin System, Almost Everywhere Unconditional Convergence, Weyl Multipliers

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Introduction

In this paper we study Weyl multipliers for almost everywhere unconditional convergence of orthogonal series with respect to general Haar and general Franklin systems generated by admissible partitions of $[0, 1]$. While for classical Haar and Franklin systems the sharp criterion for a nondecreasing sequence $\{\omega_n\}$ to be a Weyl multiplier is known to be $\sum_{n \geq 1} (n\omega_n)^{-1} < \infty$, the situation for general systems depends essentially on the geometry of the underlying partition and on possible rearrangements.

Recall that a functional series $\sum_{n=1}^{\infty} g_n(x)$ is called almost everywhere (a.e.) unconditionally convergent on a set E if, for every permutation $\{\sigma_n\}$ of the natural numbers, the series

$$\sum_{n=1}^{\infty} g_{\sigma_n}(x)$$

converges almost everywhere on E .

The series $\sum_{n=1}^{\infty} g_n(x)$ is called almost everywhere absolutely convergent on a set E if

$$\sum_{n=1}^{\infty} |g_n(x)| < \infty$$

almost everywhere on E .

It is well known that a numerical series converges absolutely if and only if it converges unconditionally. For functional series, however, the notions of almost everywhere unconditional convergence and almost everywhere absolute convergence are not equivalent, since the exceptional set of divergence may vary.

For example, if $\{R_n(x)\}_{n=1}^{\infty}$ is the Rademacher system,

$$\sum_{n=1}^{\infty} a_n^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |a_n| = \infty,$$

then the series

$$\sum_{n=1}^{\infty} a_n R_n(x)$$

converges almost everywhere unconditionally (see [16], p. 36), but diverges almost everywhere absolutely.

Recall one definition from the theory of orthogonal series (see [16]).

Definition 1 Let $\psi = \{\psi_n : n = 1, 2, \dots\}$ be an orthonormal system in $L^2(0, 1)$. A monotonically increasing sequence of positive numbers ω_n is called a Weyl multiplier for a.e. unconditional convergence (for ψ) if any series

$$\sum_{n=1}^{\infty} a_n \psi_n(x)$$

with coefficients satisfying the condition

$$\sum_{n=1}^{\infty} a_n^2 \omega_n < \infty,$$

converges unconditionally a.e.

Orlicz's theorem on Weyl multipliers for a.e. unconditional convergence (see, for example, [17]) was formulated and proved by P. L. Ulyanov in a simpler but equivalent form in the paper [22]: if the sequence $\{\omega_n\}_{n=1}^{\infty}$ satisfies the condition

$$\sum_{n=2}^{\infty} \frac{1}{\omega_n n \ln n} < \infty,$$

then the sequence $\{\omega_n \ln^2 n\}_{n=2}^\infty$ is a Weyl multiplier for a.e. unconditional convergence for any orthonormal system.

P. L. Ulyanov proved (see [22], [21]) that the sequence $\{\omega_n\}_{n=1}^\infty$ is a Weyl multiplier for a.e. unconditional convergence for the Haar system if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n\omega_n} < \infty. \quad (1)$$

In the paper [5], it is proved that $\{\omega_n\}_{n=1}^\infty$ is a Weyl multiplier for a.e. unconditional convergence for the Franklin system if and only if (1) holds. Similar results for the Ciesielski system [2] and the periodic Stromberg system [20] were obtained, respectively, in [7] and [9].

It should also be noted that S. V. Bochkarev [1] and S. Nakata [18] proved the necessity of condition (1) for $\{\omega_n\}_{n=1}^\infty$ to be a Weyl multiplier for a.e. unconditional convergence for the Walsh system. A similar result for the trigonometric system was recently obtained by G. A. Karagulyan [14]. In the paper [13], Weyl multipliers for orthonormal wavelet-type systems were studied. G. A. Karagulyan [15] published a paper on Weyl multipliers for multiple series in the trigonometric system.

While these results provide sharp characterizations for classical systems, the situation becomes more delicate for general Haar and Franklin systems, where the underlying partition geometry plays a crucial role. In the present paper, Weyl multipliers for a.e. unconditional convergence of series in general Haar and Franklin systems are investigated.

We establish results describing when the multiplier property is preserved between the corresponding general Haar and Franklin systems under weak regularity, construct rearrangements that destroy the multiplier property when $\sum_{n \geq 1} (n\omega_n)^{-1} = \infty$, and show that in the quasi-dyadic weakly regular setting the above condition remains necessary and sufficient; we also demonstrate that without weak regularity the behavior can change significantly.

1 Formulation and proof of main results

In this section, we establish our main results on Weyl multipliers for general Haar and Franklin systems. We begin by introducing the necessary notation and definitions. We will use the following notations throughout this paper:

- $\dim D$ — dimension of the finite-dimensional space D ,
- C, C_1, C_p, \dots — positive constants depending only on their indices,
- $a \sim_\gamma b$ means that $C_{1,\gamma}b \leq a \leq C_{2,\gamma}b$,
- $\mu(E)$ — Lebesgue measure of the set E ,

- $\|f\|_p$ — norm of the function f in the space $L^p[0, 1]$, $1 \leq p \leq \infty$.

To formulate the results, following the paper [3], we define the general Franklin system.

Definition 2 *A sequence of points $\mathcal{T} = \{t_n : n \geq 0\}$ is called admissible (for the general Franklin system) if $t_0 = 0$, $t_1 = 1$, $t_n \in (0, 1)$ for any $n \geq 2$, \mathcal{T} is everywhere dense in $[0, 1]$, and no point $t \in (0, 1)$ appears in \mathcal{T} more than twice.*

Let $\mathcal{T} = \{t_n : n \geq 0\}$ be an admissible sequence. For $n \geq 1$, denote $\mathcal{T}_n = \{t_i : 0 \leq i \leq n\}$. Let π_n be obtained from \mathcal{T}_n by nondecreasing permutation: $\pi_n = \{\tau_i^n : \tau_i^n \leq \tau_{i+1}^n, 0 \leq i \leq n-1\}$, $\pi_n = \mathcal{T}_n$. Then by \mathbf{S}_n we denote the space of functions defined on $[0, 1]$, left-continuous, linear on (τ_i^n, τ_{i+1}^n) , and continuous at τ_i^n if $\tau_{i-1}^n < \tau_i^n < \tau_{i+1}^n$ for any $i = 0, 1, \dots, n$. Clearly, $\dim \mathbf{S}_n = n+1$ and $\mathbf{S}_{n-1} \subset \mathbf{S}_n$. Consequently, there exists a unique (up to sign) function $f_n \in \mathbf{S}_n$ that is orthogonal to \mathbf{S}_{n-1} and $\|f_n\|_2 = 1$. This function will be called the n -th Franklin function corresponding to the partition (sequence) \mathcal{T} . We call a partition simple if each point $t \in (0, 1)$ occurs in \mathcal{T} at most once.

Definition 3 *A simple admissible sequence \mathcal{T} is called quasi-dyadic if for any $k \geq 1$, $t_{2^{k+1}} < t_{2^k+2} < \dots < t_{2^k+1}$ and between neighboring points from \mathcal{T}_{2^k} there is only one point from $\{t_n\}_{n=2^k+1}^{2^{k+1}}$.*

Definition 4 *The general Franklin system $\{f_n : n \geq 0\}$ corresponding to the partition \mathcal{T} is defined by the rule*

$$f_0(t) = 1, \quad f_1(t) = \sqrt{3}(2t - 1),$$

and for $n \geq 2$, f_n is the n -th Franklin function corresponding to the partition \mathcal{T} . It is assumed that $f_n(t_n) > 0$.

Definition 5 *A sequence of points $\mathcal{T} = \{t_n : n \geq 0\}$ is called admissible (for the general Haar system) if $t_0 = 0$, $t_1 = 1$, $t_n \in (0, 1)$ for any $n \geq 1$, \mathcal{T} is everywhere dense in $[0, 1]$, and each point $t \in (0, 1)$ occurs in \mathcal{T} at most once.*

It is clear that the sequence \mathcal{T} is admissible for the general Haar system if it is admissible for the general Franklin system and simple.

Let $\mathcal{T} = \{t_n : n \geq 0\}$ be an admissible sequence (for the Haar system). For $n \geq 2$, denote $\mathcal{T}_n = \{t_i : 0 \leq i \leq n\}$. Let π_n be obtained from \mathcal{T}_n by increasing permutation: $\pi_n = \{\tau_i^n : \tau_i^n < \tau_{i+1}^n, 0 \leq i \leq n-1\}$, $\pi_n = \mathcal{T}_n$. Then by $\bar{\mathbf{S}}_n$ we denote the space of functions defined on $[0, 1]$ and constant on $[\tau_i^n, \tau_{i+1}^n)$. Clearly, $\dim \bar{\mathbf{S}}_n = n$ and $\bar{\mathbf{S}}_{n-1} \subset \bar{\mathbf{S}}_n$. Consequently, there exists a unique (up to sign) function $\chi_n \in \bar{\mathbf{S}}_n$ that is orthogonal to $\bar{\mathbf{S}}_{n-1}$ and $\|\chi_n\|_2 = 1$. This function will be called the n -th Haar function corresponding to the partition (sequence) \mathcal{T} .

Definition 6 *The general Haar system $\{\chi_n : n \geq 1\}$ corresponding to the partition \mathcal{T} is defined by the rule $\chi_1(x) = 1$ and for $n \geq 2$, χ_n is the n -th Haar function corresponding to the partition \mathcal{T} .*

When $t_n = \frac{2m-1}{2^{k+1}}$, where $n = 2^k + m$, $k = 0, 1, 2, \dots$, $m = 1, 2, \dots, 2^k$, we obtain the classical Franklin system and the classical Haar system, equivalently defined, respectively, in [4] and [12].

As we have already noted, in general, for functional series, unconditional convergence a.e. is not equivalent to absolute convergence a.e. However, for series in certain systems, these concepts are equivalent. For the classical Haar system, the relationship between unconditional and absolute convergence a.e. is particularly well studied. In particular, the following fundamental result establishes their equivalence.

Theorem 1 (E. M. Nikishin, P. L. Ulyanov [19]) *For unconditional convergence a.e. on a set E , $\mu(E) > 0$, of the series $\sum_{n=1}^{\infty} a_n \chi_n(x)$, where $\{\chi_n(x)\}_{n=1}^{\infty}$ is the classical Haar system, it is necessary and sufficient that for a.e. $x \in E$ the sum $\sum_{n=1}^{\infty} |a_n \chi_n(x)|$ be finite.*

By repeating verbatim the proof of Theorem 1 given in [16] (see [16] Ch. 3 § 5, Theorem 15), one can prove an analogue of Theorem 1 for any general Haar system.

An analogue of Theorem 1 for the classical Franklin system is proved in [6], and for the general Franklin system in [11].

For $n \geq 2$, we define left and right neighbors of t_n as

$$\begin{aligned} t_n^+ &:= \min\{t_k : k < n \text{ and } t_k \geq t_n\}, \\ t_n^- &:= \max\{t_k : k < n \text{ and } t_k \leq t_n\}, \end{aligned} \quad (2)$$

and denote support intervals as

$$\Delta_n := (t_n^-, t_n^+), \quad \Delta_n^+ := (t_n^-, t_n), \quad \Delta_n^- := (t_n, t_n^+) \quad n \geq 2.$$

It is easy to see that the support of the function χ_n is the closure of Δ_n .

The general Haar system corresponding to a simple partition \mathcal{T} can be defined by the formulas: $\chi_1(x) = 1$ when $x \in [0, 1]$, and

$$\chi_n(x) = \begin{cases} \sqrt{\frac{|\Delta_n^-|}{|\Delta_n^+|}} \frac{1}{\sqrt{|\Delta_n|}}, & \text{when } x \in \Delta_n^+, \\ -\sqrt{\frac{|\Delta_n^+|}{|\Delta_n^-|}} \frac{1}{\sqrt{|\Delta_n|}}, & \text{when } x \in \Delta_n^-, \\ 0, & \text{when } x \notin \Delta_n. \end{cases} \quad (3)$$

The following definitions were given in the paper [10].

Definition 7 A sequence \mathcal{T} is called weakly regular (with parameter γ) if there exists $\gamma \geq 1$ such that

$$\gamma^{-1} \leq \frac{|\Delta_n^+|}{|\Delta_n^-|} \leq \gamma \quad \text{for any } n \geq 2.$$

Definition 8 A sequence \mathcal{T} is called strongly regular (with parameter γ) if there exists $\gamma \geq 1$ such that

$$\gamma^{-1} \leq \frac{\tau_{n,i} - \tau_{n,i-1}}{\tau_{n,i+1} - \tau_{n,i}} \leq \gamma \quad \text{for all } n, i.$$

When \mathcal{T} is weakly regular with parameter γ , then (see [10])

$$\|\chi_n\|_p \sim_{p,\gamma} \|f_n\|_p \sim_{p,\gamma} |\Delta_n|^{1/p-1/2} \quad \text{for all } 1 \leq p \leq \infty. \quad (4)$$

In particular,

$$\|\chi_n\|_1 \sim_\gamma \|f_n\|_1 \sim_\gamma |\Delta_n|^{1/2} \quad \text{and} \quad \|\chi_n\|_\infty \sim_\gamma \|f_n\|_\infty \sim_\gamma |\Delta_n|^{-1/2}. \quad (5)$$

In the general case, we have the following relations (see (3) and [10])

$$\|\chi_n\|_p \leq c_p |\Delta_n|^{1/p-1/2} \quad \text{and} \quad \|f_n\|_p \leq c_p |\Delta_n|^{1/p-1/2}, \quad \text{when } 1 \leq p \leq 2,$$

and

$$\|\chi_n\|_p \geq c_p |\Delta_n|^{1/p-1/2} \quad \text{and} \quad \|f_n\|_p \geq c_p |\Delta_n|^{1/p-1/2}, \quad \text{when } 2 \leq p \leq \infty.$$

It is clear that condition (1) is equivalent to the condition

$$\sum_{k=1}^{\infty} \frac{1}{\omega_{2^k}} < \infty. \quad (6)$$

The following lemma holds.

Lemma 1 For the set defined below

$$J := \{k : \omega_{2^{k+1}} < 2\omega_{2^k}\}, \quad (7)$$

the following is correct

$$\sum_{k \in J} \frac{1}{\omega_{2^k}} = \infty. \quad (8)$$

Proof. Let $j_1, j_2 \in J$ be such that $j_1 < j_2$ and $(j_1, j_2) \cap J = \emptyset$. Then for $k \in (j_1, j_2)$, we have $\omega_{2^{k+1}} \geq 2\omega_{2^k}$, and therefore,

$$\frac{1}{\omega_{2^k}} \leq \frac{1}{2\omega_{2^{k-1}}} \leq \cdots \leq \frac{1}{2^{k-j_1}\omega_{2^{j_1}}}.$$

Consequently,

$$\sum_{k=j_1+1}^{j_2-1} \frac{1}{\omega_{2^k}} \leq \frac{1}{\omega_{2^{j_1}}} \sum_{k=j_1+1}^{j_2-1} \frac{1}{2^{k-j_1}} < \frac{1}{\omega_{2^{j_1}}}. \quad (9)$$

From (9) it follows that

$$\sum_{\substack{k \notin J \\ k \geq \min J}} \frac{1}{\omega_{2^k}} < \sum_{k \in J} \frac{1}{\omega_{2^k}}. \quad (10)$$

Since (6) does not hold, from (10) it follows (8). \square

With this lemma established, we now recall several key results from [7] that relate the multiplier properties of general Haar and Franklin systems under weak regularity assumptions. The following theorems are proved in the paper [7].

Theorem 2 *Let \mathcal{T} be a weakly regular partition of the interval $[0, 1]$ and $\{\chi_n(x)\}_{n=1}^\infty$, $\{f_n(x)\}_{n=0}^\infty$ be the general Haar and Franklin systems corresponding to the partition \mathcal{T} . If the sequence ω_n is not a Weyl multiplier for a.e. unconditional convergence of series in the general Haar system, then it is not a Weyl multiplier for a.e. unconditional convergence of series in the general Franklin system.*

Theorem 3 *For any nondecreasing and unbounded sequence ω_n , there exists a strongly regular partition \mathcal{T} such that for the corresponding general Franklin (Haar) system, ω_n is a Weyl multiplier for a.e. unconditional convergence of series in the general Franklin (Haar) system.*

Theorem 4 *Suppose ω_n is a nondecreasing and unbounded sequence and for the orthonormal system $\{\varphi_n(x)\}_{n=1}^\infty$ we have $\inf_n \|\varphi_n\|_1 = 0$. Then the system $\{\varphi_n(x)\}$ can be rearranged so that ω_n will be a Weyl multiplier for a.e. unconditional convergence of series in the system $\{\varphi_n(t)\}_{n=1}^\infty$.*

Taking into account the relations (4), from Theorem 4 we obtain that for any positive and unboundedly increasing sequence ω_n , $n \in \mathbb{N}$, the general Haar and Franklin systems can be rearranged so that the sequence ω_n is a Weyl multiplier for the general Haar and Franklin systems. This means that by rearranging the general Haar and Franklin systems one can ensure that any predetermined positive and unboundedly increasing sequence ω_n , $n \in \mathbb{N}$, is a Weyl multiplier for a.e. unconditional convergence of series in the general Haar and Franklin systems.

Here we study the question of how fast a positive increasing sequence ω_n , $n \in \mathbb{N}$, can grow so that it is possible to rearrange the Haar (Franklin) system in such a way that ω_n is not a Weyl multiplier for a.e. unconditional

convergence of series in the general Haar (Franklin) system. From Orlicz's theorem it follows that $\omega_n = (\ln n)^2 (\ln \ln n)^{1+\varepsilon}$, $\varepsilon > 0$, is a Weyl multiplier for a.e. unconditional convergence for any orthonormal system.

The following theorem holds.

Theorem 5 *Let \mathcal{T} be a weakly regular partition of the interval $[0, 1]$ and $\{\chi_n(x)\}_{n=1}^\infty$ be the Haar system corresponding to the partition \mathcal{T} . If*

$$\sum_{n=1}^{\infty} \frac{1}{n\omega_n} = \infty, \quad (11)$$

then the Haar system can be rearranged so that ω_n is not a Weyl multiplier for a.e. unconditional convergence for the Haar system.

Proof. It is clear that condition (11) is equivalent to condition (6). Applying Lemma 1, we obtain the set J defined by (7) and satisfying (8).

We proceed to the rearrangement of the Haar system. Let \mathcal{T} be a weakly regular partition of the interval $[0, 1]$ and $\{\chi_n(x)\}_{n=1}^\infty$ be the Haar system corresponding to the partition \mathcal{T} . We will rearrange the sequence \mathcal{T} in such a way that the rearranged sequence \mathcal{T}_σ corresponds to the Haar system $\{\chi_{\sigma_n}\}_{n=1}^\infty$, obtained by rearranging the system $\{\chi_n\}_{n=1}^\infty$.

Let $\sigma_0 = 0$, $\sigma_1 = 1$, and $\sigma_2 = t_2$. Suppose we have already defined σ_i for $i = 0, 1, 2, \dots, 2^k$. We define σ_i for $i = 2^k + 1, 2^k + 2, \dots, 2^{k+1}$ as follows. Let

$$\{\theta_i^k\}_{i=0}^{2^k} = \{t_{\sigma_i}\}_{i=0}^{2^k}, \quad (12)$$

with $\theta_{i-1}^k < \theta_i^k$ for all $i = 1, 2, \dots, 2^k$. Then by definition

$$\sigma_{2^k+i} := \min\{j : \theta_{i-1}^k < t_j < \theta_i^k\}, \quad i = 1, 2, \dots, 2^k. \quad (13)$$

It is easy to verify that the sequence $\{\sigma_n\}_{n=0}^\infty$ obtained in this way is a permutation of the nonnegative integers. Denote $\mathcal{T}_\sigma = \{t_{\sigma_n}\}_{n=0}^\infty$, and denote the corresponding general Haar system by $\{\chi_n^\sigma\}_{n=1}^\infty$. It is clear that \mathcal{T}_σ is quasi-dyadic.

It is easy to see that (see (2), (13))

$$t_{\sigma_{2^k+i}}^+ = \theta_i^k \quad \text{and} \quad t_{\sigma_{2^k+i}}^- = \theta_{i-1}^k, \quad i = 1, 2, \dots, 2^k. \quad (14)$$

From (14) it follows that \mathcal{T}_σ is also weakly regular with the same parameter γ . From (14) it also follows that

$$\chi_{2^k+i}^\sigma(x) = \chi_j(x) \quad \text{if} \quad \sigma_{2^k+i} = j, \quad (15)$$

i.e., the system $\{\chi_n^\sigma\}_{n=1}^\infty$ is a rearrangement of the system $\{\chi_n(x)\}_{n=1}^\infty$. We denote the support of the function χ_n^σ by Δ_n^σ . Then, by virtue of (12)-(15), we have

$$\bigcup_{n=2^k+1}^{2^{k+1}} \Delta_n^\sigma = [0, 1] \quad \text{and} \quad \Delta_{2^k+i}^\sigma \cap \Delta_{2^k+i'}^\sigma = \emptyset \quad \text{if} \quad k \in J, \quad 1 \leq i < i' \leq 2^k. \quad (16)$$

Let $J = \{k_i : i \in \mathbb{N}\}$. Then from (8) it follows that there exists a sequence q_i tending to zero such that

$$\sum_{i=1}^{\infty} \frac{q_i}{\omega_{k_i}} = \infty, \quad (17)$$

$$\sum_{i=1}^{\infty} \frac{q_i^2}{\omega_{k_i}} < \infty. \quad (18)$$

Set

$$a_n = \begin{cases} \sqrt{|\Delta_n^\sigma|} \frac{q_i}{\omega_{k_i}}, & \text{if } n \in (2^{k_i}, 2^{k_i+1}], i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases} \quad (19)$$

Then, taking into account the weak regularity of \mathcal{T}_σ , from (19) and (5) we obtain that for $n \in (2^{k_i}, 2^{k_i+1}]$, $i \in \mathbb{N}$ we have

$$|a_n \chi_n^\sigma(x)| \sim_\gamma \frac{q_i}{\omega_{k_i}}, \text{ when } x \in \Delta_n^\sigma, \text{ and } |a_n \chi_n^\sigma(x)| = 0, \quad x \notin \overline{\Delta_n^\sigma}. \quad (20)$$

Consequently (see (16) and (20))

$$\sum_{n=2^{k_i+1}}^{2^{k_i+1}} |a_n \chi_n^\sigma(x)| \sim_\gamma \frac{q_i}{\omega_{k_i}} \text{ a.e. on } [0, 1]. \quad (21)$$

From (17), (19), and (21) we obtain

$$\sum_{n=1}^{\infty} |a_n \chi_n^\sigma(x)| = \infty \text{ a.e. on } [0, 1]. \quad (22)$$

As noted after the formulation of Theorem 1, Theorem 1 holds for any general Haar system. Consequently, the series $\sum_{n=1}^{\infty} a_n \chi_n^\sigma(x)$ diverges a.e. after some rearrangement. On the other hand (see (19))

$$\sum_{n=1}^{\infty} a_n^2 \omega_n = \sum_{i \in \mathbb{N}} \sum_{n=2^{k_i+1}}^{2^{k_i+1}} a_n^2 \omega_n = \sum_{i \in \mathbb{N}} \sum_{n=2^{k_i+1}}^{2^{k_i+1}} |\Delta_n^\sigma| \frac{q_i^2}{\omega_{k_i}^2} \omega_n. \quad (23)$$

Recall that if $k_i \in J$, then $\omega_{2^{k_i+1}} < 2\omega_{2^{k_i}}$. Consequently, from (23) and (16) we obtain

$$\sum_{n=1}^{\infty} a_n^2 \omega_n < 2 \sum_{i \in \mathbb{N}} \sum_{n=2^{k_i+1}}^{2^{k_i+1}} |\Delta_n^\sigma| \frac{q_i^2}{\omega_{k_i}} = 2 \sum_{i \in \mathbb{N}} \frac{q_i^2}{\omega_{k_i}} \sum_{n=2^{k_i+1}}^{2^{k_i+1}} |\Delta_n^\sigma| = 2 \sum_{i \in \mathbb{N}} \frac{q_i^2}{\omega_{k_i}}. \quad (24)$$

From (18) and (24) it follows that

$$\sum_{n=1}^{\infty} a_n^2 \omega_n < \infty. \quad (25)$$

□

A parallel result holds for the Franklin system under the same weak regularity assumption.

Theorem 6 *Let \mathcal{T} be a weakly regular partition of the interval $[0, 1]$ and $\{f_n(x)\}_{n=0}^{\infty}$ be the Franklin system corresponding to the partition \mathcal{T} . Then, if (11) holds, the Franklin system can be rearranged so that ω_n is not a Weyl multiplier for a.e. unconditional convergence for the rearranged Franklin system.*

Before proceeding to the proof, we first recall an auxiliary theorem from [8] that relates absolute convergence in general Haar and Franklin systems.

Theorem 7 *Let \mathcal{T} be a weakly regular partition of the interval $[0, 1]$ and $\{\chi_n(x)\}_{n=1}^{\infty}$, $\{f_n(x)\}_{n=0}^{\infty}$ be the general Haar and Franklin systems corresponding to the partition \mathcal{T} . If*

$$\sum_{n=1}^{\infty} |a_n \chi_n(x)| = +\infty, \quad \text{when } x \in E,$$

then

$$\sum_{n=1}^{\infty} |a_n f_n(x)| = +\infty, \quad \text{a.e. on } E.$$

Proof of Theorem 6. Let \mathcal{T} be a weakly regular partition of the interval $[0, 1]$ and $\{f_n(x)\}_{n=0}^{\infty}$ be the Franklin system corresponding to the partition \mathcal{T} . We define the set J and the permutation σ as in the proof of Theorem 5. In the case of the Haar system, the sequence $\mathcal{T}_\sigma = \{t_{\sigma_n}\}_{n=1}^{\infty}$ corresponded to the system $\{\chi_n^\sigma(x)\}_{n=1}^{\infty}$, which is a rearrangement of the system $\{\chi_n(x)\}_{n=1}^{\infty}$, i.e., $\{\chi_n^\sigma(x)\}_{n=1}^{\infty} = \{\chi_{\sigma_n}(x)\}_{n=1}^{\infty}$. In the case of the general Franklin system, the situation is different: in general, $f_n^\sigma(x) \neq f_{\sigma_n}(x)$.

Nevertheless, we define the coefficients a_n of the series in the rearranged Franklin system $\{f_{\sigma_n}\}_{n=0}^{\infty}$ by formula (19). We prove that

$$\sum_{n=0}^{\infty} |a_n f_{\sigma_n}(x)| = \infty \quad \text{a.e. on } [0, 1]. \quad (26)$$

Denote by $\tilde{\sigma}_n$ the inverse permutation of the permutation σ_n . Then from (22) and (15) it follows that

$$\sum_{n=1}^{\infty} |a_{\tilde{\sigma}_n} \chi_n(x)| = \infty \quad \text{a.e. on } [0, 1]. \quad (27)$$

Hence, from (27) and Theorem 7 it follows that

$$\sum_{n=1}^{\infty} |a_{\tilde{\sigma}_n} f_n(x)| = +\infty, \quad \text{a.e. on } [0, 1].$$

From this we obtain (26). Taking into account that also for series in the general Franklin system, a.e. absolute convergence is equivalent to a.e. unconditional convergence (see [11, Theorem 2]), from (25) and (26) we obtain the proof of the theorem. \square

Note that in Theorems 5 and 6, the “bad rearrangement” resulted in a quasi-dyadic sequence \mathcal{T}_{σ_n} .

Theorem 8 *Let \mathcal{T} be a quasi-dyadic and weakly regular sequence with parameter γ . Then an increasing and nonnegative sequence will be a Weyl multiplier for a.e. unconditional convergence for the general Haar (Franklin) system if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\omega_{2^k}} < \infty.$$

Proof. *Necessity* is contained in the proof of Theorems 5 and 6.

Sufficiency follows from the following inequalities, where $\{\varphi_n\}$ is the general Haar or Franklin system, and the coefficients a_n satisfy the condition $\sum_{n=1}^{\infty} a_n^2 \omega_n = d < \infty$ (see (5), (16))

$$\begin{aligned} \sum_{n=3}^{\infty} \|a_n \varphi_n\|_1 &= \sum_{n=3}^{\infty} |a_n \sqrt{\omega_n}| \frac{\|\varphi_n\|_1}{\sqrt{\omega_n}} \leq \left\{ \sum_{n=3}^{\infty} a_n^2 \omega_n \right\}^{1/2} \left\{ \sum_{n=3}^{\infty} \frac{\|\varphi_n\|_1^2}{\omega_n} \right\}^{1/2} \leq \\ &d \left\{ \sum_{k=1}^{\infty} \frac{1}{\omega_{2^k}} \sum_{n=2^{k+1}}^{2^{k+1}} \|\varphi_n\|_1^2 \right\}^{1/2} < C_{\gamma} d \left\{ \sum_{k=1}^{\infty} \frac{1}{\omega_{2^k}} \right\}^{1/2} < \infty. \end{aligned}$$

\square

The weak regularity assumption in Theorems 5 and 6 turns out to be essential, as the next result demonstrates. Without this regularity condition, even sequences growing arbitrarily slowly can be Weyl multipliers for appropriately constructed partitions.

The following theorem indicates that in Theorems 5 and 6, the weak regularity of the partition \mathcal{T} is an essential condition.

Theorem 9 *Let ω_n be an increasing sequence of positive numbers such that*

$$\lim_{n \rightarrow \infty} \omega_n = \infty.$$

Then there exists a quasi-dyadic sequence \mathcal{T} such that for the corresponding general Haar system $\{\chi_n(x)\}_{n=1}^{\infty}$, the sequence ω_n is a Weyl multiplier for a.e. unconditional convergence for this Haar system.

Proof. Without loss of generality, we can assume that $\omega_n \geq 1$, $n \in \mathbb{N}$. Consequently, there exist numbers k_p such that

$$\frac{1}{\omega_k} < \frac{1}{p}, \quad \text{when } k_p < k \leq 2^{k_p+1}, \quad p = 1, 2, \dots \quad (28)$$

We construct the quasi-dyadic sequence \mathcal{T} as follows. Suppose we have already chosen the numbers t_n , $1 \leq n \leq 2^k$. We choose the numbers t_n , $2^k + 1 \leq n \leq 2^{k+1}$, so that (see (3))

$$\frac{|\Delta_n^+|}{|\Delta_n^-|} = \alpha_k < 1, \quad 2^k < n \leq 2^{k+1}.$$

The numbers α_k , $k \in \mathbb{N}$, will be chosen later. The sequence \mathcal{T} will be admissible (will be dense in $[0, 1]$) if

$$\sum_{k=1}^{\infty} \alpha_k = \infty. \quad (29)$$

Then from (3) we obtain the following estimate

$$\|\chi_n\|_1 = 2\sqrt{\frac{|\Delta_n^-||\Delta_n^+|}{|\Delta_n|}} \leq 2\sqrt{\alpha_k|\Delta_n|}, \quad 2^k < n \leq 2^{k+1},$$

and

$$\sum_{n=2^k+1}^{2^{k+1}} \|\chi_n\|_1^2 \leq 4\alpha_k.$$

Further,

$$\begin{aligned} \sum_{n=3}^{\infty} \|a_n \chi_n\|_1 &= \sum_{n=3}^{\infty} |a_n \sqrt{\omega_n}| \frac{\|\chi_n\|_1}{\sqrt{\omega_n}} \leq \\ &\left(\sum_{n=3}^{\infty} a_n^2 \omega_n \right)^{1/2} \left(\sum_{n=3}^{\infty} \frac{\|\chi_n\|_1^2}{\omega_n} \right)^{1/2} \leq \left(\sum_{n=3}^{\infty} a_n^2 \omega_n \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{4\alpha_k}{\omega_{2^k}} \right)^{1/2}. \end{aligned} \quad (30)$$

Consequently, if

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{\omega_{2^k}} < \infty, \quad (31)$$

holds, then for the Haar series with coefficients satisfying $\sum_{n=1}^{\infty} a_n^2 \omega_n < \infty$, from (30) and (31) we will have

$$\sum_{n=3}^{\infty} \|a_n \chi_n\|_1 < \infty,$$

and therefore, the series $\sum_{n=1}^{\infty} a_n \chi_n(x)$ will converge a.e. on $[0, 1]$.

To complete the proof, we need to find α_k such that (29) and (31) hold. It suffices to set

$$\alpha_k = \frac{1}{p(k_{p+1} - k_p)}, \quad \text{when } k_p < k \leq k_{p+1}.$$

Indeed, in this case

$$\sum_{k=1}^{\infty} \alpha_k = \sum_{p=1}^{\infty} \sum_{k=k_p+1}^{k_{p+1}} \alpha_k = \sum_{p=1}^{\infty} \frac{1}{p} = \infty$$

and using (28) we will have

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{\omega_{2^k}} = \sum_{p=1}^{\infty} \sum_{k=k_p+1}^{k_{p+1}} \frac{\alpha_k}{\omega_{2^k}} \leq \sum_{p=1}^{\infty} \frac{1}{p^2} < \infty.$$

□

Remark 1 *Note that Theorem 9 does not follow from Theorem 3. In Theorem 3, the sequence \mathcal{T} is not quasi-dyadic.*

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