Armenian Journal of Mathematics
Volume 9, Number 2, 2017, 68–83

On a Convergence of Rational Approximations by the Modified Fourier Basis

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Abstract. We continue investigations of the modified-trigonometric-rational approximations that arise while accelerating the convergence of the modified Fourier expansions by means of rational corrections. Previously, we investigated the pointwise convergence of the rational approximations away from the endpoints and the L_2-convergence on the entire interval. Here, we study the convergence at the endpoints and derive the exact constants for the main terms of asymptotic errors. We show that the Fourier-Pade approximations are much more accurate in all frameworks than the modified expansions for sufficiently smooth functions. Moreover, we consider a simplified version of the rational approximations and explore the optimal values of parameters that lead to better accuracy in the framework of the L_2-error. Numerical experiments perform comparisons of the rational approximations with the modified Fourier expansions.

Key Words: Modified Fourier Expansion, Convergence Acceleration, Rational Approximation, Fourier-Pade Approximation
Mathematics Subject Classification 2010: 41A20, 65T40

Introduction

We consider a modified Fourier basis

$$\mathcal{H} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi (n - \tfrac{1}{2}) x : n \in \mathbb{N}\} \tag{1}$$

which was originally proposed by Krein [1] and then thoroughly investigated in a series of papers [2–10].

Obviously, for even functions on $[-1, 1]$, expansions by the modified Fourier basis coincide with the expansions by the classical Fourier basis

$$\mathcal{H}_{class} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi n x : n \in \mathbb{N}\},\ x \in [-1, 1]. \tag{2}$$

Moreover, the modified Fourier basis can be derived from the other classical basis $\mathcal{H}^*$ on $[0,1]$

$$\mathcal{H}^* = \{\cos \pi n x : n \in \mathbb{Z}_+\},\ x \in [0,1] \tag{3}$$

by means of changing the variable.

By $M_N(f,x)$ denote the truncated modified Fourier series

$$M_N(f,x) = \frac{1}{2} f_0^c + \sum_{n=1}^{N} [f_n^c \cos \pi n x + f_n^s \sin \pi (n - \tfrac{1}{2}) x], \tag{4}$$

where

$$f_n^c = \int_{-1}^{1} f(x) \cos \pi n x dx,\ \ f_n^s = \int_{-1}^{1} f(x) \sin \pi (n - \tfrac{1}{2}) x dx. \tag{5}$$

In [10], we considered convergence acceleration of the modified expansions by means of rational correction functions which led to the modified Fourier-Pade (MFP) approximations. There, we studied the pointwise convergence of the MFP-approximations on $|x| < 1$ and the convergence in the L_2-norm. We proved better accuracies of approximations in comparison with the modified expansions for enough smooth functions.

Consider a sequence of real numbers $\theta = \{\theta_k\}_{k=1}^p,\ p \ge 1$. Let $\hat{c} = \{c_n\}$ for some sequence $\{c_n\}$ of complex or real numbers. By $\Delta_n^k(\theta, \hat{c})$, we denote the following generalized finite differences

$$\begin{aligned} &\Delta_n^0(\theta, \hat{c}) = c_n, \\ &\Delta_n^k(\theta, \hat{c}) = \Delta_n^{k-1}(\theta, \hat{c}) + \theta_k \Delta_{n-1}^{k-1}(\theta, \hat{c}),\ \ k \ge 1. \end{aligned} \tag{6}$$

By $\Delta_n^k(\hat{c})$, we denote the classical finite differences which correspond to the generalized differences $\Delta_n^k(\theta, \hat{c})$ with $\theta \equiv 1$.

Let

$$\begin{aligned} R_N(f,x) &= f(x) - M_N(f,x) \\ &= R_N^c(f,x) + R_N^s(f,x), \end{aligned} \tag{7}$$

where

$$R_N^c(f,x) = \sum_{n=N+1}^{\infty} f_n^c \cos \pi n x, \tag{8}$$

and

$$R_N^s(f,x) = \sum_{n=N+1}^{\infty} f_n^s \sin \pi (n - \tfrac{1}{2}) x. \tag{9}$$

Consider two sequences of real numbers $\theta^c = \{\theta_k^c\}_{k=1}^p$ and $\theta^s = \{\theta_k^s\}_{k=1}^p$. Denote $\hat{f}^s = \{f_n^s\}_{n=1}^{\infty}$ and $\hat{f}^c = \{f_n^c\}_{n=0}^{\infty}$.

Let $\mu_t(k,\theta)$ be defined by the following identities

$$\prod_{t=1}^{k}(1+\theta_t x) = \sum_{t=0}^{k}\mu_t(k,\theta)x^t. \tag{10}$$

By means of Abel transformations (see details in [10]), we derive the following modified-trigonometric-rational (MTR-) approximations:

$$\begin{aligned} M_{N,p}(f,\theta^c,\theta^s,x) &= M_N(f,x) - \sum_{k=1}^{p}\frac{\theta_k^c\,\Delta_N^{k-1}(\theta^c,\hat{f}^c)}{\prod_{r=1}^{k}(1+2\theta_r^c\cos\pi x+(\theta_r^c)^2)} \\ &\quad\times\sum_{t=0}^{k}\mu_t(k,\theta^c)\cos\pi(N+1-t)x \\ &\quad - \sum_{k=1}^{p}\frac{\theta_k^s\,\Delta_N^{k-1}(\theta^s,\hat{f}^s)}{\prod_{r=1}^{k}(1+2\theta_r^s\cos\pi x+(\theta_r^s)^2)} \\ &\quad\times\sum_{t=0}^{k}\mu_t(k,\theta^s)\sin\pi(N+\tfrac{1}{2}-t)x \end{aligned} \tag{11}$$

with the error

$$\begin{aligned} R_{N,p}(f,\theta^c,\theta^s,x) &= f(x) - M_{N,p}(f,\theta^c,\theta^s,x) \\ &= R_{N,p}^c(f,\theta^c,x) + R_{N,p}^s(f,\theta^s,x), \end{aligned} \tag{12}$$

where

$$\begin{aligned} R_{N,p}^c(f,\theta,x) &= \frac{1}{2\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta,\hat{f}^c)e^{i\pi n x} \\ &\quad + \frac{1}{2\prod_{k=1}^{p}(1+\theta_k e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta,\hat{f}^c)e^{-i\pi n x}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} R_{N,p}^s(f,\theta,x) &= \frac{e^{-\frac{i\pi x}{2}}}{2i\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta,\hat{f}^s)e^{i\pi n x} \\ &\quad - \frac{e^{\frac{i\pi x}{2}}}{2i\prod_{k=1}^{p}(1+\theta_k e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta,\hat{f}^s)e^{-i\pi n x}. \end{aligned} \tag{14}$$

In this setting, the parameters $\theta^c = \{\theta_1^c,\ldots,\theta_p^c\}$ and $\theta^s = \{\theta_1^s,\ldots,\theta_p^s\}$ are unknown.

Different approaches are known for their determination. We know that (see [11–17]) by appropriate choice of those parameters, we will get better convergence in comparison with the classical expansions if the approximated function is sufficiently smooth.

An interesting approach (see [10, 12, 18]) leads to the Fourier-Pade type approximations, where the parameters are determined as the solutions of the following systems

$$\Delta_n^p(\theta^c, \hat{f}^c) = 0,\ n = N, N-1, \dots, N-p+1, \tag{15}$$

and

$$\Delta_n^p(\theta^s, \hat{f}^s) = 0,\ n = N, N-1, \dots, N-p+1. \tag{16}$$

We call this approach as the modified Fourier-Pade (MFP-) approximation. The MFP-approximations are essentially non-linear in the sense that

$$M_{N,p}(f+g, \theta^c, \theta^s, x) \neq M_{N,p}(f, \theta^c, \theta^s, x) + M_{N,p}(g, \theta^c, \theta^s, x) \tag{17}$$

as for each approximation we need to determine its own θ^c and θ^s vectors.

As we showed in [10], the parameters θ^c and θ^s derived as solutions of (15) and (16), respectively, has the following asymptotic expansions in terms of $1/N$

$$\theta_k^c = 1 - \frac{\tau_k^c}{N} + o(N^{-1}),\ \theta_k^s = 1 - \frac{\tau_k^s}{N} + o(N^{-1}),\ k = 1, \dots, p, \tag{18}$$

where $\tau_k^c = \tau_k^s$ are the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$.

In this paper, we continue investigations of the MFP-approximations and explore the pointwise convergence at the endpoints $x = \pm 1$. We derive the exact constants of the main terms of asymptotic errors and show that the rational and classical expansions by the modified Fourier basis have the same convergence rate. However, a comparison of the exact estimates allows to observe that rational approximations have bigger asymptotic accuracy.

Then, we consider more simplified parametric version of the rational approximations. Instead of solving the systems (15) and (16) for each N, we assume that parameters θ^c and θ^s are determined as follows (compare with (18))

$$\theta_k^c = \theta_k^s = 1 - \frac{\tau_k}{N},\ k = 1, \dots, p, \tag{19}$$

where τ_k are independent of N. Further, we determine parameters τ_k to minimize the error of the rational approximations in the L_2-norm. We call this approach as L_2-minimal MTR-approximations.

It is important that in the optimal rational approximations, the values of parameters τ_k, $k = 1, \dots, p$ depend only on p and the smoothness of f. It means that if functions f, g and $f+g$ have the same smoothness, then the optimal approach leads to linear rational approximations in the sense that

$$M_{N,p}(f+g, \theta^c, \theta^s, x) = M_{N,p}(f, \theta^c, \theta^s, x) + M_{N,p}(g, \theta^c, \theta^s, x) \tag{20}$$

with the same parameters θ^c and θ^s for all included functions.

1 Pointwise Convergence of the Modified Fourier-Pade Approximations

Let $f \in C^{2q+1}[-1,1]$, $q \geq 0$ and denote

$$A_{2k+1}(f) = \left(f^{(2k+1)}(1) - f^{(2k+1)}(-1)\right)(-1)^k, \; k = 0, \dots, q, \tag{21}$$

and

$$B_{2k+1}(f) = \left(f^{(2k+1)}(1) + f^{(2k+1)}(-1)\right)(-1)^k, \; k = 0, \dots, q. \tag{22}$$

The next result describes the pointwise convergence of the MFP-approximation on $|x| < 1$ by deriving the exact constants of the corresponding asymptotic errors.

Theorem 1 *[10] Let $f \in C^{2q+2p+2}[-1,1]$, $f^{(2q+2p+2)} \in BV[-1,1]$, $q \geq 0$, $p \geq 0$. Let θ^c, θ^s be the unique solutions of* (15) *and* (16)*, respectively. Let*

$$f^{(2k+1)}(\pm 1) = 0, \;\; k = 0, \dots, q-1, \tag{23}$$

and

$$A_{2q+1}(f) B_{2q+1}(f) \neq 0. \tag{24}$$

Then, the following estimates are valid for $x \in (-1,1)$ as $N \to \infty$

$$\begin{aligned} R^c_{N,p}(f, \theta^c, x) = {} & A_{2q+1}(f) \frac{(-1)^{N+1}(2q+p+1)!p!}{2^{2p+1}\pi^{2q+2}N^{2q+2p+2}(2q+1)!} \\ & \times \frac{\cos\frac{\pi x}{2}(2N-2p+1)}{\cos^{2p+1}\frac{\pi x}{2}} + o(N^{-2q-2p-2}), \end{aligned} \tag{25}$$

and

$$\begin{aligned} R^s_{N,p}(f, \theta^s, x) = {} & B_{2q+1}(f) \frac{(-1)^{N}(2q+p+1)!p!}{2^{2p+1}\pi^{2q+2}N^{2q+2p+2}(2q+1)!} \\ & \times \frac{\sin\frac{\pi x}{2}(2N-2p)}{\cos^{2p+1}\frac{\pi x}{2}} + o(N^{-2q-2p-2}). \end{aligned} \tag{26}$$

Note that for $p = 0$, Theorem 1 coincides with the well-known result for the modified Fourier expansion proved in [9] (Theorem 2.22, page 29):

Theorem 2 *[9] Suppose that $f \in C^{2q+2}(-1,1)$, $f^{(2q+2)} \in BV[-1,1]$ and f obeys the first q derivative conditions*

$$f^{(2r+1)}(\pm 1) = 0, \; r = 0, \dots, q-1. \tag{27}$$

Then, the following estimates are valid for $x \in (-1,1)$ as $N \to \infty$

$$f(x) - M_N(f, x) = \frac{(-1)^{N+1}}{2\pi^{2q+2}N^{2q+2}\cos\frac{\pi x}{2}} H_q(x) + o(N^{-2q-2}), \tag{28}$$

where

$$H_q(x) = A_{2q+1}(f)\cos\pi(N+1/2)x - B_{2q+1}(f)\sin\pi N x. \tag{29}$$

A comparison of Theorems 1 and 2 shows better pointwise convergence rate of the MFP-approximations on $|x| < 1$ and, the improvement is thanks to the factor $O(N^{2p})$ as $N \to \infty$.

Let

$$f_q(x) = \sin(x-1)(x^2-1)^{2q}, \ q = 0, 1, 2, \dots . \tag{30}$$

It is easy to verify that these functions obey the first q derivative conditions.

Figure 1 shows the results of the approximation of $f_1(x)$ by the modified Fourier expansion ($p = 0$) and the MFP-approximations ($p = 1, 2, 3$). We see a tremendous increase in accuracy while applying the MFP-approximations for smooth function on $[-0.7, 0.7]$. For example, in case of $p = 3$, the improvement is $3.8 \cdot 10^7$ times.

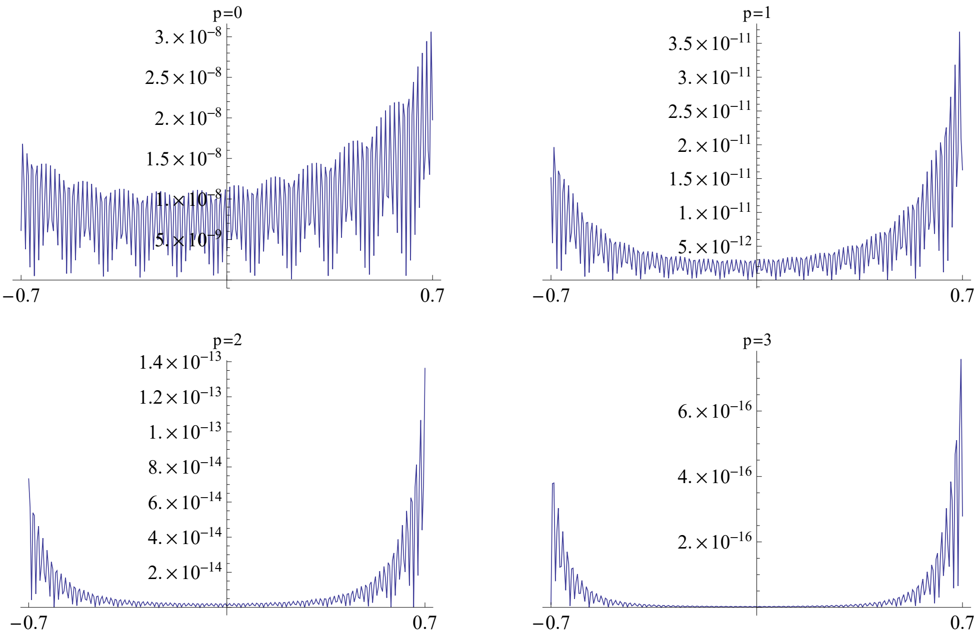

Figure 1: The graphs of $|R_{N,p}(f, \theta^c, \theta^s, x)|$ on $[-0.7, 0.7]$ for $N = 64$ while approximating $f_1(x)$ by the modified Fourier expansions ($p = 0$) and the MFP-approximations ($p = 1, 2, 3$).

Now, we investigate the pointwise convergence of the MFP-approximations at the endpoints $x = \pm 1$. We estimate (13) and (14) for $x = \pm 1$. Hence, we need asymptotic expansions for $\Delta_n^p(\theta^c, \hat{f}^c)$ and $\Delta_n^p(\theta^s, \hat{f}^s)$.

Lemma 1 *[10] Let $f \in C^{(2q+p+1)}[-1,1]$, $f^{(2q+p+1)} \in BV[-1,1]$, $q \geq 0$, and*

$$f^{(2k+1)}(\pm 1) = 0, \ k = 0, \dots, q-1, \tag{31}$$

with

$$A_{2q+1}(f) B_{2q+1}(f) \neq 0. \tag{32}$$

Let the systems (15), (16) *have unique solutions* θ^c *and* θ^s, *respectively. Then, the following estimates hold*

$$\Delta_n^p(\theta^c, \hat{f}^c) = A_{2q+1}(f)\frac{(-1)^n(2q+p+1)!}{N^p n^{2q+2}\pi^{2q+2}(2q+1)!}\left(1-\frac{N}{n}\right)^p + o(N^{-p})\frac{1}{n^{2q+2}},\ n > N, N \to \infty, \tag{33}$$

and

$$\Delta_n^p(\theta^s, \hat{f}^s) = B_{2q+1}(f)\frac{(-1)^{n+1}(2q+p+1)!}{N^p n^{2q+2}\pi^{2q+2}(2q+1)!}\left(1-\frac{N}{n-\frac{1}{2}}\right)^p + o(N^{-p})\frac{1}{n^{2q+2}},\ n > N, N \to \infty. \tag{34}$$

Theorem 3 *Assume* $f \in C^{2q+p+2}[-1,1]$, $f^{(2q+p+2)} \in BV[-1,1]$, $q \geq 0$, $p \geq 0$, *and*

$$f^{(2k+1)}(\pm 1) = 0,\ \ k = 0, \ldots, q-1, \tag{35}$$

with

$$A_{2q+1}(f)B_{2q+1}(f) \neq 0. \tag{36}$$

Let the systems (15), (16) *have unique solutions* θ^c *and* θ^s, *respectively. Then, the following estimate holds*

$$R_{N,p}(f, \theta^c, \theta^s, \pm 1) = \frac{h_{p,q}}{\pi^{2q+2}(2q+1)N^{2q+1}}\left(A_{2q+1}(f) \pm B_{2q+1}(f)\right) + o(N^{-2q-1}), \tag{37}$$

where

$$h_{p,q} = \frac{p!(2q+1)!}{(2q+p+1)!}. \tag{38}$$

Proof. As we mentioned above parameters θ^c and θ^s have asymptotic expansions as in (18), where parameters $\tau_k^c = \tau_k^s = \tau_k$, $k = 1, \ldots, p$ and τ_k are the roots of the generalized Laguerre polynomial

$$L_p^{(2q+1)}(x) = \sum_{k=0}^{p}(-1)^k\frac{(p+2q+1)!}{k!(p-k)!(2q+1+k)!}x^k. \tag{39}$$

From here and the Vieta's formula, we also have

$$\prod_{k=1}^{p}\tau_k = \frac{(2q+1+p)!}{(2q+1)!}. \tag{40}$$

In view of (13) and (14), we write

$$R_{N,p}^c(f, \theta^c, \pm 1) = \frac{N^p}{\prod_{k=1}^{p}(\tau_k + o(1))}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta^c, f_n^c)(-1)^n, \tag{41}$$

and

$$R^s_{N,p}(f,\theta^s,\pm 1) = \pm \frac{N^p}{\prod_{k=1}^p(\tau_k + o(1))} \sum_{n=N+1}^{\infty} \Delta^p_n(\theta^s, f^s_n)(-1)^n. \tag{42}$$

From (18) and Lemma 1, we get

$$\begin{aligned} R^c_{N,p}(f,\theta^c,\pm 1) = {} & \frac{A_{2q+1}(f)}{\pi^{2q+2}} \frac{(2q+p+1)!}{(2q+1)!\prod_{k=1}^p \tau_k} \sum_{n=N+1}^{\infty} \frac{1}{n^{2q+2}} \left(1 - \frac{N}{n}\right)^p \\ & + o(N^{-2q-1}). \end{aligned} \tag{43}$$

Finally, we get

$$R^c_{N,p}(f,\theta^c,\theta^s,\pm 1) = \frac{A_{2q+1}(f)}{\pi^{2q+2}N^{2q+1}} \frac{h_{p,q}}{(2q+1)} + o(N^{-2q-1}), \tag{44}$$

where

$$h_{p,q} = (2q+1)\sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{2q+k+1}. \tag{45}$$

Similarly, we show that

$$R^s_{N,p}(f,\theta^c,\theta^s,\pm 1) = \pm \frac{B_{2q+1}(f)}{\pi^{2q+2}N^{2q+1}} \frac{h_{p,q}}{(2q+1)} + o(N^{-2q-1}). \tag{46}$$

It remains to note that ([19])

$$\sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{2q+k+1} = \frac{p!(2q)!}{(2q+p+1)!}. \tag{47}$$

□

For $p = 0$, Theorem 3 coincides with the following well-known result:

Theorem 4 *[6] Suppose that $f \in C^{2q+2}(-1,1)$, $f^{(2q+2)} \in BV[-1,1]$ and f obeys the first q derivative conditions*

$$f^{(2r+1)}(\pm 1) = 0,\ r = 0,\dots,q-1. \tag{48}$$

Then,

$$\begin{aligned} f(\pm 1) - M_N(f,\pm 1) = {} & \frac{1}{\pi^{2q+2}(2q+1)N^{2q+1}}\left(A_{2q+1}(f) \pm B_{2q+1}(f)\right) \\ & + o(N^{-2q-1}). \end{aligned} \tag{49}$$

$p\backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p=1$	2	4	6	8	10	12	14
$p=2$	3	10	21	36	55	78	105
$p=3$	4	20	56	120	220	364	560
$p=4$	5	35	126	330	715	1365	2380

Table 1: The values of $1/h_{p,q}$.

A comparison of Theorems 3 and 4 shows that the expansions by the modified Fourier basis and the MFP-approximations have the same convergence rates at the endpoints $x = \pm 1$. However, a comparison of constants $h_{p,q}$ with $h_{0,q} = 1$ shows that the MFP-approximations are much more accurate than the classical expansions and asymptotic improvement is thanks to the factor $h_{0,q}/h_{p,q} = 1/h_{p,q}$. Table 1 presents the numerical values of the ratio $1/h_{p,q}$.

It would be interesting to compare the asymptotic improvement with actual improvement for moderate values of N. Table 2 shows the values of $\frac{\max|R_N(f,\pm1)|}{\max|R_{N,p}(f,\theta^c,\theta^s,\pm1)|}$ for $f_q(x)$ (see (30)) for $N = 64$. We see that for small values of p and q, the corresponding numbers of Tables 1 and 2 are rather close.

$p\backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p=1$	1.97	3.82	5.55	7.17	8.70	10.13	11.49
$p=2$	2.91	9.10	17.93	28.90	41.54	55.50	70.50
$p=3$	3.81	17.33	44.11	86.08	143.93	217.63	306.81
$p=4$	4.69	28.88	91.42	211.17	404.31	684.00	1060.94

Table 2: The values of ratio $\frac{\max|R_N(f,\pm1)|}{\max|R_{N,p}(f,\theta^c,\theta^s,\pm1)|}$ while approximating (30) for $q = 0, 1, \ldots, 6$ with $N = 64$.

Figure 2 demonstrates the values of $-\log_{10}(\max|R_{N,p}(f,\theta^c,\theta^s,\pm1)|)$ for different values of N and $p = 0, 1, 2, 3$ while approximating $f_1(x)$. The case $p = 0$ corresponds to the expansions by the modified Fourier basis.

2 L_2-minimal MTR-approximations

Next theorem reveals the behavior of the MFP-approximations in the L_2-norm.

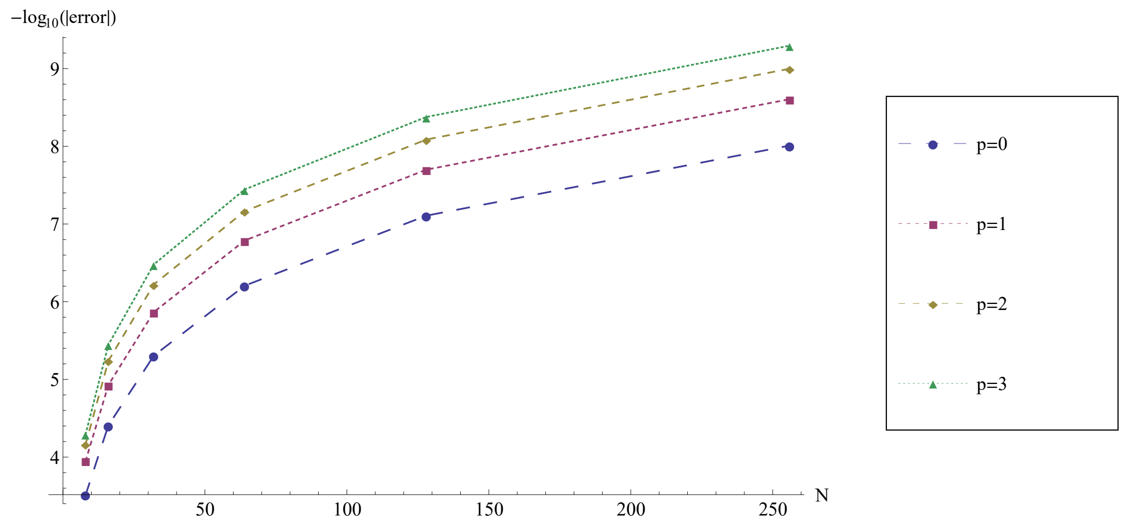

Figure 2: The values of $-\log_{10}(\max|R_{N,p}(f,\theta^c,\theta^s,\pm 1)|)$ for different N and p while approximating $f_1(x)$. The case $p=0$ corresponds to the expansion by the modified Fourier basis.

Theorem 5 *[10] Let* $f \in C^{2q+p+1}[-1,1]$, $f^{(2q+p+1)} \in BV[-1,1]$, $q \geq 0, p \geq 1$. *Let* θ^c, θ^s *be the unique solutions of* (15) *and* (16)*, respectively. If*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \tag{50}$$

and

$$A_{2q+1}(f)B_{2q+1}(f) \neq 0, \tag{51}$$

then, the following estimate holds

$$\lim_{N\to\infty} N^{2q+\frac{3}{2}}||R_{N,p}||_{L_2} = \frac{c_{p,q}}{\pi^{2q+2}\sqrt{4q+3}}\sqrt{A_{2q+1}^2(f)+B_{2q+1}^2(f)}, \tag{52}$$

where

$$\begin{aligned} c_{p,q} = {} & \frac{(p+2q+1)!\sqrt{4q+3}}{(2q+1)!} \\ & \times \left(\int_1^\infty dt \left| \int_1^t \frac{(1-x)^p}{x^{2q+p+2}} \sum_{j=1}^p \frac{e^{-\tau_j(t-x)}}{\prod_{\substack{k=1\\k\neq j}}^p(\tau_j-\tau_k)} dx \right|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{53}$$

and τ_k, $k = 1,\dots,p$ *are the roots of the generalized Laguerre polynomial* $L_p^{(2q+1)}(x)$ *(see* (39)*).*

The estimate (52) is valid also for $p=0$. This leads to the next theorem concerning the L_2-convergence of the modified Fourier expansion.

Theorem 6 *Let* $f \in C^{2q+1}[-1,1]$*,* $q \geq 0$*,* $f^{(2q+1)} \in BV[-1,1]$ *and*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \tag{54}$$

Then, the following estimate holds

$$\lim_{N\to\infty} N^{2q+\frac{3}{2}} ||R_N(f,x)||_{L_2} = \frac{1}{\pi^{2q+2}\sqrt{4q+3}} \sqrt{A^2_{2q+1}(f) + B^2_{2q+1}(f)}. \tag{55}$$

Proof. We prove that $c_{0,q} = 1$. We use (53) and after some transformations derive

$$\begin{aligned} c_{p,q} = \frac{(p+2q+1)!\sqrt{4q+3}}{(2q+1)!} &\left(\int_1^\infty \left(\frac{(-1)^p}{t^{2q+2}} - \sum_{j=1}^{p} \frac{e^{-\tau_j(t-1)}}{\prod_{\substack{i=1\\ i\neq j}}^{p}(\tau_i - \tau_j)} \right.\right. \\ &\left.\left. \times \sum_{k=0}^{p} (-1)^{k+1}\gamma_k(\tau) \sum_{m=0}^{p-k-1} \tau_j^m \frac{(2q+p-k-m)!}{(2q+1)!} \right)^2 dt \right)^{\frac{1}{2}}. \end{aligned} \tag{56}$$

By putting $p = 0$, we get

$$c_{0,q} = \sqrt{4q+3} \left(\int_1^\infty \frac{1}{t^{4q+4}} dt \right)^{\frac{1}{2}} = 1, \tag{57}$$

which concludes the proof. □

Comparison of Theorems 5 and 6 shows that the classical expansions and the MFP-approximations have the same convergence rates in the L_2-norm. However, comparison of constants $c_{p,q}$ and $c_{0,q} = 1$ shows that the rational approximations are asymptotically more accurate and, the improvement is thanks to the factor $c_{0,q}/c_{p,q} = 1/c_{p,q}$. Table 3 shows the numerical values of ratio $1/c_{p,q}$. For example, when $q = 6$ and $p = 4$, the asymptotic improvement is 5595 times.

$p\backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p=1$	3.4	6.6	9.9	13	16	19	22
$p=2$	6.3	20	41	70	107	151	203
$p=3$	9.8	46	125	265	481	791	1212
$p=4$	13	89	310	797	1706	3229	5595

Table 3: The values of the ratio $1/c_{p,q}$.

Let us see, how those asymptotic estimates could be achieved for moderate values of N. Table 4 shows the values of $\frac{||R_N||_{L_2}}{||R_{N,p}||_{L_2}}$ while approximating

$p\backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p=1$	3.3	6.3	9.1	11	15	16	18
$p=2$	6.1	18	34	55	89	106	134
$p=3$	9.2	38	96	185	343	461	647
$p=4$	12	71	217	493	1042	1565	2413

Table 4: The values of $\frac{||R_N||_{L_2}}{||R_{N,p}||_{L_2}}$ while approximating (30) with $N=64$.

$f_q(x)$ (see (30)) for $N = 64$. We see that the corresponding numbers in Tables 3 and 4 are close for small p and q.

Theorem 5 shows that the L_2-error of the MFP-approximations depends only on the τ_k parameters which are the coefficients of the second term in the asymptotic expansion of $\theta_k = \theta_k(N)$ in terms of $\frac{1}{N}$. Paper [11] showed that the accuracy of the rational approximations by the classical Fourier basis could be increased by appropriate selection of those parameters. We try the same approach for the MTR-approximations with parameters θ^c and θ^s defined by (19).

We omit the proof of the next theorem as it imitates the proof of similar theorem in [11].

Theorem 7 *Let $f \in C^{(2q+p+1)}[-1,1]$ and $f^{(2q+p+1)} \in BV[-1,1]$, $q \geq 0$ $p \geq 1$. Let*

$$f^{(2k+1)}(\pm 1) = 0,\ k = 0,\dots,q-1, \tag{58}$$

and

$$\theta_k^c = \theta_k^s = 1 - \frac{\tau_k}{N},\ \tau_k > 0,\ k = 1,\dots,p. \tag{59}$$

Then,

$$\lim_{N\to\infty} N^{2q+\frac{3}{2}}||R_{N,p}||_{L_2} = \frac{c^*_{p,q}}{\pi^{2q+2}\sqrt{4q+3}}\sqrt{A^2_{2q+1}(f) + B^2_{2q+1}(f)}, \tag{60}$$

where $c^*_{p,q}$ is defined by (53) for all $\tau_k > 0$, $k = 1,\dots,p$.

In the estimate (60), the constant $c^*_{p,q}$ will coincide with $c_{p,q}$ if the parameters τ_k are the roots of $L_p^{(2q+1)}(x)$. Our goal is the minimization of $c^*_{p,q}$ by an appropriate selection of parameters τ_k, $k = 1,\dots,p$ (see also [11] for a similar problem). The corresponding MTR-aproximation we call as L_2-minimal MTR-approximation. Table 5 shows some of the optimal values of τ_k (see [11]) with the corresponding value of $1/c^*_{p,q}$ which shows the efficiency of the L_2-minimal MTR-approximation in comparison with the classical expansions by the modified Fourier basis with the exact asymptotic constant $c_{0,q} = 1$.

q	0	1	2	3	4	5	6
$1/c_{1,q}^*$	5.7	12.2	18.7	25.3	31.8	38.3	44.9
τ_1	1.35	3.3	5.3	7.3	9.3	11.3	13.3
$1/c_{2,q}^*$	21.1	82.4	183.8	326	508	731	994
τ_1	2.8	5.4	3.3	4.9	6.5	14.9	17.1
τ_2	0.53	1.8	7.8	10.2	12.6	8.2	9.9
$1/c_{3,q}^*$	61.3	412	1297	2965	5667	9650	15167
τ_1	4.2	7.3	5.2	7.2	4.95	17.9	20.4
τ_2	0.3	3.2	10.1	12.7	9.1	6.4	7.9
τ_3	1.29	1.12	2.3	3.6	15.3	11.1	13.1
$1/c_{4,q}^*$	157	1704	7378	21442	49700	99492	179701
τ_1	2.2	4.6	6.9	5.4	11.4	5.2	10.6
τ_2	5.7	9.1	12.2	15.1	17.9	8.8	6.5
τ_3	0.1	2.1	3.7	9.2	7.1	20.6	23.3
τ_4	0.67	0.74	1.6	2.7	3.9	13.7	15.9

Table 5: Numerical values of $1/c_{p,q}^*$ for $p = 1, 2, 3, 4$ with the optimal values of τ_k, $k = 1, \ldots, p$ that minimize $c_{p,q}^*$.

A comparison of Tables 3 and 5 shows that the L_2-error of the L_2-minimal MTR-approximation is smaller than the L_2-error of the corresponding MFP-approximation. For example, in case of $p = 2$ and $q = 6$, the asymptotic improvement is thanks to the factor 4.9.

Table 6 presents the values of the ratio $\frac{||R_N||_{L_2}}{||R_{N,p}^*||_{L_2}}$ while approximating $f_q(x)$ with $N = 64$. Here, $R_{N,p}^*$ corresponds to the error of the L_2-minimal MTR-approximation. We see that in case of the L_2-minimal approximations it is harder to achieve the theoretical estimates with moderate values of N.

$p\backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p = 1$	5.72	11.78	16.35	17.46	17.70	13.49	11.33
$p = 2$	20.01	53.39	60.92	60.78	68.71	64.43	70.81
$p = 3$	31.32	38.96	108.66	157.55	269.63	436.33	1016.48
$p = 4$	10.88	56.53	148.14	367.79	1476.83	1823.77	864.76

Table 6: The values of $\frac{||R_N||_{L_2}}{||R_{N,p}^*||_{L_2}}$ while approximating (30) with $N = 64$.

Now, let us compare the pointwise convergence of the rational approximations for $|x| < 1$. Figure 3 shows the behaviors of the L_2-minimal MTR-approximations on $[-0.7, 0.7]$. A comparison with Figure 1 shows that the MFP-approximations are substantially more accurate on $|x| < 1$ than the L_2-minimal MTR-approximations.

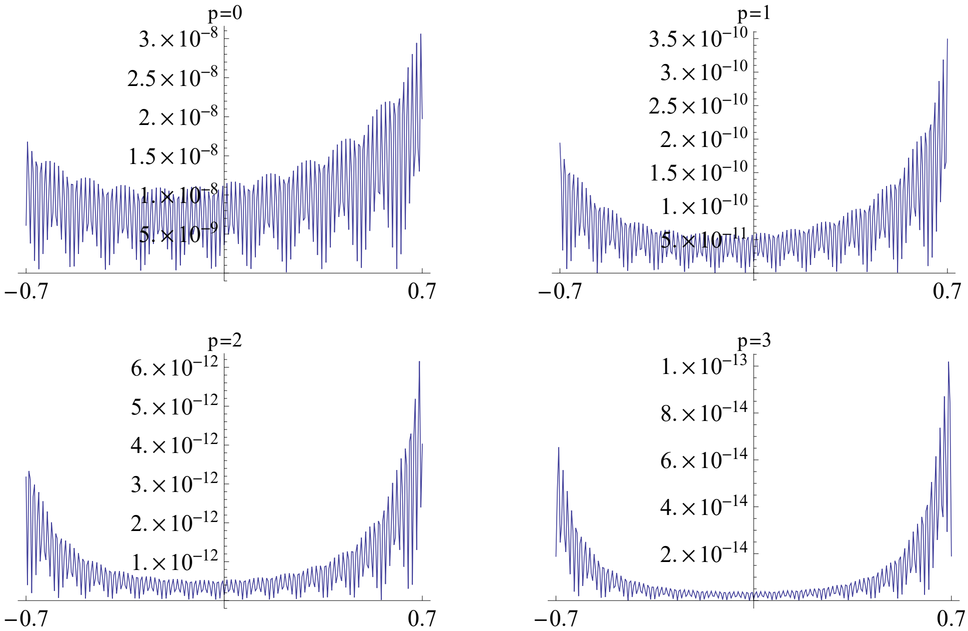

Figure 3: The graphs of $R^*_{N,p}$ for $N = 64$ while approximating $f_1(x)$ on $[-0.7, 0.7]$.

3 Conclusion

In this article, we explored convergence of the modified-trigonometric-rational (MTR-) approximations in different frameworks. These approximations arose while accelerating the convergence of expansions by the modified Fourier basis by means of parametrized rational corrections. Different approaches are known for the parameter determination.

One approach led to the modified Fourier-Pade (MFP-) approximations. Previously, we investigated the pointwise convergence of the MFP-approximations away from the endpoints of an interval of approximation and the L_2-convergence on the entire interval.

Here, we continued those investigations considering the pointwise convergence at the endpoints of the interval by deriving the exact constant of the main term of the asymptotic error. Although the classical modified expansions and the MFP-approximations have the same convergence rates, the comparison of the exact constants outlines the bigger asymptotic accuracy of the rational approximations.

As a result, we observed that the MFP-approximations are much more accurate in comparison with the classical expansions in all considered frameworks for sufficiently smooth functions.

Another approach for parameter determination leads to MTR-approximations optimal in different frameworks. Here, we considered optimality in the sense of the L_2-norm and found the optimal values of parameters minimizing the corresponding constant of the main term of the asymptotic error. We called this approach as L_2-minimal MTR-approximation.

Although the MFP and the L_2-minimal MTR-approximations have the same convergence rate in the L_2-norm, the comparison of the asymptotic estimates shows better accuracy of the optimal approach. However, optimality in the L_2-norm degrades the pointwise accuracy away from the endpoints in comparing with the MFP-approximations.

Acknowledgements

The research was supported by SCS grant 16A-1a40.

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Please, cite to this paper as published in
Armen. J. Math., V. **9**, N. 2(2017), pp. 68–83