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Numerical Blow-up Analysis for Fractional Ordinary Differential Equations with Linearly Implicit L1 Method

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Abstract. In this paper, we consider the blow-up behavior of linearly implicit L1 method for fractional ordinary differential equations. Based on Nakagawa's criteria, a suitable adaptive step strategy is introduced. The existence of global numerical solution is proved. Moreover, we show that the finite blow-up behaviors are replicated for any positive solution. Finally, some numerical examples are given to test the main result.

Key Words: Fractional Ordinary Differential Equations, Linearly Implicit L1 Method, Global Existence, Blow-up
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Introduction

In the past few decades, fractional ordinary differential equations (FODEs) have attracted many authors (see, for example, [19,25,30]). It is caused both by the intensive development of the theory of fractional calculus itself and by its broader applications such as physics, chemistry, biology, economics, control theory, and many different aspects [2, 12, 21, 23, 24]. There are a great number of papers dealing with the existence, stability, smoothness, periodicity, chaotic and asymptotic behaviors of initial or boundary value problem for some FODEs. For details and examples, see [4,5,13,15,16,20,22] and the references therein.

Though the exact solutions of some special cases can be obtained, many FODEs can not be solved analytically. Therefore, numerical methods such as finite difference methods [17,18], finite element methods [10,14] and spectral methods [29] have been established for solving FODEs. In these numerical methods, finite difference methods have played a very important role.

Recently, much attention has been paid to the consideration of numerical blow-up for ordinary differential equations [6], partial differential equations [1,3] and integro-differential equations [28]. During the reproduction of finite blow-up time, many techniques have been developed to implement the time-stepping, such as the adaptive time-stepping approaches (see [9,26]) and the uniform time-stepping (see [7, 8]). Since the uniform time-stepping is not suitable for simulating blow-up solutions (see [26]), the key point is how to built a suitable adaptive time-stepping strategy.

Up to now, limited work has been done in the numerical blow-up for FODEs except for [11]. In [11], the blow-up behaviors of FODEs are simulated by the explicit and implicit schemes with a uniform mesh. Motivated by the above mentioned works, we studied the global existence and blow-up of numerical solutions of FODEs by the explicit L1-scheme in [27].

It is well-known that when solving FODEs, implicit schemes often offer higher computational efficiency and reliability than explicit schemes in cases involving fine grids, fractional orders approaching 2, or long-time integration. Different from [27], to achieve flexibility in step-size selection and absolute stability in numerical computation, in this paper, we are interested in the blow-up analysis of numerical solutions to FODEs on bounded domains by using the linearly implicit L1 method. We adapt the step to bound the numerical solutions and prove that a global numerical solution to FODEs exists when the initial value is sufficiently small. Furthermore, by the introduction and analysis of an adaptive step strategy, the replicative finite blow-up results are proved under certain conditions.

The structure of this paper is organized as follows. In Section 1, we apply the linearly implicit L1 method to FODEs. In Section 2, the global existence of numerical solutions is studied. In Section 3, the numerical blow-up is proved after providing an adaptive step strategy. Some examples are given to illustrate the main results in Section 4.

1 Linearly implicit L1 method

We consider the following nonlinear FODEs:

$$\begin{aligned} &{}^C\mathcal{D}_t^\alpha u = -\lambda u + u^p, \quad \lambda > 0, \\ &u(0) = u_0 > 0, \end{aligned} \tag{1}$$

where $0 < \alpha < 1$, $p > 1$ and ${}^C\mathcal{D}_t^\alpha$ is the Caputo fractional derivative of fractional order α which is defined by

$$({}^C\mathcal{D}_t^\alpha u)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds.$$

Let $\Delta t_n = t_{n+1} - t_n$ be a step, it can be adapted during the numerical processes, then the linearly implicit L1 method for (1) is given by

$$\sum_{l=0}^{n} \frac{u_{l+1} - u_l}{\Delta t_l} \omega_{n,l} = -\lambda u_{n+1} + (u_n)^p, \tag{2}$$

where

$$\omega_{n,l} = \frac{1}{\Gamma(1-\alpha)} \int_{t_l}^{t_{l+1}} (t_{n+1} - s)^{-\alpha} ds, \quad j = 0, 1, \ldots, n. \tag{3}$$

Let

$$\gamma_{n,l} = \frac{\omega_{n,l}}{\Delta t_l},$$

then from (2) we have

$$\left(1 + \frac{\lambda}{\gamma_{n,n}}\right) u_{n+1} = \sum_{l=0}^{n} C_{n,l} u_l + \frac{1}{\gamma_{n,n}} (u_n)^p, \tag{4}$$

where

$$C_{n,l} = \frac{\gamma_{n,l} - \gamma_{n,l-1}}{\gamma_{n,n}}, \quad l = 0, 1, \ldots, n$$

and $\gamma_{n,-1} = 0$.

Notice that

$$\gamma_{n,n} = \frac{1}{\Delta t_n^{\alpha} \Gamma(2-\alpha)},$$

then (4) gives

$$u_{n+1} = \frac{\sum\limits_{l=0}^{n} C_{n,l} u_l + \Delta t_n^{\alpha} \Gamma(2-\alpha) (u_n)^p}{1 + \lambda \Delta t_n^{\alpha} \Gamma(2-\alpha)}. \tag{5}$$

Before the deep analysis, we present the preservation of the positivity of numerical solutions. This is important both for the theoretical analysis and for the practical application.

Lemma 1 *For $l = 1, 2, \ldots, n$, it holds $\gamma_{n,l} - \gamma_{n,l-1} > 0$.*

Proof. The proof is obvious by the mean value theorem for definite integral. □

Lemma 2 *The numerical solutions $u_n > 0$ for all $n > 0$ whenever $u_0 > 0$.*

Proof. Since $1 + \lambda/\gamma_{n,n} > 0$ in (4), the proof follows from Lemma 1 and the method of mathematical induction. □

Roughly speaking, the increments of numerical solutions must be linearly controllable, i.e.,

$$\sup_{n\geqslant 0}\frac{u_{n+1}}{u_n}<\infty. \tag{6}$$

For convenience, in this paper we only consider the blow-up behavior of numerical solutions such that the criteria (6) holds.

Definition 1 *A numerical solution to* (5) *blows up in finite time if there is an adaptive step-size such that the criteria* (6) *holds,* $\lim\limits_{n\to\infty}u_n=\infty$ *and* $\lim\limits_{n\to\infty}t_n<\infty$.

2 Global existence for numerical solutions

In this section, we present the global existence of numerical solution.

Definition 2 *A numerical solution exists globally if* $\lim\limits_{n\to\infty}t_n=\infty$ *for sufficiently small initial value* u_0.

Theorem 1 *Assume that* $\alpha\in(0,1)$ *and* $p>1$. *Then for any given step* Δt_n*, the numerical solution* u_n *satisfy* $0<u_n\le u_0$ *for all* $n=1,2,\ldots$ *whenever*

$$0<u_0\le\lambda^{\frac{1}{p-1}}. \tag{7}$$

Proof. This theorem will be proved by the method of mathematical induction.

When $k=1$, from (7) we have $u_0^p\le\lambda u_0$, thus, by (5) with $n=0$ we obtain

$$u_1=\frac{u_0+\Delta t_0^\alpha\Gamma(2-\alpha)u_0^p}{1+\lambda\Delta t_0^\alpha\Gamma(2-\alpha)}\le\frac{u_0+\Delta t_0^\alpha\Gamma(2-\alpha)\lambda u_0}{1+\lambda\Delta t_0^\alpha\Gamma(2-\alpha)}=u_0.$$

Assume that $0<u_k\le u_0$ for $k=1,2,\ldots,n$. In view of (5) we obtain

$$\begin{aligned}u_{n+1}=&\frac{\sum\limits_{l=0}^{n}C_{n,l}u_l+\Delta t_n^\alpha\Gamma(2-\alpha)u_n^p}{1+\lambda\Delta t_n^\alpha\Gamma(2-\alpha)}\\ \le&\frac{\sum\limits_{l=0}^{n}C_{n,l}u_0+\Delta t_n^\alpha\Gamma(2-\alpha)u_0^p}{1+\lambda\Delta t_n^\alpha\Gamma(2-\alpha)}\\ \le&\frac{u_0+\Delta t_n^\alpha\Gamma(2-\alpha)\lambda u_0}{1+\lambda\Delta t_n^\alpha\Gamma(2-\alpha)}\\ =&u_0.\end{aligned}$$

The proof is complete. □

3 Adaptive step strategy and blow-up analysis

In this section, we present an adaptive step strategy. Moreover, we prove that numerical solutions exhibit the exact finite blow-up behavior.

For any given $\tau > 0$, the adaptive step from t_n to t_{n+1} is defined by

$$\Delta t_n = \tau \min\left\{1, \left(\frac{1}{(u_n)^{p-1}}\right)^{\frac{1}{\alpha}}\right\}. \tag{8}$$

Next, we show the monotonicity of the numerical solution. For convenience, from (3) we can rewrite $\gamma_{n,l}$ as follows:

$$\begin{aligned}\gamma_{n,l} = \frac{\omega_{n,l}}{\Delta t_l} &= \frac{1}{\Delta t_l \Gamma(1-\alpha)}\int_{t_l}^{t_{l+1}} (t_{n+1}-s)^{-\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^1 (t_{n+1}-t_l-\theta\Delta t_l)^{-\alpha} d\theta.\end{aligned} \tag{9}$$

Lemma 3 *Let $\tau > 0$ and the adaptive step be defined by* (8)*. Then the numerical solution u_n is an increasing sequence if*

$$u_0 > \lambda^{\frac{1}{p-1}}. \tag{10}$$

Proof. The lemma is proved by the method of mathematical induction.

By (5) we have

$$u_1 = \frac{u_0 + \Delta t_0^\alpha \Gamma(2-\alpha) u_0^p}{1+\lambda \Delta t_0^\alpha \Gamma(2-\alpha)} > \frac{u_0 + \Delta t_0^\alpha \Gamma(2-\alpha)\lambda u_0}{1+\lambda \Delta t_0^\alpha \Gamma(2-\alpha)} = u_0.$$

Now suppose that u_k is increasing for all $k < n$. From (4) we get

$$u_n^p - \lambda u_{n+1} = \sum_{l=0}^{n} \gamma_{n,l}(u_{l+1}-u_l),$$

thus,

$$u_{n-1}^p - \lambda u_n = \sum_{l=0}^{n-1} \gamma_{n-1,l}(u_{l+1}-u_l). \tag{11}$$

Then by (9) and (11) we have

$$\begin{aligned} u_{n-1}^p - \lambda u_n &= \sum_{l=0}^{n-1} \gamma_{n-1,l}(u_{l+1} - u_l) \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{l=0}^{n-1} (u_{l+1} - u_l) \int_0^1 (t_n - t_l - \theta \Delta t_l)^{-\alpha} d\theta \\ &> \frac{1}{\Gamma(1-\alpha)} \sum_{l=0}^{n-1} (u_{l+1} - u_l) \int_0^1 (t_{n+1} - t_l - \theta \Delta t_l)^{-\alpha} d\theta \\ &= \sum_{l=0}^{n-1} \gamma_{n,l}(u_{l+1} - u_l). \end{aligned} \tag{12}$$

Further, by (4) and (12) we have

$$\begin{aligned} u_{n+1} &= \sum_{l=0}^{n} \frac{\gamma_{n,l} - \gamma_{n,l-1}}{\gamma_{n,n}} u_l + \frac{1}{\gamma_{n,n}} (u_n^p - \lambda u_{n+1}) \\ &\geq \sum_{l=0}^{n} \frac{\gamma_{n,l} - \gamma_{n,l-1}}{\gamma_{n,n}} u_l + \frac{1}{\gamma_{n,n}} (u_{n-1}^p - \lambda u_{n+1}) \\ &> \sum_{l=0}^{n} \frac{\gamma_{n,l} - \gamma_{n,l-1}}{\gamma_{n,n}} u_l + \frac{1}{\gamma_{n,n}} \left(\sum_{l=0}^{n-1} \gamma_{n,l}(u_{l+1} - u_l) + \lambda(u_n - u_{n+1}) \right) \\ &= u_n - \frac{1}{\gamma_{n,n}} \sum_{l=0}^{n-1} \gamma_{n,l}(u_{l+1} - u_l) + \frac{1}{\gamma_{n,n}} \left(\sum_{l=0}^{n-1} \gamma_{n,l}(u_{l+1} - u_l) + \lambda(u_n - u_{n+1}) \right) \\ &= u_n + \frac{\lambda}{\gamma_{n,n}} (u_n - u_{n+1}), \end{aligned}$$

therefore,

$$\left(1 + \frac{\lambda}{\gamma_{n,n}}\right)(u_{n+1} - u_n) > 0,$$

that is,

$$u_{n+1} > u_n.$$

The proof is completed. □

In the following, we show that numerical solutions exhibit the exact finite blow-up behavior.

Theorem 2 *Assume that* $0 < \alpha < 1$*,* $p > 1$*,* $u_0 > 0$ *and* (10) *holds. Then for any given* $\tau > 0$*, the numerical solution with the adaptive step* (8) *blows up in finite time if all the following conditions hold:*

1) $\lim_{n \to \infty} u_n = \infty$*,* *2)* $\lim\limits_{n\to\infty} \Delta t_n = 0$,

3) there exist constants n_τ *and* $\rho > 0$ *such that*

$$u_{n+n_\tau} \geq (1+\rho\tau)u_n,$$

4) $\lim\limits_{n\to\infty} t_n = \sum\limits_{n=0}^{\infty} \Delta t_n < \infty$.

Proof. 1) Assume that there exists a subsequence u_{n_l} such that $\sup\limits_l u_{n_l} < \infty$, then $\inf\limits_l \Delta t_{n_l} > 0$. In view of Lemma 3, we have $\sup\limits_n u_n < \infty$. Therefore, $\lim\limits_{n\to\infty} u_n = u_* < \infty$. Then there is a sufficiently small $\epsilon > 0$ such that

$$\Gamma(2-\alpha)(\inf_l \Delta t_{n_l}^{\alpha})((u_*-\epsilon)^p - \lambda(u_*-\epsilon)) > 2\epsilon, \tag{13}$$

and there is $N_0 > 0$ such that $u_n > u_* - \epsilon > u_0$ for $n > N_0$. Then for sufficiently large n, we have

$$\begin{aligned}
\omega_n &= \sum_{l=0}^{n} C_{n,l}u_l \geq \sum_{l=N_0}^{n} C_{n,l}u_l > (u_*-\epsilon)\sum_{l=N_0}^{n} \frac{\gamma_{n,l}-\gamma_{n,l-1}}{\gamma_{n,n}} \\
&= (u_*-\epsilon)\left(1 - \Delta t_n^{\alpha}(1-\alpha)\int_0^1 (t_{n+1} - t_{N_0-1} - \theta\Delta t_{N_0-1})^{-\alpha} d\theta\right) \\
&\geq (u_*-\epsilon)\left(1 - \tau^{\alpha}(1-\alpha)\int_0^1 (t_{n+1} - t_{N_0-1} - \theta\Delta t_{N_0-1})^{-\alpha} d\theta\right).
\end{aligned}$$

On the other hand, for sufficiently large n, since $0 < \alpha < 1$ and $\lim\limits_{n\to\infty} t_n = \infty$, it holds $\omega_n > u_* - 2\epsilon$.

Thus, for sufficiently large l, by (13) we obtain

$$\begin{aligned}
u_* > u_{n_l+1} &= \omega_{n_l} + \Delta t_{n_l}^{\alpha}\Gamma(2-\alpha)((u_{n_l})^p - \lambda u_{n_l+1}) \\
&> u_* - 2\epsilon + \Gamma(2-\alpha)(\inf_l \Delta t_{n_l}^{\alpha})((u_*-\epsilon)^p - \lambda(u_*-\epsilon)) \\
&> u_* - 2\epsilon + 2\epsilon \\
&= u_*,
\end{aligned}$$

which is a contradiction.

2) This is a direct result from (8) and the proof of 1).

3) Let $\tau > 0$ and n_τ be defined by

$$(1+n_\tau)^{1-\alpha} - n_\tau^{1-\alpha} < \frac{1}{4}\tau\Gamma(2-\alpha). \tag{14}$$

From Lemma 3 and (14) we have

$$\begin{aligned}
\omega_{n+n_\tau-1} &= \sum_{l=0}^{n+n_\tau-1} \frac{\gamma_{n+n_\tau-1,l}-\gamma_{n+n_\tau-1,l-1}}{\gamma_{n+n_\tau-1,n+n_\tau-1}} u_l \\
&\geq \sum_{l=n}^{n+n_\tau-1} \frac{\gamma_{n+n_\tau-1,l}-\gamma_{n+n_\tau-1,l-1}}{\gamma_{n+n_\tau-1,n+n_\tau-1}} u_l \\
&\geq \frac{\gamma_{n+n_\tau-1,n+n_\tau-1}-\gamma_{n+n_\tau-1,n-1}}{\gamma_{n+n_\tau-1,n+n_\tau-1}} u_n \\
&\geq \left(1-\frac{1}{4}\tau\Gamma(2-\alpha)\right) u_n.
\end{aligned} \tag{15}$$

For sufficiently large n, we assume that

$$\lambda u_{n+n_\tau} < \frac{1}{2}(u_{n+n_\tau-1})^p, \Delta t_{n+n_\tau-1}^\alpha (u_{n+n_\tau-1})^p \geq \tau u_n.$$

Thus, from (14) and (15) we obtain

$$\begin{aligned}
u_{n+n_\tau} &= \omega_{n+n_\tau-1} + \Delta t_{n+n_\tau-1}^\alpha \Gamma(2-\alpha)(-\lambda u_{n+n_\tau} + u_{n+n_\tau-1}^p) \\
&\geq \left(1-\frac{1}{4}\tau\Gamma(2-\alpha)\right) u_n + \frac{1}{2}\Delta t_{n+n_\tau-1}^\alpha \Gamma(2-\alpha) u_{n+n_\tau-1}^p \\
&\geq \left(1-\frac{1}{4}\tau\Gamma(2-\alpha)\right) u_n + \frac{1}{2}\tau\Gamma(2-\alpha) u_n \\
&= (1+\rho\tau)u_n,
\end{aligned}$$

where $\rho = (\Gamma(2-\alpha))/4 > 0$.

4) For sufficiently large n, we assume that $\Delta t_n^\alpha (u_n)^p \geq \tau^\alpha u_n$. From (8) and the proof of 1), 2) and 3), we have

$$\begin{aligned}
\Delta t_{n+n_\tau} &\leq \tau \left(\frac{1}{u_{n+n_\tau}^{p-1}}\right)^{\frac{1}{\alpha}} \leq \tau \left(\frac{1}{(1+\rho\tau)^{p-1} u_n^{p-1}}\right)^{\frac{1}{\alpha}} \\
&\leq \tau \left(\frac{1}{(1+\rho\tau)^{p-1} \frac{\tau^\alpha}{\Delta t_n^\alpha}}\right)^{\frac{1}{\alpha}} = \Delta t_n \left(\frac{1}{1+\rho\tau}\right)^{\frac{1}{\alpha(p-1)}},
\end{aligned}$$

thus,

$$\sup_n \frac{\Delta t_{n+n_\tau}}{\Delta t_n} \leq \left(\frac{1}{1+\rho\tau}\right)^{\frac{1}{\alpha(p-1)}} < 1.$$

The proof is complete. □

4 Numerical experiments

To illustrate the main results of the paper, we present some examples in this section.

Consider the following FODEs:

$$\begin{aligned} &{}^{C}\mathcal{D}_t^{\frac{3}{5}}u = -11u + u^2, \quad t > 0, \\ &u(0) = u_0 > 0. \end{aligned} \tag{16}$$

In Figure 1, we draw the curves of u_n for $\alpha = 3/5$, $p = 2$, $\tau = 0.002$ and different initial values under the double logarithmic coordinates. It can be seen that the initial data has important influence on blow-up of numerical solution. The blow-up and global existence of numerical solution are various with respect to the initial values, which is in agreement with Theorems 1 and 2.

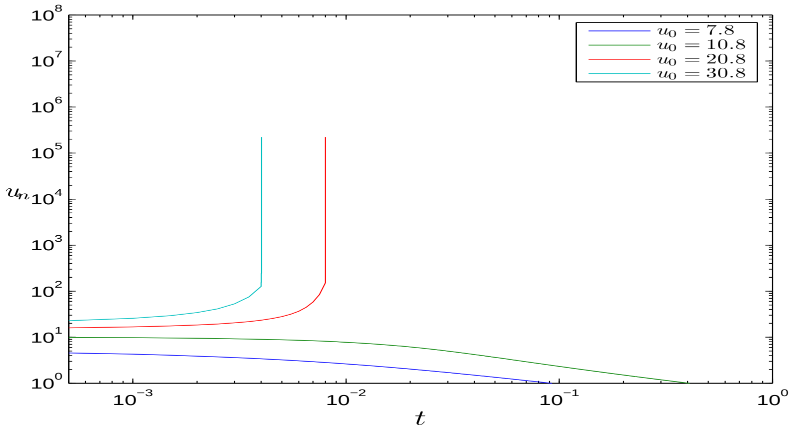

Figure 1: The numerical solutions of (16) with $\alpha = 3/5$, $p = 2$, $\tau = 0.002$ and different initial values

To show the influence of fractional exponents to the blow-up times, we consider the following FODEs:

$$\begin{aligned} &{}^{C}\mathcal{D}_t^{\alpha}u = -12u + u^2, \quad t > 0, \\ &u(0) = 26. \end{aligned} \tag{17}$$

In Figure 2, we draw the curves of u_n for $p = 2$, $\tau = 0.002$, $u_0 = 26$ and different fractional exponents under the double logarithmic coordinates. It can be seen that all the numerical solutions blow up in finite time. Specifically, the blow-up times are increasing with respect to α.

To assess the influence of the power to the blow-up behaviors, we consider

$$\begin{aligned} &{}^{C}\mathcal{D}_t^{\frac{3}{5}}u = -28u + u^p, \quad t > 0, \\ &u(0) = 9.8. \end{aligned} \tag{18}$$

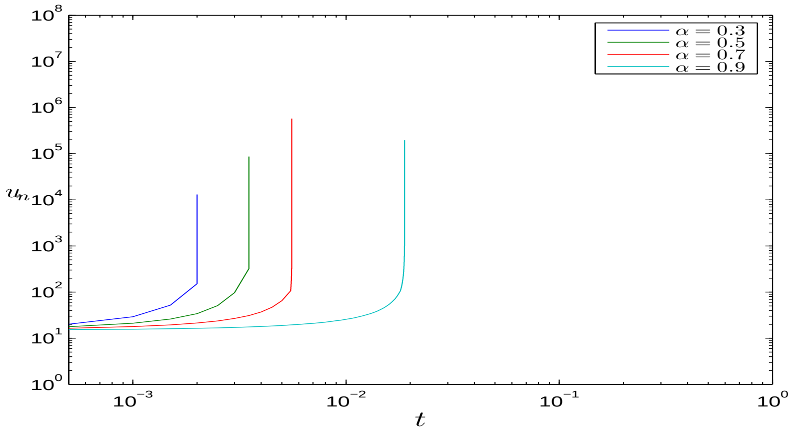

Figure 2: The numerical solutions of (17) with $p = 2$, $\tau = 0.002$, $u_0 = 26$ and different fractional exponents

In Figure 3, we draw the curves of u_n for $\alpha = 3/5$, $\tau = 0.002$, $u_0 = 9.8$ and different powers under the double logarithmic coordinates. From this figure we can see that a global solution exists for smaller power $p = 2.4$ and numerical solutions blow up in finite time for other lager powers. Furthermore, we notice that the numerical solutions blow up faster as p is increasing. This phenomena is consistent with Theorem 2.

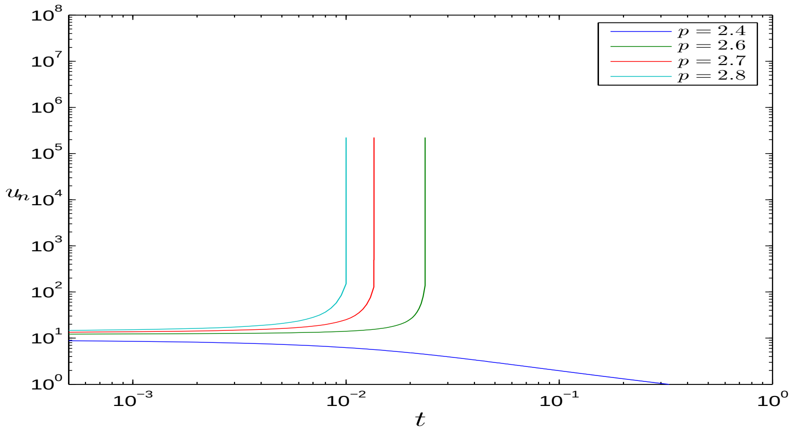

Figure 3: The numerical solutions of (18) with $\alpha = 3/5$, $\tau = 0.002$, $u_0 = 9.8$ and different powers

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