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Quantitative Uncertainty Principles Associated with the Sturm–Liouville Wavelet Transform

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Abstract. The Sturm–Liouville wavelet transform (SLWT) is a novel addition to the class of Sturm–Liouville transforms, which has gained a respectable status in the realm of time-frequency signal analysis within a short span of time. Given that the study of time-frequency analysis is of both theoretically interesting and practically useful, the aim of this paper is to explore a class of quantitative uncertainty principles (UP) associated with SLWT, including Faris local-type UP, Shannon-type UP, Donoho–Stark-type UP and Benedicks-type UP.

Key Words: Sturm–Liouville Operator, Sturm–Liouville Wavelet Transform, Quantitative Uncertainty Principles
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1 Introduction

The Fourier transform stands out as a significant discovery in mathematical sciences, that plays a crucial role in modern scientific and technological advancements. In signal processing, extensive research has utilized the Fourier transform to analyze stationary signals or processes with statistically invariant properties over time. Although, Fourier transforms have many successful applications that fascinated the mathematical, physical and engineering communities over decades, they still have numerous shortcomings.

One of the significant disadvantages of the Fourier transforms is that they do not give any information about the occurrence of the frequency component at a particular time. They only enable us to analyse the signals either in time domain or frequency domain, but not simultaneously in both domains [7, 24].

To get over this problem, Gabor [10] introduced the short time Fourier transform (STFT). The author considered a nonzero function $g \in L^2(\mathbb{R}^d)$ called a window and defined the short time Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ on the so-called time-frequency plan. Even though the short time Fourier transform solved the localization problem, a given window could not be well adapted to study every multi-frequency signals. To solve this issue, Grossman and Morlet [11] introduced the wavelet transform. The classical wavelet transform of a function $f \in L^2(\mathbb{R}^d)$ is defined on the so-called time-scale plan $\mathbb{R}_+^* \times \mathbb{R}^d$. The classical wavelet transform is closely related to signal theory, for more details, we refer the reader to [6, 35]. In recent years, many extensions and variants of the wavelet transform have been suggested, see, for instance, [2, 13, 15, 30]. Integral transforms have recently been the subject of extensive research within harmonic analysis [17–19, 21, 22]. A further important tool in signal theory, and the main object of the present work, is the Sturm–Liouville wavelet transform (SLWT).

We consider the Sturm–Liouville operator defined on $\mathbb{R}_+^*$ by

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'}{A}(x)\frac{\mathrm{d}}{\mathrm{d}x},$$

where A is a nonnegative function satisfying certain conditions, see Section 2. This operator has been extensively examined (or investigated) in harmonic analysis, as demonstrated in [3,4,36,39]. Specifically, we consider the Sturm–Liouville transform (SLT)

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x) A(x)\mathrm{d}x, \quad \lambda \in \mathbb{R}_+,$$

where φ_λ represents the Sturm–Liouville function described in Section 2 below. The SLT can be viewed as a generalization of various Fourier transforms [9, 12, 23]. Several uncertainty principles have been established for the SLT, for example, Bouattour and Trimèche [1] proved the Cowling–Price theorem, Daher et al. [5] established the Miyachi theorem, Ma [16] proved a Heisenberg uncertainty principle and Soltani [26, 27] proved a local uncertainty principle and a Donoho–Stark uncertainty principle. Additional details on the Sturm–Liouville transform is presented in [28–30].

A function $g \in L^2(\mathbb{R}_+, A(x)\mathrm{d}x)$ is called a Sturm–Liouville wavelet (SLW) if it satisfies the admissibility condition

$$0 < \omega_g := \int_{\mathbb{R}_+} |\mathscr{F}(g)(\lambda)|^2 \frac{\mathrm{d}\lambda}{\lambda} < \infty. \tag{1}$$

Let $g \in L^2(\mathbb{R}_+, A(x)\mathrm{d}x)$ be a SLW. For $f \in L^2(\mathbb{R}_+, A(x)\mathrm{d}x)$, we define the Sturm–Liouville wavelet transform (SLWT) by

$$\Phi_g(f)(a,b) := \int_{\mathbb{R}_+} f(x)\overline{\tau_b g_a(x)} A(x)\mathrm{d}x, \quad (a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+,$$

where τ_b is the Sturm–Liouville traslation introduced by Chébli [4] and g_a is the dilation function given by the relation

$$\mathscr{F}(g_a)(\lambda) = \mathscr{F}(g)(a\lambda), \quad \lambda \in \mathbb{R}_+.$$

In the classical context, Wilczok [38] first introduced the notion of quantitative uncertainty principles for the continuous wavelet transform. This work was subsequently extended to include the Hankel wavelet transform [2, 13, 15].

Our paper continues the investigation of the harmonic analysis associated with the Sturm–Liouville wavelet transform, building upon earlier works [30–33]. Our main objective is to explore several versions of quantitative uncertainty principles for this transform.

By analyzing the concentration property of the transform Φ_g, we first establish a dispersion inequality. This inequality is then used to establish a Faris-type local uncertainty principle for Φ_g. Then we demonstrate Shannon-type and Donoho–Stark uncertainty principles by utilizing the crucial orthogonal property of Φ_g. Finally, we apply a Benedicks-type uncertainty principle to the transform Φ_g, by examining two orthogonal projections P_g and P_U.

Recently, similar results were obtained by Soltani and Zarrougui (see [34]) for the Sturm–Liouville–Stockwell transform; the authors of the mentioned paper used a new convolution and modulation operator techniques. The quantitative uncertainty principles for the Sturm–Liouville–Stockwell transform are given for $f, g \in L^2\left(\mathbb{R}_+, \dfrac{\mathrm{d}\lambda}{2\pi|c(\lambda)|^2}\right)$, where $c(\lambda)$ is the Harish–Chandra function. However, in this paper, our quantitative uncertainty principles for the Sturm–Liouville wavelet transform are established for $f, g \in L^2(\mathbb{R}_+, A(x)\mathrm{d}x)$ with g satisfying an admissibility condition.

The paper is organized as follows. Sections 2 and 3 recall the necessary background on the Sturm–Liouville transform and the Sturm–Liouville wavelet transform. In Section 4, we establish several quantitative uncertainty principles associated with the SLST.

2 Sturm–Liouville transform $\mathscr{F}$

In this section, we recall some results about the Sturm–Liouville transform and we establish some properties of the Sturm–Liouville wavelet transform. We consider the Sturm–Liouville operator Δ defined on $\mathbb{R}_+^*$ by

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'}{A}(x)\frac{\mathrm{d}}{\mathrm{d}x},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. We assume that A satisfies the following conditions:

(i) A is increasing and $A(x) \to \infty$ as $x \to \infty$,

(ii) $\frac{A'}{A}$ is decreasing and $\frac{A'}{A}(x) \to 2\rho \geq 0$ as $x \to \infty$,

(iii) there exists a constant $\delta > 0$ such that

$$\frac{A'}{A}(x) = 2\rho + e^{-\delta x} D(x) \quad \text{if } \rho > 0,$$

$$\frac{A'}{A}(x) = \frac{2\alpha + 1}{x} + e^{-\delta x} D(x) \quad \text{if } \rho = 0,$$

where D is a C^∞-function on $\mathbb{R}_+^*$, bounded and with bounded derivatives on all interval $[x_0, \infty)$ for $x_0 > 0$.

For all $\lambda \in \mathbb{C}$, the equation

$$\Delta u = -(\lambda^2 + \rho^2)u, \quad u(0) = 1, \; u'(0) = 0,$$

admits a unique solution, denoted by φ_λ, and called Sturm–Liouville function (see [4, 36]). The kernel φ_λ satisfies the following properties.

(i) The function $\lambda \mapsto \varphi_\lambda(x)$, $x \in \mathbb{R}_+$, is analytic on $\mathbb{C}$.

(ii) The function $x \mapsto \varphi_\lambda(x)$, $\lambda \in \mathbb{C}$, is even and C^∞ on $\mathbb{R}$.

(iii) The function φ_λ possesses the following property (see [1]):

$$-1 \leq \varphi_\lambda(x) \leq 1, \quad \lambda, x \in \mathbb{R}_+. \tag{2}$$

Example 1 *[12] (The Bessel case). Let $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$ and $\rho = 0$. The Sturm–Liouville operator Δ is the Bessel operator denoted by Δ_α:*

$$\Delta_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x}\frac{d}{dx}.$$

The SL-function $\varphi_\lambda(x)$ is the spherical Bessel function $j_\alpha(\lambda x)$.

Example 2 *[9, 23] (The Jacobi case). Let $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, $\alpha > \beta \geq -1/2$ and $\rho = \alpha + \beta + 1 > 0$. The Sturm–Liouville operator Δ is the Jacobi operator denoted by $\Delta_{\alpha,\beta}$:*

$$\Delta_{\alpha,\beta} = \frac{d^2}{dx^2} + [(2\alpha + 1)\coth(x) + (2\beta + 1)\tanh(x)]\frac{d}{dx}.$$

The SL-function $\varphi_\lambda(x)$ is the Jacobi function denoted by $\phi_\lambda^{(\alpha,\beta)}(x)$:

$$\phi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1, -\sinh^2(x)),$$

where ${}_2F_1(a, b, c, z)$ is the hypergeometric function.

We denote by μ the measure defined on $\mathbb{R}_+$ by $\mathrm{d}\mu(x) := A(x)\mathrm{d}x$, and by $L^p(\mu)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}_+$ such that

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{R}_+} |f(x)|^p \mathrm{d}\mu(x)\right]^{1/p} < \infty, \quad p \in [1, \infty),$$
$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}_+} |f(x)| < \infty.$$

Further, denote by ν the measure defined on $\mathbb{R}_+$ by $\mathrm{d}\nu(\lambda) := \dfrac{\mathrm{d}\lambda}{2\pi |c(\lambda)|^2}$, and by $L^p(\nu)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}_+$ such that $\|f\|_{L^p(\nu)} < \infty$. Here $c(\lambda)$ is the Harish–Chandra function [37].

The Sturm–Liouville transform is defined for $f \in L^1(\mu)$ by

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x) \mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+.$$

The Sturm–Liouville transform satisfies the inequality

$$\|\mathscr{F}(f)\|_\infty \leq \|f\|_{L^1(\mu)}, \quad f \in L^1(\mu).$$

In the theorem below, we state some known properties of SLT, see [3, 4, 36].

Theorem 1 *Let $\mathscr{F}$ be a SLT. Then*

(i) $\mathscr{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular, we have the Plancherel formula for $\mathscr{F}$:

$$\|f\|_{L^2(\mu)} = \|\mathscr{F}(f)\|_{L^2(\nu)}.$$

(ii) For all $f, g \in L^2(\mu)$, the Parseval identity holds for $\mathscr{F}$:

$$\langle f, g\rangle_{L^2(\mu)} = \langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{L^2(\nu)}.$$

(iii) Let $f \in L^1(\mu)$ be such that $\mathscr{F}(f) \in L^1(\nu)$. Then

$$f(x) = \int_{\mathbb{R}_+} \varphi_\lambda(x) \mathscr{F}(f)(\lambda) d\nu(\lambda), \quad a.e. \;\; x \in \mathbb{R}_+.$$

The Sturm–Liouville function φ_λ satisfies the product formula [4, 36]

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}_+} \varphi_\lambda(z) w(x, y, z) \mathrm{d}\mu(z), \quad x, y \in \mathbb{R}_+, \tag{3}$$

where $w(x, y, .)$ is a measurable positive function on $\mathbb{R}_+$, with support in $[|x - y|, x + y]$, satisfying

$$\int_{\mathbb{R}_+} w(x, y, z) \mathrm{d}\mu(z) = 1, \tag{4}$$

$$w(x,y,z) = w(y,x,z) \quad \text{for } z \in \mathbb{R}_+, \tag{5}$$

$$w(x,y,z) = w(x,z,y) \quad \text{for } z \in \mathbb{R}_+^*. \tag{6}$$

The definition of the Sturm–Liouville translation operator is induced by (3). For $f \in L^1(\mu)$, this operator is defined by

$$\tau_y f(x) := \int_{\mathbb{R}_+} f(z) w(x,y,z) \mathrm{d}\mu(z), \quad x,y \in \mathbb{R}_+. \tag{7}$$

As a first remark, we note that the relations (4), (5) and (6) mean that

$$\tau_y f(x) = \tau_x f(y), \quad x,y \in \mathbb{R}_+,$$

and

$$\int_{\mathbb{R}_+} \tau_y f(x) \mathrm{d}\mu(x) = \int_{\mathbb{R}_+} f(x) \mathrm{d}\mu(x), \quad f \in L^1(\mu).$$

Theorem 2 *[29, 31] We have*

(i) For all $y \geq 0$ and $f \in L^p(\mu)$, $p \in [1,\infty]$, we have

$$\|\tau_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

(ii) For $f \in L^2(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathscr{F}(\tau_y f)(\lambda) = \varphi_\lambda(y) \mathscr{F}(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

Let $f, g \in L^2(\mu)$. The Sturm–Liouville convolution $f * g$ of f and g is defined by

$$f * g(x) := \int_{\mathbb{R}_+} f(y) \tau_x g(y) \mathrm{d}\mu(y), \quad x \in \mathbb{R}_+. \tag{8}$$

The convolution $*$ is commutative, associative and satisfies the Young inequality (see [20]). Let $p, q, r \in [1,\infty]$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Then for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ we have

$$\|f * g\|_{L^r(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Theorem 3 *[20, 37] We have*

(i) For $f, g \in L^2(\mu)$, the function $f * g$ belongs to $L^\infty(\mu)$, and

$$f * g = \mathscr{F}^{-1}(\mathscr{F}(f)\mathscr{F}(g)),$$

where $\mathscr{F}^{-1}$ is the inverse of the transform $\mathscr{F}$.

(ii) Let $f, g \in L^2(\mu)$. Then $f*g$ belongs to $L^2(\mu)$ if and only if $\mathscr{F}(f)\mathscr{F}(g)$ belongs to $L^2(\nu)$. In the L^2 case,

$$\mathscr{F}(f * g) = \mathscr{F}(f)\mathscr{F}(g).$$

(iii) Let $f, g \in L^2(\mu)$. Then

$$\int_{\mathbb{R}_+} |f * g(x)|^2 \, d\mu(x) = \int_{\mathbb{R}_+} |\mathscr{F}(f)(\lambda)|^2 |\mathscr{F}(g)(\lambda)|^2 \, d\nu(\lambda),$$

where both sides are finite or infinite.

3 Sturm–Liouville wavelet transform Φ_g

In this section, we first recall some fundamental results on the Sturm–Liouville wavelet transform. This transform has been investigated in depth in [20, 37], where precise definitions, examples, and a more complete discussion of its properties can be found. By using the harmonic analysis associated with the operator Δ, we show that the range of this integral transform is a reproducing kernel Hilbert space (RKHS).

We begin by the following lemma, see [20, 37].

Lemma 1 *Let $g \in L^2(\mu)$ and $a > 0$. Then there exists a function g_a in $L^2(\mu)$, such that*

$$\mathscr{F}(g_a)(\lambda) = \mathscr{F}(g)(a\lambda), \quad \lambda \in \mathbb{R}_+, \tag{9}$$

and satisfies

$$\|g_a\|_{L^2(\mu)} \leq \frac{k(a)}{\sqrt{a}} \|g\|_{L^2(\mu)}, \tag{10}$$

where

$$k(a) = \sup_{\lambda > 0} \frac{|c(\lambda)|}{|c(\lambda/a)|}.$$

For a function $g \in L^2(\mu)$ and for $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+$, we denote by $g_{a,b}$ the function defined on $\mathbb{R}_+$ by

$$g_{a,b}(x) := \tau_b g_a(x),$$

where τ_b are the generalized translation operators given by (7).

From Theorem 2 (i) and (10) it follows that function $g_{a,b}$ satisfies

$$\|g_{a,b}\|_{L^2(\mu)} \leq \frac{k(a)}{\sqrt{a}} \|g\|_{L^2(\mu)}. \tag{11}$$

Let $g \in L^2(\mu)$ be a SLW. For $f \in L^2(\mu)$, we define the Sturm–Liouville wavelet transform by

$$\Phi_g(f)(a,b) := \int_{\mathbb{R}_+} f(x)\overline{g_{a,b}(x)}\mathrm{d}\mu(x), \tag{12}$$

which can also be written in the form

$$\Phi_g(f)(a,b) = f * \overline{g_a}(b), \tag{13}$$

where $*$ is the generalized convolution product given by (8).

From (11) and (12) with Hölder's inequality we obtain

$$|\Phi_g(f)(a,b)| \leq \frac{k(a)}{\sqrt{a}}\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}, \quad (a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+. \tag{14}$$

Example 3 *The function g given by*

$$g(x) := \int_{\mathbb{R}_+} \lambda^2 e^{-\lambda^2}\varphi_\lambda(x)d\nu(\lambda), \quad x \in \mathbb{R}_+,$$

is a SLW and $\omega_g = 1/8$.

Denote by γ the measure defined on $\mathbb{R}_+^2$ by

$$\mathrm{d}\gamma(a,b) := \mathrm{d}\mu(b)\frac{\mathrm{d}a}{a},$$

and by $L^p(\gamma)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+^2$, such that $\|f\|_{L^p(\gamma)} < \infty$.

The following theorem establishes Plancherel and Parseval formulas for Φ_g.

Theorem 4 *Let $g \in L^2(\mu)$ be a SLW.*

(i) For $f \in L^2(\mu)$, we have

$$\|f\|^2_{L^2(\mu)} = \frac{1}{\omega_g}\|\Phi_g(f)\|^2_{L^2(\gamma)}.$$

(ii) For $f, h \in L^2(\mu)$, we have

$$\langle f, h\rangle_{L^2(\mu)} = \frac{1}{\omega_g}\langle \Phi_g(f), \Phi_g(h)\rangle_{L^2(\gamma)}.$$

Proof. (i) Using the Fubini theorem, Theorem 3 (iii), and relation (13), we obtain

$$\begin{aligned}
\frac{1}{\omega_g}\|\Phi_g(f)\|^2_{L^2(\gamma)} &= \frac{1}{\omega_g}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f * \overline{g_a}(b)|^2 \mathrm{d}\mu(b)\frac{\mathrm{d}a}{a} \\
&= \frac{1}{\omega_g}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |\mathscr{F}(f)(\lambda)|^2|\mathscr{F}(\overline{g_a})(\lambda)|^2\mathrm{d}\nu(\lambda)\frac{\mathrm{d}a}{a} \\
&= \int_{\mathbb{R}_+} |\mathscr{F}(f)(\lambda)|^2\left[\frac{1}{\omega_g}\int_{\mathbb{R}_+} |\mathscr{F}(g)(a\lambda)|^2\frac{\mathrm{d}a}{a}\right]\mathrm{d}\nu(\lambda).
\end{aligned}$$

By relation (1) we have

$$\frac{1}{\omega_g}\int_{\mathbb{R}_+}|\mathscr{F}(g)(a\lambda)|^2\frac{\mathrm{d}a}{a}=1.$$

Then we deduce the desired result from Theorem 1 (i).

(ii) The result is easily deduced from (i). □

Theorem 5 *Let $g\in L^2(\mu)$ be a SLW. Then for $f\in L^2(\mu)$, we have*

$$\Phi_g(f)(a,b)=\int_{\mathbb{R}_+}\mathscr{F}(f)(\lambda)\overline{\mathscr{F}(g)(a\lambda)}\varphi_\lambda(b)\,d\nu(\lambda).$$

Proof. Let $g\in L^2(\mu)$ be a SLW, and let $f\in L^2(\mu)$. By (12), Theorem 1 (ii), Theorem 2 (ii), (9) and (2), we have

$$\begin{aligned}\Phi_g(f)(a,b)&=\int_{\mathbb{R}_+}f(x)\overline{\tau_b g_a(x)}\mathrm{d}\mu(x)\\&=\int_{\mathbb{R}_+}\mathscr{F}(f)(\lambda)\overline{\mathscr{F}(\tau_b g_a)(\lambda)}\mathrm{d}\nu(\lambda)\\&=\int_{\mathbb{R}_+}\mathscr{F}(f)(\lambda)\overline{\mathscr{F}(g)(a\lambda)}\varphi_\lambda(b)\mathrm{d}\nu(\lambda).\end{aligned}$$

The theorem is proved. □

Theorem 6 *Let $g\in L^2(\mu)$ be a SLW. The space $\Phi_g(L^2(\mu))$ is a reproducing kernel Hilbert space with kernel function*

$$W_g((a,b);(a',b'))=\frac{1}{\omega_g}\langle g_{a,b},g_{a',b'}\rangle_{L^2(\mu)}.$$

Moreover, the kernel W_g is pointwise bounded, and for all $(a,b),(a',b')\in\mathbb{R}_+^*\times\mathbb{R}_+$,

$$|W_g((a,b);(a',b'))|\leq\frac{k(a)k(a')}{\omega_g\sqrt{aa'}}\|g\|^2_{L^2(\mu)}.$$

Proof. From (12) we have

$$W_g((a,b);(a',b'))=\frac{1}{\omega_g}\Phi_g(g_{a,b})(a',b'). \tag{15}$$

Then, from (11) for every $(a,b)\in\mathbb{R}_+^*\times\mathbb{R}_+$, the function $g_{a,b}$ belongs to $L^2(\mu)$, and therefore, function $W_g((a,b);.)$ belongs to $\Phi_g(L^2(\mu))$.
Moreover by (15) and Theorem 4 (i), we obtain

$$\|W_g((a,b);.)\|^2_{L^2(\gamma)}=\frac{1}{\omega_g^2}\|\Phi_g(g_{a,b})\|^2_{L^2(\gamma)}=\frac{1}{\omega_g}\|g_{a,b}\|^2_{L^2(\mu)}<\infty.$$

On the other hand, from (12) and Theorem 4 (ii), we have

$$\begin{aligned}\Phi_g(f)(a,b) &= \langle f, g_{a,b}\rangle_{L^2(\mu)}\\ &= \frac{1}{\omega_g}\langle \Phi_g(f), \Phi_g(g_{a,b})\rangle_{L^2(\gamma)}\\ &= \langle \Phi_g(f), W_g((a,b);.)\rangle_{L^2(\gamma)}.\end{aligned}$$

We conclude that $W_g((a,b);(a',b'))$ is a reproducing kernel of the Hilbert space $\Phi_g(L^2(\mu))$.
Finally, from Hölder's inequality and (11), we obtain

$$|W_g((a,b);(a',b'))| \leq \frac{1}{\omega_g}\|g_{a,b}\|_{L^2(\mu)}\|g_{a',b'}\|_{L^2(\mu)} \leq \frac{k(a)k(a')}{\omega_g\sqrt{aa'}}\|g\|^2_{L^2(\mu)}.$$

□

Next let us establish concentration property of Φ_g, and deduce dispersion inequality for Φ_g. Denote by η the measure defined on $\mathbb{R}^2_+$ by

$$\mathrm{d}\eta(a,b) := \frac{k^2(a)}{a}\mathrm{d}\gamma(a,b) = \frac{k^2(a)}{a^2}\mathrm{d}\mu(b)\mathrm{d}a,$$

and by $L^p(\eta)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}^2_+$, such that $\|f\|_{L^p(\eta)} < \infty$.

Theorem 7 *Let $g \in L^2(\mu)$ be a SLW and let $U \subset \mathbb{R}^2_+$ with*

$$0 < \eta(U) < \frac{\omega_g}{\|g\|^2_{L^2(\mu)}}.$$

Then for all $f \in L^2(\mu)$,

$$\|(1-\chi_U)\Phi_g(f)\|_{L^2(\gamma)} \geq \sqrt{\omega_g - \eta(U)\|g\|^2_{L^2(\mu)}}\|f\|_{L^2(\mu)},$$

where χ_U is the characteristic function of the set U. If $\Phi_g(f)$ is supported in U, then $f = 0$.

Proof. From Theorem 4 (i), we have

$$\|f\|^2_{L^2(\mu)} = \frac{1}{\omega_g}\|\Phi_g(f)\|^2_{L^2(\gamma)} = \frac{1}{\omega_g}\left(\|\chi_U\Phi_g(f)\|^2_{L^2(\gamma)} + \|(1-\chi_U)\Phi_g(f)\|^2_{L^2(\gamma)}\right).$$

On the other hand, from (14), we have

$$\|\chi_U\Phi_g(f)\|^2_{L^2(\gamma)} \leq \eta(U)\|g\|^2_{L^2(\mu)}\|f\|^2_{L^2(\mu)}. \tag{16}$$

□

For $r > 0$, denote by $B_+(0,r)$ the open ball of $\mathbb{R}^2_+$ given by

$$B_+(0,r) := \{(a,b) \in \mathbb{R}^2_+ : |(a,b)| = \sqrt{a^2+b^2} < r\}.$$

The following dispersion inequality holds.

Theorem 8 *Let $s > 0$. Then there exists a positive constant $C_g^{(1)}(s)$ such that for all $f \in L^2(\mu)$,*

$$\| |(a,b)|^s \Phi_g(f)(a,b)\|_{L^2(\gamma)} \geq C_g^{(1)}(s) \|f\|_{L^2(\mu)}.$$

Proof. Let $r > 0$ be such that

$$0 < \eta(B_+(0,r)) < \frac{\omega_g}{\|g\|^2_{L^2(\mu)}}.$$

By applying Theorem 7 with $U = B_+(0,r)$, we obtain

$$\begin{aligned}\left(\omega_g - \eta(B_+(0,r))\|g\|^2_{L^2(\mu)}\right) \|f\|^2_{L^2(\mu)} &\leq \int_{|(a,b)|\geq r} |\Phi_g(f)(a,b)|^2 \mathrm{d}\gamma(a,b) \\ &\leq \frac{1}{r^{2s}} \int_{|(a,b)|\geq r} |(a,b)|^{2s} |\Phi_g(f)(a,b)|^2 \mathrm{d}\gamma(a,b) \\ &\leq \frac{1}{r^{2s}} \| |(a,b)|^s \Phi_g(f)(a,b)\|^2_{L^2(\gamma)}.\end{aligned}$$

It remains to put

$$C_g^{(1)}(s) := r^s \sqrt{\omega_g - \eta(B_+(0,r))\|g\|^2_{L^2(\mu)}}.$$

□

4 Uncertainty principles for the transform Φ_g

In this section, we establish different quantitative uncertainty principles associated with SLWT, including Faris local UP, Shannon's UP, Donoho–Stark UP and Benedicks UP.

a) Faris local UP for Φ_g. We use the dispersion inequality for Φ_g to prove the following result.

Theorem 9 *Let $s > 0$. Then there exists a positive constant $C_g^{(2)}(s)$ such that, for all $f \in L^2(\mu)$ and every subset $U \subset \mathbb{R}^2_+$, $0 < \eta(U) < \infty$, it holds*

$$\|\chi_U \Phi_g(f)\|_{L^2(\gamma)} \leq C_g^{(2)}(s) \sqrt{\eta(U)} \| |(a,b)|^s \Phi_g(f)(a,b)\|_{L^2(\gamma)}.$$

Proof. From (16) we have

$$\|\chi_U \Phi_g(f)\|_{L^2(\gamma)} \leq \sqrt{\eta(U)}\|g\|_{L^2(\mu)}\|f\|_{L^2(\mu)}.$$

We obtain the result from Theorem 8, with

$$C_g^{(2)}(s) = \frac{\|g\|_{L^2(\mu)}}{C_g^{(1)}(s)}.$$

□

b) Shannon's UP for Φ_g. Let ρ be a probability density function on $\mathbb{R}_+^2$ such that

$$\int_{\mathbb{R}_+^2} \rho(a,b)\mathrm{d}\eta(a,b) = 1.$$

Following Shannon [25], the entropy of a probability density function ρ on $\mathbb{R}_+^2$ is defined by

$$E(\rho) := -\int_{\mathbb{R}_+^2} \ln(\rho(a,b))\rho(a,b)\mathrm{d}\eta(a,b). \tag{17}$$

Henceforth, we extend the definition of the entropy of a nonnegative measurable function ρ on $\mathbb{R}_+^2$ whenever the previous integral on the right-hand side is well defined.

Theorem 10 *For all* $f \in L^2(\mu)$, *it holds*

$$E\left(\frac{a}{k^2(a)}|\Phi_g(f)|^2(a,b)\right) \geq -2\omega_g\|f\|_{L^2(\mu)}^2 \ln\left(\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}\right).$$

Proof. Assume that $\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)} = 1$. By (14) we have

$$0 \leq \frac{\sqrt{a}}{k(a)}|\Phi_g(f)(a,b)| \leq 1, \quad (a,b) \in \mathbb{R}_+^2.$$

Then

$$-\frac{a}{k^2(a)}|\Phi_g(f)|^2(a,b)\ln\left(\frac{a}{k^2(a)}|\Phi_g(f)|^2(a,b)\right) \geq 0,$$

and therefore,

$$E\left(\frac{a}{k^2(a)}|\Phi_g(f)|^2(a,b)\right) \geq 0.$$

Next, let us drop the above assumption, and let

$$\varphi := \frac{f}{\|f\|_{L^2(\mu)}} \quad \text{and} \quad \psi := \frac{g}{\|g\|_{L^2(\mu)}}.$$

Then, $\varphi, \psi \in L^2(\mu)$ and $\|\varphi\|_{L^2(\mu)}\|\psi\|_{L^2(\mu)} = 1$. Therefore,

$$E\left(\frac{a}{k^2(a)}|\Phi_\psi(\varphi)|^2(a,b)\right) \geq 0.$$

Moreover,

$$\Phi_\psi(\varphi) = \frac{1}{\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}}\Phi_g(f),$$

thus, by (17) and Theorem 4 (i), we obtain

$$E\left(\frac{a}{k^2(a)}|\Phi_\psi(\varphi)|^2(a,b)\right) = \frac{E\left(\frac{a}{k^2(a)}|\Phi_g(f)|^2(a,b)\right)}{\|f\|^2_{L^2(\mu)}\|g\|^2_{L^2(\mu)}} + \frac{2\omega_g \ln\left(\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}\right)}{\|g\|^2_{L^2(\mu)}}.$$

Using the fact that $E\left(\frac{a}{k^2(a)}|\Phi_\psi(\varphi)|^2(a,b)\right) \geq 0$, we deduce that

$$E\left(\frac{a}{k^2(a)}|\Phi_g(f)|^2(a,b)\right) \geq -2\omega_g\|f\|^2_{L^2(\mu)} \ln\left(\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}\right).$$

The theorem is proved. □

c) Donoho–Stark UP for Φ_g. In the following, we establish Donoho–Stark uncertainty principle [8, 27] for the transform Φ_g.

Let E be a measurable subset of $\mathbb{R}_+$. We say that a function $f \in L^2(\mu)$ is ε-concentrated on E in $L^2(\mu)$-norm if

$$\|f - \chi_E f\|_{L^2(\mu)} \leq \varepsilon\|f\|_{L^2(\mu)}. \tag{18}$$

Let U be a measurable subset of $\mathbb{R}^2_+$ and let $f \in L^2(\mu)$. We say that $\Phi_g(f)$ is σ-concentrated on U in $L^2(\gamma)$-norm if

$$\|\Phi_g(f) - \chi_U\Phi_g(f)\|_{L^2(\gamma)} \leq \sigma\|\Phi_g(f)\|_{L^2(\gamma)}. \tag{19}$$

Theorem 11 *Let E be a measurable subset of $\mathbb{R}_+$, U be a measurable subset of $\mathbb{R}^2_+$ and let $f \in L^2(\mu)$. If f is ε-concentrated on E in $L^2(\mu)$-norm, $\Phi_g(f)$ is σ-concentrated on U in $L^2(\gamma)$-norm and $\varepsilon + \sigma < 1$, then*

$$\eta(U) \geq (1-\sigma-\varepsilon)^2\frac{\omega_g}{\|g\|^2_{L^2(\mu)}}.$$

Proof. Let $f \in L^2(\mu)$ and $\varepsilon + \sigma < 1$. Assume that $\eta(U) < \infty$. From (18), (19) and Theorem 4 (i) it follows that

$$\begin{aligned}
&\|\Phi_g(f) - \chi_U \Phi_g(\chi_E f)\|_{L^2(\gamma)} \\
&\qquad \leq \|\Phi_g(f) - \chi_U \Phi_g(f)\|_{L^2(\gamma)} + \|\chi_U \Phi_g(f - \chi_E f)\|_{L^2(\gamma)} \\
&\qquad \leq \sigma \|\Phi_g(f)\|_{L^2(\gamma)} + \|\Phi_g(f - \chi_E f)\|_{L^2(\gamma)} \\
&\qquad \leq (\sigma + \varepsilon)\sqrt{\omega_g}\|f\|_{L^2(\mu)}.
\end{aligned}$$

Then the triangle inequality and (16) show that

$$\begin{aligned}
\|\Phi_g(f)\|_{L^2(\gamma)} &\leq \|\chi_U \Phi_g(\chi_E f)\|_{L^2(\gamma)} + \|\Phi_g(f) - \chi_U \Phi_g(\chi_E f)\|_{L^2(\gamma)} \\
&\leq \sqrt{\eta(U)}\|g\|_{L^2(\mu)}\|f\|_{L^2(\mu)} + (\sigma + \varepsilon)\sqrt{\omega_g}\|f\|_{L^2(\mu)} \\
&\leq \left[\sqrt{\eta(U)}\|g\|_{L^2(\mu)} + (\sigma + \varepsilon)\sqrt{\omega_g}\right]\|f\|_{L^2(\mu)}.
\end{aligned}$$

By applying Theorem 4 (i), we obtain

$$\sqrt{\eta(U)} \geq (1 - \sigma - \varepsilon)\frac{\sqrt{\omega_g}}{\|g\|_{L^2(\mu)}},$$

which gives the desired result. □

d) Benedicks-type UP for Φ_g. In order to prove the Benedicks-type uncertainty principle for the transform Φ_g, we need the following definitions.

Let $g \in L^2(\mu)$. The orthogonal projection $P_g : L^2(\gamma) \longrightarrow L^2(\gamma)$ is defined by

$$\begin{aligned}
P_g(F)(a,b) &:= \langle F, W_g((a,b);.)\rangle_{L^2(\gamma)} \\
&= \int_{\mathbb{R}_+^2} F(a',b')\overline{W_g((a,b);(a',b'))}\mathrm{d}\gamma(a',b'), \quad (20)
\end{aligned}$$

where W_g is the kernel given by (15).

Let $U \subset \mathbb{R}_+^2$ with $\eta(U) < \infty$. We define the orthogonal projection $P_U : L^2(\gamma) \longrightarrow L^2(\gamma)$ by

$$P_U(F)(a,b) := \chi_U(a,b)F(a,b). \quad (21)$$

Lemma 2 *The operator $P_U P_g$ is a Hilbert–Schmidt operator and*

$$\|P_U P_g\|_{HS} \leq \sqrt{\frac{\eta(U)}{\omega_g}}.$$

Proof. From (20) and (21), we have

$$P_U P_g(F)(a,b) = \int_{\mathbb{R}^2_+} \chi_U(a,b) F(a',b') \overline{W_g((a,b);(a',b'))} \mathrm{d}\gamma(a',b').$$

Then

$$\begin{aligned}
\|P_U P_g\|^2_{HS} &= \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \chi_U(a,b) |W_g((a,b);(a',b'))|^2 \mathrm{d}\gamma(a',b') \mathrm{d}\gamma(a,b) \\
&= \frac{1}{\omega_g^2} \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \chi_U(a,b) |\Phi_g(g_{a,b})(a',b')|^2 \mathrm{d}\gamma(a',b') \mathrm{d}\gamma(a,b) \\
&= \frac{1}{\omega_g^2} \int_U \left(\int_{\mathbb{R}^2_+} |\Phi_g(g_{a,b})(a',b')|^2 \mathrm{d}\gamma(a',b') \right) \mathrm{d}\gamma(a,b) \\
&= \frac{1}{\omega_g} \int_U \|g_{a,b}\|^2_{L^2(\mu)} \mathrm{d}\gamma(a,b).
\end{aligned}$$

Hence, by (11) we deduce that

$$\|P_U P_g\|^2_{\mathrm{HS}} \leq \frac{\eta(U)}{\omega_g}.$$

Thus, $P_U P_g$ is an integral operator with Hilbert–Schmidt kernel. Hence, $P_U P_g$ is a Hilbert–Schmidt operator, and therefore, it is a compact operator. □

Lemma 3 *[38] Let H_1 and H_2 be two closed subspaces of a Hilbert space H satisfying $H_1 \cap H_2 = \{0\}$. Let P_{H_1} and P_{H_2} be the corresponding orthogonal projections, and suppose that the product $P_{H_1} P_{H_2}$ is a compact operator. Then, there exists a constant $C > 0$ such that for any $f \in H$,*

$$\|P_{H_1^\perp} f\|_H + \|P_{H_2^\perp} f\|_H \geq C \|f\|_H. \tag{22}$$

We use Lemma 3 to prove the following theorem.

Theorem 12 *Let $g \in L^2(\mu)$ and $U \subset \mathbb{R}^2_+$ be such that $0 < \eta(U) < \infty$. If*

$$P_g(L^2(\gamma)) \cap P_U(L^2(\gamma)) = \{0\},$$

then there exists a positive constant $C := C(g,U)$ such that for all $f \in L^2(\mu)$,

$$\|(1-\chi_U)\Phi_g(f)\|_{L^2(\gamma)} \geq C \|f\|_{L^2(\mu)}. \tag{23}$$

Proof. Define

$$H_1 := P_U(L^2(\gamma)), \quad H_2 := P_g(L^2(\gamma)) \quad \text{and} \quad H := L^2(\gamma).$$

From Lemma 2, $P_U P_g$ is a Hilbert–Schmidt operator. Therefore, $P_U P_g$ is a compact operator, and hence, by Lemma 3, there exists a constant $C > 0$ such that (22) holds for $P_{H_1} := P_U$ and $P_{H_2} := P_g$. Since

$$P_{H_2^\perp}(\Phi_g(f)) = (I - P_g)\Phi_g(f) = 0,$$

this leads to (23). □

Let $g \in L^2(\mu)$ and $U \subset \mathbb{R}^2_+$ such be that $0 < \eta(U) < \infty$.

We say that U is weakly annihilating, if any function $f \in L^2(\mu)$ vanishes when its Sturm–Liouville-wavelet transform $\Phi_g(f)$ with respect to the wavelet g is supported in U.

We say that U is strongly annihilating, if there exists a constant $C(U) > 0$ such that for any function $f \in L^2(\mu)$, we have

$$\|(1 - \chi_U)\Phi_g(f)\|_{L^2(\gamma)} \geq C(U)\|f\|_{L^2(\mu)}.$$

The constant $C(U)$ is called the annihilation constant of U.

Remark 1 *(i) It is clear that every strongly annihilating set is also a weakly.*

(ii) From Theorem 7, we see that any set $U \subset \mathbb{R}^2_+$ with

$$0 < \eta(U) < \frac{\omega_g}{\|g\|^2_{L^2(\mu)}}$$

is strongly annihilating.

(iii) The operator $P_U P_g$, being a Hilbert–Schmidt operator, hence is compact. Then from [14] it follows that if U is weakly annihilating, it is also strongly annihilating.

(iv) If $\|P_U P_g\| < 1$, then for all $f \in L^2(\mu)$, we have

$$\|(1 - \chi_U)\Phi_g(f)\|_{L^2(\gamma)} \geq \sqrt{1 - \|P_U P_g\|^2}\|f\|_{L^2(\mu)}.$$

(v) Following the result established in a general context in [14, page 88], we conclude that if U is strongly annihilating, then $\|P_U P_g\| < 1$.

5 Conclusion and perspective

In this paper, we have examined various quantitative uncertainty principles associated with SLWT, which are Faris local-type UP, Shannon-type UP, Donoho-Start-type and Benedicks-type UP. It is our hope that this work motivates the researchers to study the SLWT and such qualitative uncertainty principles as Hardy's, Morgan's, Beurling's, Miyachi's and others.

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