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Numerical Solution of a Wave Partial Differential Equation With the Caputo-Fabrizio Time-Fractional Derivative Using the Finite Element Method Under Non-Homogenous Dirichlet and Neumann Boundary Conditions

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Abstract. In the present article, we propose a numerical resolution of the Caputo–Fabrizio temporal fractional *wave* equation with non-homogenous Dirichlet and Neumann functional conditions. We derive and analyze the semi- and fully discrete approximations using the introduced finite difference scheme for the time Caputo–Fabrizio derivative and the finite element scheme for the spacial derivative. Result of the existence and uniqueness of the solution is discussed, stability and error estimates are established. To support the theoretical studies, a numerical example is given.

Key Words: Finite Element Method, Caputo–Fabrizio Derivative, Time Fractional Wave Equation, Error Estimates, Stability, Convergence Analysis
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Introduction

The theory of fractional calculus, or fractional derivatives, has a long and rich history and has found numerous applications in various branches of science, including physics, chemistry, engineering, and economics. Several technical applications, such as inductance separation and related engineering processes, can be found in the literature (see, for instance, [4] and [5]).

Although the study of fractional-order derivatives dates back to the early development of classical differential calculus, significant progress has been achieved in recent decades. Many researchers have contributed to the development of fractional calculus through both theoretical analysis and numerical methods for solving fractional differential equations. Among the most widely used numerical approaches are the finite difference method [21–23], finite volume methods [14, 27], finite element methods [11, 12, 15, 18, 24, 25], spectral methods [16, 17, 26].

As a consequence of the continuous development of fractional calculus theory, Caputo and Fabrizio [10] recently introduced a new definition of the fractional derivative. This definition assumes two distinct representations for the temporal and spatial variables and has attracted considerable attention due to the absence of a singular kernel, which makes it particularly appealing for modeling various physical processes.

Antanga *et al.* in [3] presented a numerical solution of the RLC circuit model using the fractional derivative without singular kernel. In [1], Antanga introduced several useful analytical tools related to this new definition and applied them to the nonlinear Fisher reaction–diffusion equation. The authors in [13] proposed an alternative representation of the diffusion and diffusion–advection equations based on the Caputo–Fabrizio definition to approximate both temporal and spatial derivatives. Furthermore, Atangana and Alqahtani [2] developed a numerical approximation for the space–time Caputo–Fabrizio fractional derivative and applied it to the groundwater pollution equation.

Lui *et al.* [19] investigated the fractional Cattaneo equation using the new fractional derivative and proposed a second-order Crank–Nicolson scheme for its numerical solution. In a subsequent work [20], they developed a second-order finite difference scheme for solving the quasilinear time Caputo–Fabrizio fractional parabolic equation.

Boutiba *et al.* in [6] investigated the theoretical and numerical aspects of fractional diffusion partial differential equations using the finite element method. In [7], they studied the existence and uniqueness of solutions and implemented three different time discretization methods: the Hammard–Diethelm quadrature formula, the Riemann–Liouville and Caputo link, and the Grunwald–Letnikov approximation, supported by detailed numerical examples. In [8], they considered two different representations for the temporal and spatial variables using the Caputo–Fabrizio fractional derivative and proposed two numerical methods, establishing the stability and convergence order with illustrative numerical results. Finally, in [9], they applied the finite element method to solve a space–time fractional reaction–diffusion equation and proved the existence and uniqueness of the solution together with error estimates. In all these studies, only homogeneous and nonhomogeneous Dirichlet boundary conditions were considered

In the present work, we extend these investigations by considering nonhomogeneous Dirichlet and Neumann functional conditions for the space–time

Caputo–Fabrizio fractional wave partial differential equation. This equation is obtained by replacing the classical second-order temporal derivative in the standard wave equation with the Caputo–Fabrizio fractional derivative, leading to the following model:

$$\begin{cases} {}_0^{CF}\mathcal{D}_t^{\lambda} u(x,t) - \dfrac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), & (x,t) \in [a,b] \times [0,T], \\ u(a,t) = \psi_a(t), u(b,t) = \psi_b(t), & t \in [0,T], \\ u(x,0) = g_0(x), \dfrac{\partial u(x,0)}{\partial t} = g_1(x), & x \in [a,b], \end{cases} \tag{1}$$

where ${}_0^{CF}\mathcal{D}_t^{\lambda}$ denotes the time fractional derivative, introduced recently by Caputo and Fabrizio [1, 10], and defined by

$${}_0^{CF}\mathcal{D}_t^{\lambda} u(x,t) = \frac{1}{2-\lambda} \int_0^t \frac{\partial^2}{\partial \xi^2} u(x,\xi) \exp\left[\frac{1-\lambda}{2-\lambda}(t-\xi)\right] d\xi$$

for $1 < \lambda < 2$.

The organization of this paper is as follows. In Section 1, we introduce the time discretization of the problem and study the existence and uniqueness of the weak solution. We also analyze the unconditional stability and derive error estimates for the semi-discrete scheme. In Section 2, we present the fully discrete scheme and establish the corresponding error estimates for problem (1). Finally, in Section 3, a numerical example is provided to illustrate the effectiveness of the proposed method and to support the theoretical results.

1 Time discretization

In this section, we derive the semi-discrete variational formulation of problem (1). We then discuss the existence and uniqueness of the solution, followed by a stability and convergence analysis of the proposed scheme.

1.1 Finite difference scheme

First, in order to approximate the temporal fractional derivative, the space time need to be discretize as: $t_j = j\Delta t$, $j = 0, 1, \ldots, J$, where $\Delta t = T/J$ is the time step. The second order derivative at $t = t_{j+1}$ is approximated by using the center difference scheme.

Then, the Caputo-Fabrizio time fractional derivative is estimated as following:

$$\begin{aligned}
{}_0^{CF}\mathcal{D}_t^{\lambda} u(x,t_{j+1}) &= \frac{1}{2-\lambda}\int_0^{t_{j+1}} \frac{\partial^2}{\partial \xi^2} u(x,\xi) \exp\left[\frac{1-\lambda}{2-\lambda}(t_{j+1}-\xi)\right] d\xi \\
&= \frac{1}{2-\lambda}\sum_{k=0}^{j}\int_{t_k}^{t_{k+1}} \frac{\partial^2}{\partial \xi^2} u(x,\xi) \exp\left[\frac{1-\lambda}{2-\lambda}(t_{j+1}-\xi)\right] d\xi \\
&= \frac{1}{2-\lambda}\sum_{k=0}^{j}\int_{t_k}^{t_{k+1}} \frac{u(x,t_{k+1})-2u(x,t_k)+u(x,t_{k-1})}{\Delta t^2} \times \\
&\qquad \times \exp\left[\frac{1-\lambda}{2-\lambda}(t_{j+1}-\xi)\right] d\xi + E_{\Delta t}^{j+1} \\
&= \frac{1}{\lambda-1}\sum_{k=0}^{j} u_{tt}(x,t_k)\mathcal{M}_{j-k} + E_{\Delta t}^{j+1}, \qquad (2)
\end{aligned}$$

such that

$$\begin{cases}
u_{tt}(x,t_k) = \dfrac{u(x,t_{k+1})-2u(x,t_k)+u(x,t_{k-1})}{\Delta t^2}, \\[2ex]
\mathcal{M}_{j-k} = \exp\left[\dfrac{1-\lambda}{2-\lambda}(j-k)\Delta t\right] - \exp\left[\dfrac{1-\lambda}{2-\lambda}(j-k+1)\Delta t\right],
\end{cases}$$

and $E_{\Delta t}^{j+1}$ is the truncation error witch is estimated by

$$E_{\Delta t}^{j+1} \leq C_u \Delta t^2$$

(see [20]). Particularly, for $k=0$, we denote

$$u(x,t_{-1}) \approx u^{-1} = u(x,0) - \Delta t u^{(1)}(x,0) = g_0 - \Delta t g_1.$$

Now, for the brevity sake, we set

$$S_t^{\lambda} u(x,t_{j+1}) = \frac{1}{\lambda-1}\sum_{k=0}^{j} u_{tt}(x,t_k)\mathcal{M}_{j-k}.$$

Then (2) becomes

$${}_0^{CF}\mathcal{D}_t^{\lambda} u(x,t_{j+1}) = S_t^{\lambda} u(x,t_{j+1}) + E_{\Delta t}^{j+1}.$$

We will use $S_t^{\lambda} u(x,t_{j+1})$ as an approximation of the time Caputo–Fabrizio fractional derivative ${}_0^{CF}\mathcal{D}_t^{\lambda} u(x,t_{j+1})$, and this leads us to the next finite difference scheme of the problem (1):

$$S_t^{\lambda} u^{j+1} - \Delta u^{j+1} = f^{j+1}, \quad j = 0,1,\ldots,J.$$

After some adjustment, we have the following scheme:

$$u^{j+1} - d\Delta u^{j+1} = \sum_{k=-1}^{j} \mathcal{N}_{j,k} u^k + df^{j+1} \tag{3}$$

in which $d = \Delta t^2(\lambda - 1)/\mathcal{M}_0$ and

$$\mathcal{M}_0.\mathcal{N}_{j,k} = \begin{cases} -\mathcal{M}_j^\lambda & k = -1, \\ 2\mathcal{M}_j - \mathcal{M}_{j-1}^\lambda & k = 0, \\ -\mathcal{M}_{j-k+1}^\lambda + 2\mathcal{M}_{j-k}^\lambda - \mathcal{M}_{j-k-1}^\lambda & 1 \leq k \leq j-1, \\ 2\mathcal{M}_0^\lambda - \mathcal{M}_1^\lambda & k = j. \end{cases}$$

1.2 Existence and uniqueness of the variational solution

To introduce the variational form of (3), we need to define some functional spaces with their norms:

$$H^1(\Omega) = \left\{ w \in L^2(\Omega), \frac{dw}{dx} \in L^2(\Omega) \right\}, \quad H_0^1(\Omega) = \left\{ w \in H^1(\Omega), w|_{\partial\Omega} = 0 \right\},$$

where $L^2(\Omega)$ is the space of measurable function whose square integrable in Ω. The inner products of $L^2(\Omega)$ and $H_0^1(\Omega)$ are defined respectively by

$$(u, w) = \int_\Omega uw dx, \quad (u, w)_1 = (u, w) + \left(\frac{du}{dx}, \frac{dw}{dx} \right),$$

and the corresponding standard norms are

$$\|w\| = (w, w)^{1/2}, \quad \|w\|_1 = (w, w)_1^{1/2} = \left(\|w\|^2 + d \left\| \frac{dw}{dx} \right\|^2 \right)^{1/2}.$$

Then the fractional wave equation (1) can be changed into a semi-discrete variational problem

$$(u^{j+1}, w) + d \left(\frac{\partial u^{j+1}}{\partial x}, \frac{\partial w}{\partial x} \right) = \sum_{k=-1}^{j} \mathcal{N}_{j,k}(u^k, w) + d(f^{j+1}, w), \ w \in H_0^1(\Omega). \tag{4}$$

In the first place, we denote

$$\mathfrak{B}(u^{j+1}, w) = (u^{j+1}, w) + d \left(\frac{\partial u^{j+1}}{\partial x}, \frac{\partial w}{\partial x} \right),$$

$$f_1 = \sum_{k=-1}^{j} \mathcal{N}_{j,k} u^k + df^{j+1},$$

and $\mathcal{L}(w) = (f_1, w)$. Then we give the variational form in its concise form by

$$\mathfrak{B}(u^{j+1}, w) = \mathcal{L}(w).$$

Lemma 1 *For a sufficiently small step size time Δt, there exists a unique solution u^{j+1} satisfying* (3) *and such that*

$$\left\|u^{j+1}\right\|_1 \leq \|f_1\|_{-1} .$$

Proof. First, we need to prove the coercivity of the bilinear form $\mathfrak{B}$ over $H_0^1(\Omega)$. We have

$$\begin{aligned}
\mathfrak{B}(u^{j+1}, u^{j+1}) &= (u^{j+1}, u^{j+1}) + d\left(\frac{\partial u^{j+1}}{\partial x}, \frac{\partial u^{j+1}}{\partial x}\right), \\
&= \left\|u^{j+1}\right\|^2 + d\left\|\frac{\partial u^{j+1}}{\partial x}\right\|^2, \\
&= \left\|u^{j+1}\right\|_1^2 .
\end{aligned}$$

Next we prove the continuity of the bilinear form $\mathfrak{B}$ over $H_0^1(\Omega) \times H_0^1(\Omega)$. We can write

$$\begin{aligned}
\mathfrak{B}(u^{j+1}, w) &= (u^{j+1}, w) + d\left(\frac{\partial u^{j+1}}{\partial x}, \frac{\partial w}{\partial x}\right), \\
&\leq \left\|u^{j+1}\right\| \|w\| + d\left\|\frac{\partial u^{j+1}}{\partial x}\right\| \left\|\frac{\partial w}{\partial x}\right\|, \\
&\leq \left\|u^{j+1}\right\|_1 \|w\|_1 .
\end{aligned}$$

Moreover, we can prove that the linear form $\mathcal{L}(.)$ is continuous over $H_0^1(\omega)$. Indeed, since $f_1 \in H_0^1(\Omega) \subset H^{-1}(\Omega)$, we have

$$|\mathcal{L}(w)| = \|f_1(w)\| = \|f_1\|_{-1} \|w\|_1 .$$

□

1.3 Stability analysis and error estimates

For further considerations, we need the following result.

Lemma 2 *The semi discrete scheme* (4) *is unconditionally stable, in the sense that for all $\Delta t > 0$, it holds*

$$\left\|u^{j+1}\right\|_1 \leq C\left(\|g_0\| + \Delta t \|g_1\|\right)$$

for all $j = 0, 1, \ldots, J-1$.

Proof. For the sake of simplicity and without loss of generality, we suppose that $f \equiv 0$.

We will prove the lemma by mathematical induction. In (4), for $j=0$, we get

$$\left(u^1, w\right) + d\left(\frac{\partial u^1}{\partial x}, \frac{\partial w}{\partial x}\right) = \sum_{k=-1}^{0} \mathcal{N}_{j,k}\left(u^k, w\right).$$

By taking $w = u^1$ and using $\|u^1\| \leq \|u^1\|_1$, we immediately obtain

$$\left\|u^1\right\|_1 \leq C\left(\|g_0\| + \Delta t\,\|g_0\|\right).$$

Now, suppose that

$$\left\|u^k\right\|_1 \leq C\left(\|g_0\| + \Delta t\,\|g_1\|\right)$$

holds for $k = 0, 1, \ldots, j$, and let us prove that

$$\left\|u^{j+1}\right\|_1 \leq C\left(\|g_0\| + \Delta t\,\|g_1\|\right).$$

In the semi-discrete scheme (4) taking $w = u^{j+1}$, we have

$$\begin{aligned}
\left\|u^{j+1}\right\|_1^2 &\leq \sum_{k=-1}^{j} \left|\mathcal{N}_{j,k}\right| . \left\|u^k\right\| . \left\|u^{j+1}\right\|, \\
&\leq \sum_{k=-1}^{j} \left|\mathcal{N}_{j,k}\right| C\left(\|g_0\| + \Delta t\,\|g_1\|\right) \left\|u^{j+1}\right\|, \\
&\leq C\left(\|g_0\| + \Delta t\,\|g_1\|\right) \left\|u^{j+1}\right\|_1.
\end{aligned}$$

From here it follows that

$$\left\|u^{j+1}\right\|_1 \leq C\left(\|g_0\| + \Delta t\,\|g_1\|\right).$$

□

Now we carry out the error analysis for our semi-discrete scheme.

Theorem 1 *Assume that problem* (1) *has an exact solution* $u(t_{j+1})$ *at* $t = t_{j+1}$*, and let* $\{u^{j+1}\}_{j=0}^{J-1}$ *be the solution of the semi-discrete form* (4) *subject to the initial and boundary condition. Then we have the following error estimates for* $1 < \lambda < 2$*:*

$$\left\|u(t_{j+1}) - u^{j+1}\right\|_1 \leq C(\Delta t)^2 \left(\|u_{tt}\|_{\infty,0} + \|u_{ttt}\|_{\infty,0}\right).$$

Proof. First, denote

$$\varepsilon = u(t_{j+1}) - u^{j+1}$$

at $t = t_{j+1}$ for $t_j = 0, 1, \ldots, J-1$. Since the exact solution $u(t_{j+1})$ also satisfy the semi-discrete scheme, for all $w \in H_0^1(\Omega)$, we have

$$(u(t_{j+1}), w) + d\left(\frac{\partial u(t_{j+1})}{\partial x}, \frac{\partial w}{\partial x}\right) = \sum_{k=-1}^{j} \mathcal{N}_{j,k}\left(u(t_k), w\right) - d\left(E_{\Delta t}^{j+1}, w\right). \quad (5)$$

Thus, subtracting (5) from (4), we obtain

$$\left(\varepsilon^{j+1}, w\right) + d\left(\frac{\partial \varepsilon^{j+1}}{\partial x}, \frac{\partial w}{\partial x}\right) = \sum_{k=-1}^{j} \mathcal{N}_{j,k}\left(\varepsilon^{k}, w\right) - d\left(E_{\Delta t}^{j+1}, w\right), \tag{6}$$

and substituting $w = \varepsilon^{j+1}$ in (6), we get

$$\left(\varepsilon^{j+1}, \varepsilon^{j+1}\right) + d\left(\frac{\partial \varepsilon^{j+1}}{\partial x}, \frac{\partial \varepsilon^{j+1}}{\partial x}\right) = \sum_{k=-1}^{j} \mathcal{N}_{j,k}\left(\varepsilon^{k}, \varepsilon^{j+1}\right) - d\left(E_{\Delta t}^{j+1}, \varepsilon^{j+1}\right). \tag{7}$$

The left hand side of (7) is equivalent to $\left\|\varepsilon^{j+1}\right\|_1$, hence,

$$\left\|\varepsilon^{j+1}\right\|_1^2 \leq \sum_{k=-1}^{j} \left|\mathcal{N}_{j,k}\right| . \left\|\varepsilon^{k}\right\| . \left\|\varepsilon^{j+1}\right\| + d\left\|E_{\Delta t}^{j+1}\right\| . \left\|\varepsilon^{j+1}\right\|. \tag{8}$$

Now we will use the mathematical induction to get the error estimates

$$\left\|\varepsilon^{j+1}\right\|_1 \leq C(\Delta t)^2 \left(\|u_{tt}\|_{\infty,0} + \|u_{ttt}\|_{\infty,0}\right).$$

First, for $j = 0$ in (8), using the fact that $\left\|\varepsilon^{0}\right\| = 0$ and

$$\begin{aligned} \varepsilon^{-1} &= u(t_{-1}) - u^{-1} = u(t_{-1}) - u(0) + \Delta t u_t(0) \\ &= \frac{(\Delta t)^2}{2!} u_{tt}(0) + \mathcal{O}(\Delta t)^3, \end{aligned} \tag{9}$$

we have

$$\begin{aligned} \left\|\varepsilon^{1}\right\|_1^2 &\leq \sum_{k=-1}^{0} \left|\mathcal{N}_{j,k}\right| . \left\|\varepsilon^{k}\right\| . \left\|\varepsilon^{1}\right\| + d\left\|E_{\Delta t}^{1}\right\| . \left\|\varepsilon^{1}\right\| \\ &\leq \left(\left|\mathcal{N}_{0.-1}\right| . \left\|\varepsilon^{-1}\right\| + \left|\mathcal{N}_{0.0}\right| . \left\|\varepsilon^{0}\right\| + d(\Delta t)^2 . \|u_{ttt}\|_{\infty,0}\right) \left\|\varepsilon^{1}\right\| \\ &\leq C(\Delta t)^2 \left(\|u_{tt}\|_{\infty,0} + \|u_{ttt}\|_{\infty,0}\right) \left\|\varepsilon^{1}\right\|_1 . \end{aligned}$$

Now we suppose that

$$\left\|\varepsilon^{k}\right\|_1 \leq C(\Delta t)^2 \left(\|u_{tt}\|_{\infty,0} + \|u_{ttt}\|_{\infty,0}\right)$$

holds for $k = 0, 1, \ldots, j$. According to (7) and (9), we get

$$\begin{aligned} \left\|\varepsilon^{j+1}\right\|_1^2 &\leq \sum_{k=-1}^{j} \left|\mathcal{N}_{j,k}\right| . \left\|\varepsilon^{k}\right\| . \left\|\varepsilon^{j+1}\right\| + d\left\|E_{\Delta t}^{j+1}\right\| . \left\|\varepsilon^{j+1}\right\| \\ &\leq \left(\sum_{k=1}^{j} \left|\mathcal{N}_{j,k}\right| . \left\|\varepsilon^{k}\right\| + \left|\mathcal{N}_{0.-1}\right| . \left\|\varepsilon^{-1}\right\| + d(\Delta t)^2 . \|u_{ttt}\|_{\infty,0}\right) \left\|\varepsilon^{j+1}\right\| . \end{aligned}$$

Then

$$\|\varepsilon^{j+1}\|_1^2 \leq \left(\sum_{k=1}^{j} |\mathcal{N}_{j,k}| (\Delta t)^2 \left(\|u_{tt}\|_{\infty,0} + \|u_{ttt}\|_{\infty,0}\right) + C\left(\|u_{tt}\|_{\infty,0} + \|u_{ttt}\|_{\infty,0}\right)\right) \|\varepsilon^{j+1}\|_1 . \tag{10}$$

Since $0 < \mathcal{M}_k < 1$ for all non negative integers k, we obtain

$$\left\|\varepsilon^{j+1}\right\|_1 \leq C(\Delta t)^2 \left(\|u_{tt}\|_{\infty,0} + \|u_{ttt}\|_{\infty,0}\right),$$

and this complete the prove. □

2 Full discretization: Finite element scheme

We first present the full discrete scheme and then discuss the error estimates.

Define Ω_h as a uniform partition of Ω, which is given by

$$a = x_0 < x_1 < \ldots < x_m = b$$

where m is a positive integer. Let $h = 1/m = x_i - x_{i-1}$ and $\Omega_i = [x_{i-1}, x_i[$ for $i = 1, 2, \ldots, m$.

Define the space χ_h as a set of piecewise polynomials of order n, $n \in \mathbb{N}$, on Ω_h as

$$\chi_h = \left\{ w \in H_0^1(\Omega) \cap C(\bar{\Omega}) : w|_{\Omega_i} \in P_n(\Omega) \right\}.$$

Let u_h^{j+1} be the finite element solution at $t = t_{j+1}$. Then we have the following full discrete scheme of the problem (1) for $1 < \lambda < 2$:

$$\left(u_h^{j+1}, w_h\right) + d\left(\frac{\partial u_h^{j+1}}{\partial x}, \frac{\partial w_h}{\partial x}\right) = \sum_{k=-1}^{j} \mathcal{N}_{j,k}\left(u_h^k, w_h\right) + d\left(f^{j+1}, w_h\right) \tag{11}$$

for all $w_h \in \chi_h$.

We can demonstrate that the full discrete scheme (11) has a unique solution u_h^j at each time step j. Since $\chi_h \subset H_0^1(\Omega)$, similar to Lemma 1 we can show that (11) also satisfy the conditions of Lax-Milgram Lemma. Thus, we can conclude that we have the existence and uniqueness of the solution u_h^{j+1}.

Now we present the principal convergence theorem.

Theorem 2 *Let $u^{j+1} = u(x, t_{j+1})$ be the exact solution of problem* (1) *and u_h^{j+1} be the finite element approximate solution of the full discrete scheme* (11). *Assume that u^{j+1} satisfies*

$$u_t^{j+1} \in L^2(0, T, H^{n+1}(\Omega)) \cap L^\infty(0, T, H^{n+1}(\Omega))$$

and

$$u_{tt}^{j+1} \in L^2(0,t,L^2(\Omega))$$

with $g_0, g_1 \in H^{n+1}(\Omega)$. *Then the following error estimates holds:*

$$\left\| u^{j+1} - u_h^{j+1} \right\| \leq C \left(h^{n+1} \left\| u \right\|_{\infty,n+1} + (\Delta t)^2 \left\| u_{tt} \right\|_{\infty,0} \right).$$

Proof. For $t = t_{j+1}$ define

$$\bar{\varepsilon}^{j+1} = u^{j+1} - u_h^{j+1}.$$

Similar to the proof of Theorem 1, we can show that the exact solution at $t = t_{j+1}$ satisfies

$$(u(t_{j+1}), w) + d \left(\frac{\partial u(t_{j+1})}{\partial x}, \frac{\partial w}{\partial x} \right) = \sum_{k=-1}^{j} \mathcal{N}_{j,k}(u(t_k), w) - d \left(E_{\Delta t}^{j+1}, w \right). \quad (12)$$

Thus, subtracting (11) from (12), we have

$$(\bar{\varepsilon}^{j+1}, w) + d \left(\frac{\partial \bar{\varepsilon}^{j+1}}{\partial x}, \frac{\partial w}{\partial x} \right) = \sum_{k=-1}^{j} \mathcal{N}_{j,k}(\bar{\varepsilon}^{k}, w) - d \left(E_{\Delta t}^{j+1}, w \right). \quad (13)$$

Taking $w = \bar{\varepsilon}^{j+1}$ in (13) and using the following estimate obtained in [24]

$$\left\| \bar{\varepsilon}^{-1} \right\| \leq C \left(h^{n+1} \left\| u \right\|_{\infty,0} + (\Delta t)^2 \left\| u_{tt} \right\|_{\infty,0} \right),$$

we get

$$\begin{aligned}
\left\| \bar{\varepsilon}^{j+1} \right\|_1^2 &\leq \left(\sum_{k=-1}^{j} \left| \mathcal{N}_{j,k} \right| . \left\| \bar{\varepsilon}^{k} \right\| + d \left\| E_{\Delta t}^{j+1} \right\| \right) \left\| \bar{\varepsilon}^{j+1} \right\| \\
&\leq C \left(\sum_{k=1}^{j} \left| \mathcal{N}_{j,k} \right| . \left\| \bar{\varepsilon}^{k} \right\| + \left| \mathcal{N}_{j,-1} \right| . \left\| \bar{\varepsilon}^{-1} \right\| + \left| \mathcal{N}_{j,0} \right| . \left\| \bar{\varepsilon}^{0} \right\| + d \left\| E_{\Delta t}^{j+1} \right\| \right) \left\| \bar{\varepsilon}^{j+1} \right\|_1 \\
&\leq C \left(\sum_{k=1}^{j} \left| \mathcal{N}_{j,k} \right| . \left\| \bar{\varepsilon}^{k} \right\| + h^{n+1} \left\| u \right\|_{\infty,0} + (\Delta t)^2 \left\| u_{tt} \right\|_{\infty,0} \right) \left\| \bar{\varepsilon}^{j+1} \right\|_1 .
\end{aligned}$$

Since $0 < M_k < 1$, we have

$$\left\| \bar{\varepsilon}^{j+1} \right\|_1 \leq C \left(\sum_{k=1}^{j} \left\| \bar{\varepsilon}^{k} \right\| + h^{n+1} \left\| u \right\|_{\infty,0} + (\Delta t)^2 \left\| u_{tt} \right\|_{\infty,0} \right).$$

Similar to the proof of Theorem 1, using the mathematical induction one can show that

$$\left\| \bar{\varepsilon}^{k} \right\|_1 \leq C \left(h^{n+1} \left\| u \right\|_{\infty,0} + (\Delta t)^2 \left\| u_{tt} \right\|_{\infty,0} \right)$$

for all $k = 0, 1, \ldots, j+1$. $\square$

3 Numerical Validation

In this section, we present the implementation of the proposed method and provide a numerical example to support our theoretical results.

3.1 Implementation

For completeness sake, we will describe the implementation here. First, let S_h be the uniform partition of $[a, b]$ and choose χ_h to be the space of piecewise linear functions on S_h

$$\chi_h = \left\{ w \in H_0^1(\Omega) \cap C(\bar{\Omega}) : w|_{\Omega_i} \in P_1(\Omega) \right\}.$$

Then we consider the following test functions $\phi_0, \phi_1, \ldots, \phi_m$ on χ_h defined by

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{elsewhere}, \end{cases}$$

where $i = 1, 2, \ldots, m-1$, and

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{h}, & x \in [x_0, x_1], \\ 0, & \text{elsewhere}, \end{cases} \qquad \phi_m(x) = \begin{cases} \dfrac{x - x_{m-1}}{h}, & x \in [x_{m-1}, x_m], \\ 0, & \text{elsewhere}. \end{cases}$$

It is not difficult to see that the following result holds true.

Lemma 3 *For $i = 1, 2, \ldots, m-1$, we have*

$$(\phi_l(x), \phi_i(x)) = \frac{h}{6} \begin{cases} 1, & |l - i| = 1, \\ 4, & l = i, \\ 0, & \textit{elsewhere}, \end{cases}$$

and

$$\left(\frac{\partial \phi_l(x)}{\partial x}, \frac{\partial \phi_i(x)}{\partial x} \right) = \frac{1}{h} \begin{cases} -1, & |l - i| = 1, \\ 2, & l = i, \\ 0, & \textit{elsewhere}. \end{cases}$$

We consider problem (11). Let us express u_h^{j+1} by

$$u_h^{j+1} = \sum_{l=0}^{m} u_l^{j+1} \phi_l(x).$$

For $i = 1, 2, \ldots, m-1$, we have

$$\left(\sum_{l=0}^{m} u_l^{j+1}\phi_l(x), \phi_i(x)\right) + d\left(\sum_{l=0}^{m} u_l^{j+1}\frac{\partial\phi_l(x)}{\partial x}, \frac{\partial\phi_i(x)}{\partial x}\right)$$
$$= \sum_{k=-1}^{k} \mathcal{N}_{j,k}\left(\sum_{l=0}^{m} u_l^{k}\phi_l(x), \phi_i(x)\right) + d\left(f^{j+1}, \phi_i(x)\right).$$

Applying Lemma 3, we obtain

$$\frac{h}{6}\left(u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i-1}^{j+1}\right) + \frac{d}{h}\left(u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}\right)$$
$$= \frac{h}{6}\sum_{k=-1}^{j} \mathcal{N}_{j,k}\left(u_{i-1}^{k} + 4u_i^{k} + u_{i+1}^{k}\right) + d\left(f^{j+1}, \phi_i(x)\right).$$

Thus, we arrive at the following matrix statement of (11):

$$(M + dK)\, U^{j+1} = \sum_{k=-1}^{j} \mathcal{N}_{j,k} K U^k + dF^{j+1},$$

where

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{pmatrix}, \; K = \frac{1}{h}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix},$$

$$F^{j+1} = \begin{pmatrix} (f^{j+1}, \phi_1(x)) \\ (f^{j+1}, \phi_2(x)) \\ \vdots \\ (f^{j+1}, \phi_{m-1}(x)) \end{pmatrix} \quad \text{and} \quad U^{j+1} = \begin{pmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ u_{m-1}^{j+1} \end{pmatrix}.$$

3.2 Numerical Example

Consider the following problem:

$$\begin{cases} {}_0^{CF}\mathcal{D}_t^{\lambda} u(x,t) - \dfrac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), & (x,t) \in [0,1] \times [0,1], \\ u(0,t) = 0, u(1,t) = t^2, & t \in [0,1], \\ u(x,0) = 0, \dfrac{\partial u(x,0)}{\partial t} = 0, & x \in [0,1], \end{cases}$$

where

$$f(x,t) = \frac{2x^3}{1-\lambda}\left(1 - exp\left[\frac{(1-\lambda)t}{2-\lambda}\right]\right) - 6xt^2.$$

The exact solution to this problem is $u(x,t) = x^3t^2$.

The following tables show the approximation errors and convergence order of the finite element method for a different values of λ. In Table 1, we set $k = \Delta x = 0.1$. In Table 2, we take $k = \Delta x = 0.01$, a value small enough to check the space error and convergence order, and in Table 3, we set $k = \Delta x = 0.001$.

Table 1: Error and convergence order for different values of λ and $\Delta x = 1/10$.

Δt	$\lambda = 1.2$		$\lambda = 1.5$		$\lambda = 1.7$		$\lambda = 1.9$	
	err(E-3)	order	err(E-3)	order	err(E-3)	order	err(E-4)	order
1/10	6.164	-	9.626	-	14.16	-	5.659	-
1/20	3.721	0.73	5.988	0.68	9.476	0.58	1.939	1.55
1/40	2.040	0.87	3.331	0.85	5.169	0.79	0.5511	1.82
1/80	1.067	0.93	1.756	0.92	2.936	0.90	0.1448	1.93

Table 2: Error and convergence order for different values of λ and $\Delta x = 1/100$.

Δt	$\lambda = 1.2$		$\lambda = 1.5$		$\lambda = 1.7$		$\lambda = 1.9$	
	err(E-2)	order	err(E-2)	order	err(E-2)	order	err(E-4)	order
1/10	9.281	-	14.23	-	22.56	-	11.30	-
1/20	2.426	1.94	3.737	0.68	5.954	1.92	2.832	2.00
1/40	0.6098	1.99	0.9479	0.85	1.522	1.97	0.7013	2.01
1/80	0.1426	2.10	0.2292	0.92	0.3774	2.01	0.1741	2.01

Table 3: Error and convergence order for different values of λ and $\Delta x = 1/1000$.

Δt	$\lambda = 1.2$		$\lambda = 1.5$		$\lambda = 1.7$		$\lambda = 1.9$	
	err(E-4)	order	err(E-4)	order	err(E-4)	order	err(E-4)	order
1/10	19.34	-	19.31	-	17.03	-	5.041	-
1/20	9.612	1.01	9.440	1.03	8.097	1.07	1.442	1.81
1/40	4.767	1.01	4.626	1.03	3.884	1.06	0.3739	1.95
1/80	2.329	1.03	2.223	1.06	1.806	1.10	0.09316	2.00

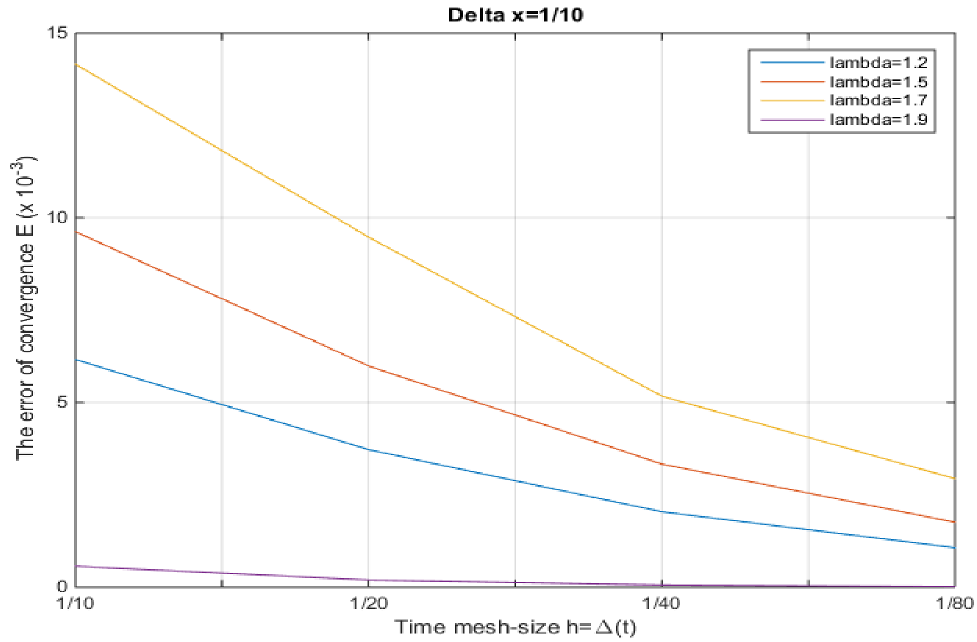

Figure 1: Convergence errors for $\Delta x = 1/10$, $\Delta x = 1/100$ and $\Delta x = 1/1000$.

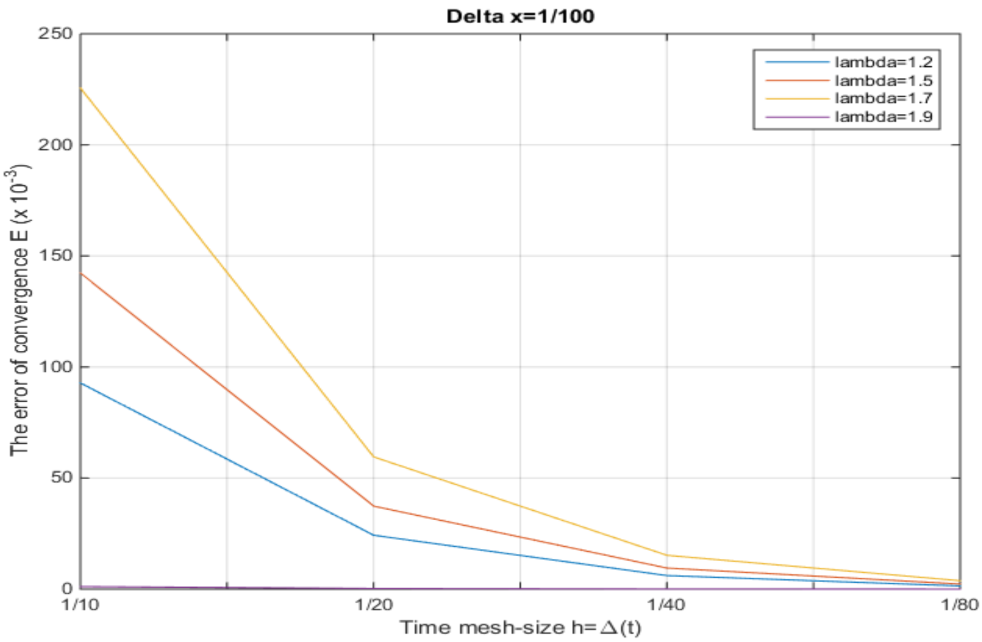

Figure 2: Convergence errors for $\Delta x = 1/10$, $\Delta x = 1/100$ and $\Delta x = 1/1000$.

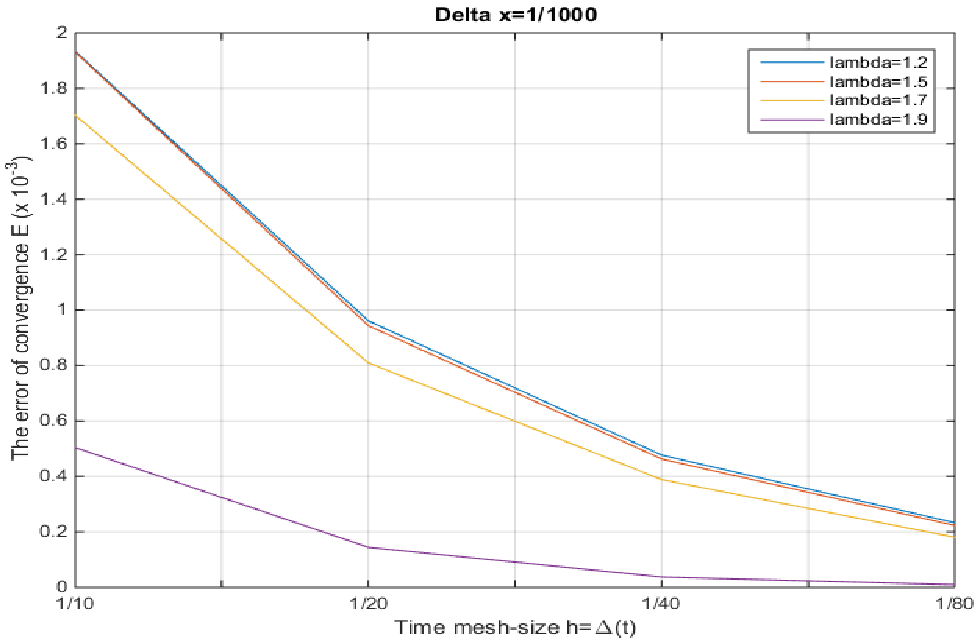

Figure 3: Convergence errors for $\Delta x = 1/10$, $\Delta x = 1/100$ and $\Delta x = 1/1000$.

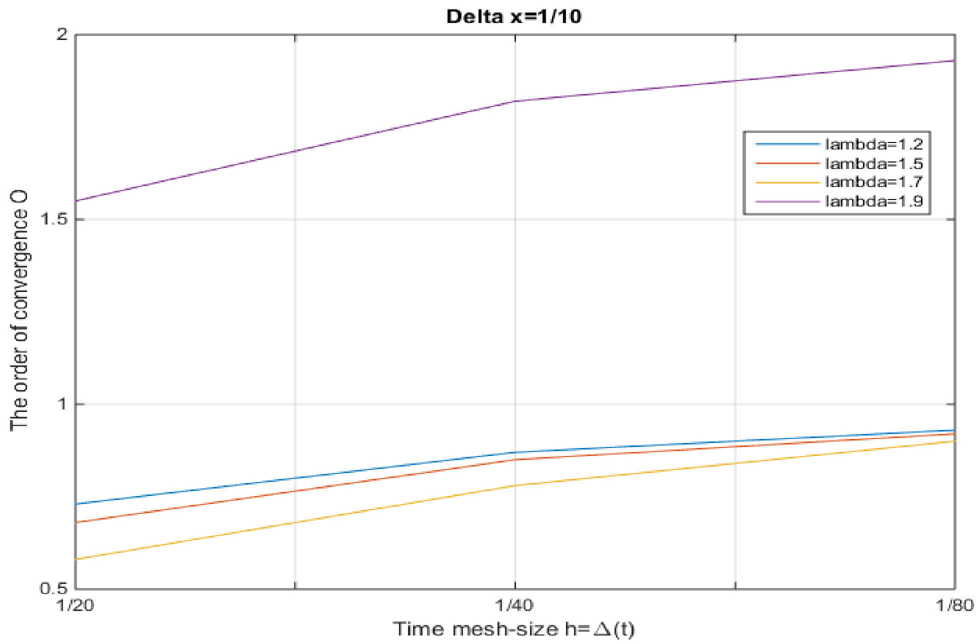

Figure 4: Convergence orders for $\Delta x = 1/10$, $\Delta x = 1/100$ and $\Delta x = 1/1000$.

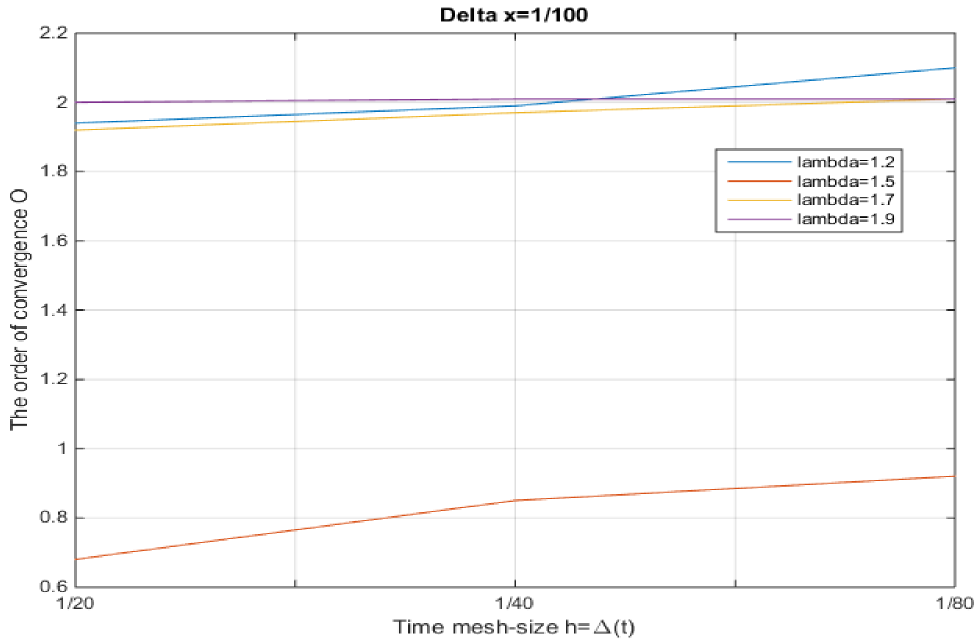

Figure 5: Convergence orders for $\Delta x = 1/10$, $\Delta x = 1/100$ and $\Delta x = 1/1000$.

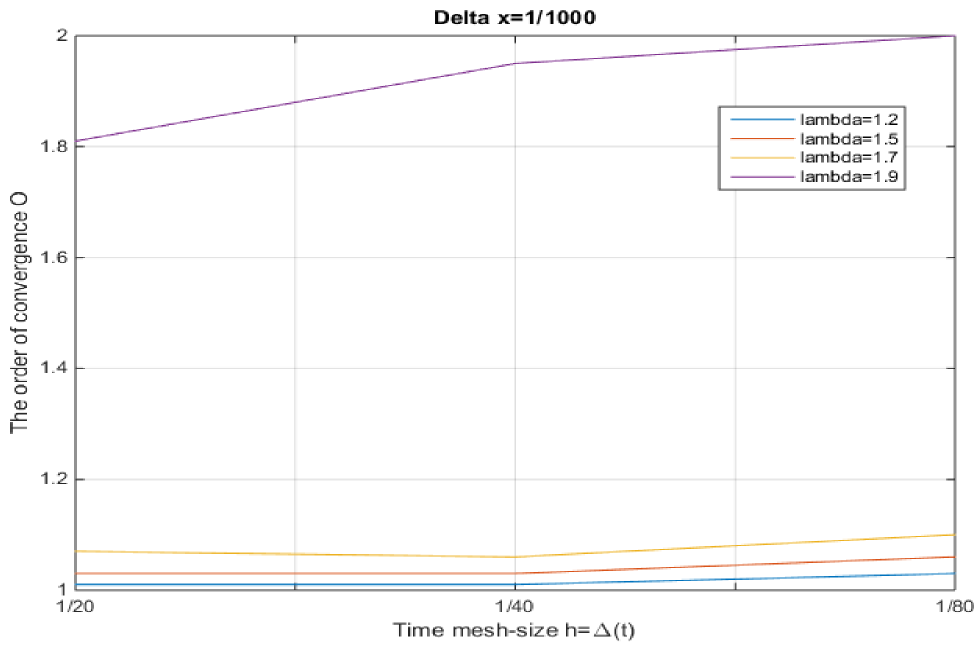

Figure 6: Convergence orders for $\Delta x = 1/10$, $\Delta x = 1/100$ and $\Delta x = 1/1000$.

4 Conclusion

In this paper, to solve the Caputo–Fabrizio temporal fractional wave equation, the finite difference method is used in the temporal direction while we used the finite element method in the space direction. We proved the existence and uniqueness of the weak solution, stability and convergence analysis of the semi and fully discrete schemes are established.

References

[1] A. Atangana, On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation. Appl. Math. Comput., **273** (2016), pp. 948–956. http://dx.doi.org/10.1016/j.amc.2015.10.021

[2] A. Atangana and R.T. Alqahtani, Numerical approximation of the space-time Caputo-Fabrizio fractional derivative and application to groundwater pollution equation. Adv. Differ. Equ., **2016** (2016), article number 156, 13 pp. https://doi.org/10.1186/s13662-016-0871-x

[3] A. Atangana and J.J. Nieto, Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel. Adv. Mech. Eng., **7** (2015), no. 10, pp. 1–7. https://doi.org/10.1177/1687814015613758

[4] S.E. Bendimerad, A. Ayad, A. Bettache, S. Baghli and A.N.E.I. Ayad, A novel formula for calculating repulsive forces in horizontally aligned nonferrous cylindrical particles within an Eddy current separator. Powder Technology, **448** (2024), article number 120300, 10 pp. https://doi.org/10.1016/j.powtec.2024.120300

[5] S.E. Bendimerad, S. Baghli, A. Ayad and A. Tilmatine, Mutual inductance of two coaxial rectangular PCB coils incorporating magnetic layer. Journal of Electrical Engineering: The Journal of Slovak University of Technology, **72** (2021), no. 5, pp. 287–296. https://doi.org/10.2478/jee-2021-0041

[6] M. Boutiba and S. Baghli-Bendimerad, *Finite Element Approach for Some Fractional Diffusion Partial Differential Equations*, Adlis Bellazma Publishing and Distribution, Algeria (Batna), 2026.

[7] M. Boutiba, S. Baghli-Bendimerad and A. Benaïssa, Three approximations of numerical solution's by finite element method for resolving space-time partial differential equations involving fractional derivative's order. Math. Model. Eng. Probl., **9** (2022), no. 5, pp. 1179–1186. https://doi.org/10.18280/mmep.090503

[8] M. Boutiba, S. Baghli-Bendimerad and N.E.H. Bouzara-Sahraoui, Numerical solution by finite element method for time Caputo-Fabrizio fractional partial diffusion equation. Advanced Mathematical Models & Applications, **9** (2024), no. 2, pp. 316–328. https://doi.org/10.62476/amma9316

[9] M. Boutiba, S. Baghli-Bendimerad and M. Feçkan, Numeric FEM's solution for space-time diffusion partial differential equations with Caputo-Fabrizion and Riemann-Liouville fractional order's derivatives. Ann. Math. Sil., **37** (2023), no. 2, pp. 204–223. https://doi.org/10.2478/amsil-2023-0009

[10] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel. Progr. Fract. Differ. Appl, **1** (2015), no. 2, pp. 73–85.

[11] L.B. Feng, P. Zhuang, F. Liu and Y.T. Gu, Finite element method for space-time fractional diffusion equation. Numer. Algor., **72** (2016), pp. 749–767. https://doi.org/10.1007/s11075-015-0065-8

[12] N. J. Ford, J. Xiao and Y. Yan, A finite element method for time fractional partial differential equations. Fract. Calc. App. Anal., **14** (2011), no. 3, pp. 454–474. https://doi.org/10.2478/s13540-011-0028-2

[13] J.F. Gomez-Aguilar, M.G. Lopez-Lopez, V.M. Alvarado-Martinez, J. Reyes-Reyes and M. Adam-Medina, Modeling diffusive transport with a fractional derivative without singular kernel. Physica A: Stat. Mech. Appl., **447** (2016), pp. 467–481. https://doi.org/10.1016/j.physa.2015.12.066

[14] H. Hejazi, T. Moroney and F. Liu, A finite volume method for solving the two-sided time-space fractional advection-dispersion equation. Cent. Eur. J. Phys., **11** (2013), no. 10, pp. 1275–1283. https://doi.org/10.2478/s11534-013-0317-y

[15] Y. Jiang and J. Ma, High-order finite element methods for time-fractional differential equations. J. Comput. App. Math., **235** (2011), no. 11, pp. 3285–3290. https://doi.org/10.1016/j.cam.2011.01.011

[16] Y. Li and C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys., **225** (2007), no. 2, pp. 1533–1552. https://doi.org/10.1016/j.jcp.2007.02.001

[17] X. Li and C. Xu, A space-time spectral method for the time fractional diffusion equation. SIAM J. Numer. Anl., **47** (2009), no. 3, pp. 2108–2131. https://doi.org/10.1137/080718942

[18] C. Li, Z. Zhao and Y. Chen, Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion. Comput. Math. Appl., **62** (2011), no. 3, pp. 855–875. https://doi.org/10.1016/j.camwa.2011.02.045

[19] Z. Liu, A. Cheng and X. Li, A second order Crank-Nicolson scheme for fractional Cattaneo equation based on new fractional derivative. Appl. Math. and Comput., **311** (2017), pp. 361–374. https://doi.org/10.1016/j.amc.2017.05.032

[20] Z. Liu, A. Cheng and X. Li, A second-order finite difference scheme for quasilinear time fractional parabolic equation based on new fractional derivative. International Journal of Computer Mathematics, **95** (2017), no. 2, pp. 396–411. http://dx.doi.org/10.1080/00207160.2017.1290434

[21] M.M. Meerschaert and C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations. J. Comput. Appl. Math., **172** (2004), no. 1, pp. 65–77. https://doi.org/10.1016/j.cam.2004.01.033

[22] E. Sousa, A second order explicit finite difference method for fractional advection diffusion equation. J. Comput. Math. App., **64** (2012), no. 10, pp. 3141–3152.

[23] E. Sousa, Finite difference approximation for a fractional advection diffusion problem. J. Comput. Phys., **228** (2009), no. 11, pp. 4038–4054.

[24] X. Zhang, P. Huang, X. Feng and L. Wei, Finite element method for two-dimensional time-fractional Tricomi-type equations, *Numer. Methods Partial Differ. Equ.*, **29** (2013), no. 4, pp. 1081–1096. https://doi.org/10.1002/num.21745

[25] Z. Zhao and C. Li, Fractional difference/finite element approximations for the time-space fractional telegraph equation. Appl. Math. Comput., **219** (2012), no. 6, pp. 2975–2988. https://doi.org/10.1016/j.amc.2012.09.022

[26] M. Zheng, F. Liu, I. Turner and V. Anh, A novel high order space-time spectral method for the time fractional Fokker-Planck equation. SIAM J. Sci. Comput., **37** (2015), no. 2, pp. A701–724. https://doi.org/10.1137/140980545

[27] P. Zhuang, F. Liu, I. Turner and Y. T. Gu, Finite volume and finite element methods for solving a one-dimensional space-fractional Boussinesq equation. Appl. Math. Modell., **38** (2014), no. 15-16, pp. 3860–3870. http://dx.doi.org/10.1016/j.apm.2013.10.008

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