Biharmonic Helices in Three-Dimensional Lorentzian Egorov and ε-Spaces

F. Z. Benzahra-Belkacem and N. Sidhoumi

Abstract. We consider three-dimensional Lorentzian Egorov and ε-spaces and show that the existence of proper biharmonic helices in these manifolds is highly constrained and closely tied to the causal character of the Frenet frame. In particular, we prove that no proper biharmonic helices exist in Egorov spaces unless the space is flat and the normal vector is spacelike. Furthermore, we establish that no proper biharmonic helices exist in three-dimensional ε-spaces.

1 Introduction

The theory of harmonic and biharmonic maps has become a foundational theme in differential geometry, with deep connections to the calculus of variations and mathematical physics.

A smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is harmonic if it is a critical point of the energy functional

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 \, d\nu_g = \frac{1}{2} \int_M \sum_{i=1}^m h(d\varphi(e_i), d\varphi(e_i)) d\nu_g,$$

where $\{e_i\}$ is a local orthonormal frame on M. Equivalently, φ is harmonic if and only if its tension field vanishes:

$$\tau(\varphi) = tr_g \nabla^\varphi d\varphi = \sum_{i=1}^m (\nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i)) = 0,$$

where $\nabla\varphi$ is the pullback connection (the induced covariant derivative along φ).

Introduced by Eells and Sampson [8] and further developed by Jiang [13], biharmonic maps are critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M \|\tau(\varphi)\|^2 \, d\nu_g,$$

and satisfy a fourth-order partial differential equation. The associated Euler–Lagrange equation is the vanishing of the bitension field:

$$\tau_2(\varphi) = \Delta^\varphi \tau(\varphi) - tr_g(R^N(d\varphi(.), \tau(\varphi))d\varphi(.)) = 0,$$

where $\Delta^\varphi = -tr_g(\nabla^\varphi)^2$ is the rough Laplacian along φ, and R^N is the curvature operator of the target manifold N.

While every harmonic map is automatically biharmonic, the converse does not hold in general. The study of proper biharmonic maps, those that satisfy the biharmonic condition without being harmonic, has become an active area of interest, highlighting various geometric constraints related to the curvature and symmetry of the ambient space.

In [3], Caddeo et al. provided a classification of non-harmonic biharmonic submanifolds in the three-dimensional unit sphere S^3. They showed that such submanifolds are limited to circles, spherical helices, and parallel spheres.

Among the most elementary yet structurally rich examples of biharmonic maps are biharmonic curves: smooth maps $\gamma : I \to (M^n, g)$ from a one-dimensional domain into a Riemannian or pseudo-Riemannian manifold whose bitension field vanishes. Several studies have addressed their classification in various geometric contexts.

In [4], Caddeo et al. investigated proper biharmonic curves on surfaces and proved that a non-geodesic biharmonic curve must lie on a surface with constant positive Gauss curvature equal to the square of the geodesic curvature. Furthermore, if the Gauss curvature is nonpositive, then all biharmonic curves are geodesics.

Considerable attention has also been devoted to biharmonic curves in Riemannian homogeneous manifolds and symmetric spaces. In [6], the authors provided a complete classification and explicit parametrization of biharmonic curves in three-dimensional Cartan-Vranceanu spaces. This study was later extended to higher-dimensional analogues in [10]. Additional contributions by Inoguchi [12] and Fetcu and Oniciuc [11] further illustrate how the existence of proper biharmonic curves is deeply influenced by the underlying curvature and symmetry.

In a related setting, Caddeo et al. [5] classified proper biharmonic curves in the Riemannian Heisenberg group, demonstrating how its geometric structure constrains such curves. In the Lorentzian case, Lee [14] extended this analysis to the three-dimensional Lorentzian Heisenberg group. There, it

was shown that proper biharmonic spacelike curves must be pseudo-helices with spacelike normal vectors. These curves are also characterized as slant curves, and their parametric forms were explicitly determined. This study revealed that the existence of proper biharmonic helices, Frenet curves with constant curvature and torsion, depends critically on the causal character of the Frenet frame, a phenomenon that does not arise in the Riemannian context.

While biharmonic curves have been extensively studied in Riemannian manifolds, their extension to Lorentzian geometry introduces distinct challenges due to the presence of null directions and the indefinite nature of the metric. The causal character of vectors and the complexity of the curvature operator in Lorentzian manifolds significantly influence the behavior and classification of such curves.

In this context, three-dimensional Lorentzian manifolds with large isometry groups provide a particularly interesting setting. These spaces, classified according to the dimension of their isometry groups, include notable examples such as Egorov spaces and ε-spaces, which belong to the third stratum in the classification developed by Patrangenaru [16]. These manifolds are known for their rich structure [1]: they are conformally flat, admit parallel null vector fields, and exhibit harmonic curvature properties. Such geometric features make them natural candidates for further investigation in Lorentzian geometry.

The present work is devoted to the analysis of biharmonic helices in the three-dimensional Egorov and ε-spaces. Our approach is motivated by the classification of Lorentzian manifolds with large isometry groups and their role in the geometric study of curvature and symmetry. We aim to determine whether proper biharmonic helices, those which are biharmonic but not geodesic, can exist in these settings, and under what geometric and causal constraints.

The structure of the paper is as follows. In Section 2, we recall key geometric properties of Egorov and ε-spaces, including explicit expressions for their Levi-Civita connections and curvature tensors with respect to pseudo-orthonormal frames. Section 3 begins with a review of the Frenet-Serret apparatus in Lorentzian three-manifolds and the definition of biharmonic curves. We then formulate and analyze the biharmonic equation in each of the two families of spaces under consideration.

Our results reveal that the existence of proper biharmonic helices is highly constrained. Specifically, we prove that in Egorov spaces, such curves may exist only in the flat case and only when the normal vector is spacelike. In contrast, we show that no proper biharmonic helices exist in ε-spaces, regardless of the causal configuration of the Frenet frame. These results highlight the strong dependence of biharmonicity on the causal character of the Frenet frame and the curvature properties of the ambient space.

2   Lorentzian Manifolds with Large Isometry Group and their curvature

In Riemannian geometry, an n-dimensional manifold admitting an isometry group of dimension $n(n-1)/2 + 1$ must be either of constant sectional curvature or a product of an $(n-1)$-dimensional space of constant curvature with a line or a circle. The Lorentzian case, however, admits greater diversity due to the presence of invariant null submanifolds under isometry group actions.

Let $l_0(n) > l_1(n) > \dots$ be the strictly decreasing sequence of all possible dimensions of isometry groups $\mathrm{Iso}(M, g)$ that an n-dimensional Lorentzian manifold (M, g) can admit, and consists of all diffeomorphisms of M that preserve the Lorentzian metric g. The maximal value is $l_0(n) = n(n+1)/2$ realized precisely by spaces of constant sectional curvature (as in the Riemannian setting).

Following [16], a connected Lorentzian manifold of dimension n is said to belong to the j-th stratum if its isometry group has dimension exactly $l_j(n)$ and it admits an effective isometric action by a Lie group of that dimension. This defines a stratification of Lorentzian manifolds according to symmetry "size", with lower j indicating higher symmetry.

The first three strata have dimensions:

Startum 0: $l_0(n) = n(n+1)/2$.

Startum 1: $l_1(n) = n(n-1)/2 + 2$.

Startum 2: $l_2(n) = n(n-1)/2 + 1$.

These strata were studied and classified in detail in [16]. A key insight in this context is that pseudo-Riemannian manifolds admitting large isometry groups are likely to be homogeneous. Patrangenaru [16] demonstrated that Lorentzian manifolds in the first two strata 0 and 1 are indeed homogeneous with constant curvature. Furthermore, the classification in the third stratum (for $n > 5$ and $n \neq 7$) shows that besides constant curvature spaces $M_1^n(c)$, and product manifolds such as $M^{n-1}(c) \times \mathbb{R}_1$ or $\mathbb{R} \times M_1^{n-1}(c)$, additional examples include:

Egorov spaces: These are Lorentzian manifolds, first introduced by I.P. Egorov in [9], of the form $(\mathbb{R}^n, g_f)$, where f is a positive function of a real variable and the metric is defined by

$$g_f = f(x_n) \sum_{i=1}^{n-2} (dx_i)^2 + 2dx_{n-1}dx_n.$$

ε-spaces: These are Lorentzian manifolds $(\mathbb{R}^n, g_\varepsilon)$, equipped with the metric

$$g_\varepsilon = \sum_{i=1}^{n-2} (dx_i)^2 - dx_{n-1}dx_n + \varepsilon \sum_{i=1}^{n-2} (x_i)^2 (dx_{n-1})^2.$$

ε-spaces provide examples of irreducible Lorentzian symmetric spaces and also serve as models for certain non-symmetric spaces [7, 17].

In [1], curvature properties of Lorentzian n-dimensional manifolds, admitting a group of isometries of dimension at least $n(n-1)/2+1$, are completely described. Interesting behaviors are found such as local symmetry, Einstein conditions, conformal flatness, and Einstein-like conditions were thoroughly analyzed. Their role in the theory of Ricci solitons was further explored in [2].

Among all Lorentzian manifolds in the third stratum of this classification, Egorov spaces stand out as the only examples that may fail to be symmetric or even homogeneous. As shown in [1], an Egorov space is locally symmetric if and only if its defining function satisfies

$$f'^2 - 2ff''^2 = k$$

for some real constant k. Moreover, it is locally homogeneous if and only if either it is locally symmetric, or satisfies

$$f'^2 - 2ff'' = \frac{c_n}{(x_n + d)^2} f^2,$$

for some real constants $c_n \neq 0$ and $d \in \mathbb{R}$. Furthermore, Egorov spaces possess a parallel null vector field ∂_{n-1}, placing them within the class of Walker manifolds [1].

In this work, we restrict our attention to the three-dimensional case ($n = 3$). Throughout this paper, the Levi-Civita connection associated with the Lorentzian metrics g_f and g_ε is denoted by ∇. We adhere to the convention in which the curvature tensor R is computed as $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

In this section, we present a detailed description of the geometry of the three-dimensional Egorov spaces $(\mathbb{R}^3, g_f)$ and ε-spaces $(\mathbb{R}^3, g_\varepsilon)$. In particular, we compute the components of their Levi-Civita connections and curvature tensors with respect to a suitably chosen pseudo-orthonormal frame.

2.1 Connection and Curvature of Egorov Spaces $(\mathbb{R}^3, g_f)$

Consider the three-dimensional Egorov space $(\mathbb{R}^3, g_f)$, where the Lorentzian metric is given by

$$g_f = f(x_3)dx_1^2 + 2dx_2 dx_3. \tag{1}$$

We define the pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ as

$$e_1 = \frac{1}{2}\partial_{x_2} + \partial_{x_3}, \qquad e_2 = \frac{1}{\sqrt{f}}\partial_{x_1}, \qquad e_3 = -\frac{1}{2}\partial_{x_2} + \partial_{x_3}, \tag{2}$$

with metric relations

$$g_f(e_1, e_1) = g_f(e_2, e_2) = 1, \qquad g_f(e_3, e_3) = -1,$$

and Lie bracket structure

$$[e_1, e_2] = -\frac{f'}{2f}e_2, \qquad [e_2, e_3] = \frac{f'}{2f}e_2, \qquad [e_1, e_3] = 0.$$

The Levi-Civita connection ∇ for $(\mathbb{R}^3, g_f)$ is determined by

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \nabla_{e_3}e_2 = \nabla_{e_3}e_3 = 0, \qquad (3)$$

$$\nabla_{e_2}e_1 = \nabla_{e_2}e_3 = \frac{f'}{2f}e_2, \quad \nabla_{e_2}e_2 = -\frac{f'}{2f}(e_1 - e_2).$$

The non-vanishing curvature components with respect to $\{e_1, e_2, e_3\}$ are

$$R(e_1, e_2)e_1 = R(e_1, e_2)e_3 = -R(e_2, e_3)e_1 = -R(e_2, e_3)e_3 = \frac{(f')^2 - 2ff''}{4f^2}e_2,$$

$$R(e_1, e_2)e_2 = -R(e_2, e_3)e_2 = -\frac{(f')^2 - 2ff''}{4f^2}(e_1 - e_3), \qquad (4)$$

along with those obtained from the standard symmetries of the curvature tensor.

2.2   Connection and Curvature of ε-Spaces $(\mathbb{R}^3, g_\varepsilon)$

Now consider the ε-space $(\mathbb{R}^3, g_\varepsilon)$, equipped with the Lorentzian metric

$$g_\varepsilon = dx_1^2 - dx_2 dx_3 + \varepsilon x_1^2 dx_2^2. \qquad (5)$$

We define the pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ as

$$e_1 = \partial_{x_3} + \left(\varepsilon x_1^2 - 1\right)\partial_{x_3}, \quad e_2 = \partial_{x_1}, \quad e_3 = \partial_{x_2} + \left(\varepsilon x_1^2 + 1\right)\partial_{x_3}, \quad (6)$$

with metric relations

$$g_\varepsilon(e_1, e_1) = g_\varepsilon(e_2, e_2) = 1, \qquad g_\varepsilon(e_3, e_3) = -1,$$

and Lie bracket structure

$$[e_1, e_2] = -[e_2, e_3] = \varepsilon x_1 (e_1 - e_3), \qquad [e_1, e_3] = 0.$$

The Levi-Civita connection ∇ for $(\mathbb{R}^3, g_\varepsilon)$ is given by

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \nabla_{e_3}e_3 = -\varepsilon x_1 e_2, \qquad (7)$$

$$\nabla_{e_1}e_2 = \nabla_{e_3}e_2 = \varepsilon x_1 (e_1 - e_3),$$

with all other covariant derivatives vanishing.

The non-zero curvature components (up to symmetry) are

$$R(e_1, e_2)e_1 = R(e_1, e_2)e_3 = -R(e_2, e_3)e_1 = -R(e_2, e_3)e_3 = -\varepsilon e_2,$$

$$R(e_1, e_2)e_2 = -R(e_2, e_3)e_2 = \varepsilon (e_1 - e_3).$$

3   Biharmonic Curves

In this section, we begin by recalling some fundamental results concerning the Frenet-Serret frame in the setting of three-dimensional Lorentzian manifolds. We then introduce the notions of biharmonic curves and helices in this context and proceed to examine the conditions under which such curves can exist in Egorov and ε-spaces.

3.1   Frenet-Serret Equations in Lorentzian 3-Manifolds

Let $\gamma : I \subset \mathbb{R} \to M^3$ be a smooth unit-speed curve in a three-dimensional Lorentzian manifold (M^3, g), where the tangent vector γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$, indicating its causal character: spacelike when $\varepsilon_1 = 1$, and timelike when $\varepsilon_1 = -1$. A unit-speed curve γ is called a Frenet curve if the covariant derivative $\nabla_{\gamma'}\gamma'$ is non-null, i.e., $g(\nabla_{\gamma'}\gamma', \nabla_{\gamma'}\gamma') \neq 0$.

For such curves, we can define an orthonormal Frenet frame $\{T, N, B\}$ along γ, where:

$T = \gamma'$ is the unit tangent vector,

$N = \dfrac{\nabla_{\gamma'}\gamma'}{\|\nabla_{\gamma'}\gamma'\|}$ is the principal normal vector,

B is the binormal vector, determined uniquely (up to orientation) by requiring $\{T, N, B\}$ to form a positively oriented pseudo-orthonormal frame.

The construction of the Frenet frame and the Frenet-Serret equations in Lorentzian signature has been described, for example, in [18] (Section 2.2, p. 5), and is carried out explicitly in the context of Lorentzian 3–manifolds (see, for instance, [15] and [14]).

The Frenet-Serret equations in this setting are

$$\begin{aligned}
\nabla_T T &= \varepsilon_2 \kappa N, \\
\nabla_T N &= -\varepsilon_1 \kappa T + \varepsilon_3 \tau B, \\
\nabla_T B &= -\varepsilon_2 \tau N,
\end{aligned}$$

where $\kappa = \|\nabla_{\gamma'}\gamma'\|$ and τ denote the geodesic curvature and torsion of γ, respectively. The constants ε_2 and ε_3 are determined by the causal characters of N and B, such that $g(N, N) = \varepsilon_2$, $g(B, B) = \varepsilon_3$, and they satisfy the identity $\varepsilon_1 \varepsilon_2 = -\varepsilon_3$.

A Frenet curve is a geodesic if and only if $\kappa = 0$. Furthermore, if both κ and τ are constant, the curve is referred to as a pseudo-helix. A pseudo-circle is a special case of a Frenet curve where $\tau = 0$ and κ is constant.

To describe the Frenet frame explicitly, we consider an oriented Lorentzian manifold (M^3, g) of signature $(+, +, -)$, and let $\{e_1, e_2, e_3\}$ be a local positively oriented pseudo-orthonormal frame, where e_1, e_2 are spacelike and e_3

is timelike. For vector fields

$$X = \sum_{i=1}^{3} X_i e_i, \qquad Y = \sum_{i=1}^{3} Y_i e_i,$$

the Lorentzian cross product $X \times Y$ is defined by

$$X \times Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{vmatrix},$$

or equivalently,

$$X \times Y = (X_2 Y_3 - X_3 Y_2)\, e_1 - (X_1 Y_3 - X_3 Y_1)\, e_2 - (X_1 Y_2 - X_2 Y_1)\, e_3.$$

The minus sign in the third column reflects the Lorentzian signature $(+, +, -)$. The vector field $X \times Y$ is the unique vector field satisfying

$$g(X \times Y, Z) = \det(X, Y, Z) \quad \text{for all } Z,$$

where the determinant is computed with respect to any local positively oriented pseudo-orthonormal frame.

Applying this to the Frenet frame $\{T, N, B\}$, the following cyclic relations hold:

$$T \times N = \varepsilon_3 B, \quad N \times B = \varepsilon_1 T, \quad B \times T = \varepsilon_2 N. \tag{8}$$

Finally, decomposing the Frenet vectors with respect to $\{e_1, e_2, e_3\}$

$$T = T_1 e_1 + T_2 e_2 + T_3 e_3, N = N_1 e_1 + N_2 e_2 + N_3 e_3, B = B_1 e_1 + B_2 e_2 + B_3 e_3,$$

where T_i, N_i, B_i are smooth real-valued functions. Using (8), we obtain the component relations:

$$\begin{aligned} \varepsilon_3 B_1 &= T_2 N_3 - T_3 N_2, \ \varepsilon_3 B_2 = T_3 N_1 - T_1 N_3, \ \varepsilon_3 B_3 = T_2 N_1 - T_1 N_2, \\ \varepsilon_1 T_1 &= N_2 B_3 - N_3 B_2, \ \varepsilon_1 T_2 = N_3 B_1 - N_1 B_3, \ \varepsilon_1 T_3 = N_2 B_1 - N_1 B_2, \\ \varepsilon_2 N_1 &= B_2 T_3 - B_3 T_2, \ \varepsilon_2 N_2 = B_3 T_1 - B_1 T_3, \ \varepsilon_2 N_3 = B_2 T_1 - B_1 T_2. \end{aligned} \tag{9}$$

3.2 Biharmonic Curves in Lorentzian Manifolds

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two pseudo-Riemannian manifolds. The tension field of φ, denoted by τ_φ, measures how far is from being a harmonic map, and is defined as

$$\tau_\varphi = \text{trace}\nabla^\varphi d\varphi = \sum_{i=1}^{m} \varepsilon_i \left(\nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i) \right),$$

where $\{e_i\}$ is a local pseudo-orthonormal frame on (M^m, g), and ∇^φ is the pullback connection induced by the Levi-Civita connection on (N^n, h). A map φ is said to be harmonic if its tension field vanishes identically, i.e., $\tau_\varphi = 0$.

A natural generalization of harmonic maps is given by biharmonic maps, which are critical points of the bienergy functional:

$$E_2(\varphi) = \int_M \|\tau_\varphi\|^2 \, dv_g.$$

The Euler-Lagrange equation for E_2 leads to the bitension field $\tau_2(\varphi)$, given by

$$\tau_2(\varphi) = \sum_{i=1}^{m} \varepsilon_i \left(\left(\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla_{e_i} e_i} \right) \tau_\varphi - R^N(\tau_\varphi, d\varphi(e_i)) d\varphi(e_i) \right),$$

where R^N denotes the curvature tensor on the target manifold. A map φ is called proper biharmonic if it is biharmonic but not harmonic, i.e., $\tau_\varphi \neq 0$ and $\tau_2(\varphi) = 0$.

In the case of a curve, that is, a smooth map $\gamma : I \subset \mathbb{R} \to (M, g)$ defined on a one-dimensional interval and parametrized by arc-length, γ is an isometric immersion of I into M. In this setting, the tension field reduces to $\tau_\gamma = \varepsilon_1 \nabla_{\gamma'} \gamma'$, and the biharmonic condition simplifies to the vanishing of the bitension field:

$$\tau_2(\gamma) = \nabla^3_{\gamma'} \gamma' - R(\nabla_{\gamma'} \gamma', \gamma')\gamma' = 0.$$

This condition characterizes biharmonic curves in any pseudo-Riemannian manifold. For curves with vanishing tension field, the equation reduces to that of a geodesic, highlighting that geodesics are trivial solutions of the biharmonic condition.

3.3 Biharmonic Helices in Egorov Spaces $(\mathbb{R}^3, g_f)$

Building on the general framework for biharmonic curves in Lorentzian manifolds, we now turn our attention to the three-dimensional Egorov space $(\mathbb{R}^3, g_f)$, where the Lorentzian metric takes the form (1). These manifolds exhibit a distinctive geometric structure, characterized by the presence of a parallel null vector field and non-constant curvature. Unlike standard models with uniform curvature (such as Minkowski, de Sitter, or anti-de Sitter space), Egorov spaces allow for richer behavior of curves due to their dependence on the defining function $f(x_3)$ (see [1]).

In this section, we examine the existence and characterization of biharmonic helices, i.e., Frenet curves with constant geodesic curvature and torsion, within this particular class of Lorentzian manifolds.

Our goal is to determine the conditions under which such helices exist, when they are proper biharmonic (i.e., biharmonic but not geodesic), and how the form of the function $f(x_3)$ influences their geometry.

To proceed, we consider a Frenet curve $\gamma : I \subset \mathbb{R} \to (\mathbb{R}^3, g_f)$, parametrized by arc-length and satisfying $g_f(\gamma', \gamma') = \varepsilon_1 = \pm 1$. Along this curve, we define a Frenet frame field $\{T = \gamma', N, B\}$, where $g_f(N, N) = \varepsilon_2 = \pm 1$ and $g_f(B, B) = \varepsilon_3 = \pm 1$.

Using the Frenet-Serret equations, the bitension field simplifies as follows

$$\tau_2(\gamma) = \nabla_T^3 T + \varepsilon_2 \kappa R(T, N)T.$$

In order to characterize the conditions under which a Frenet curve in $(\mathbb{R}^3, g_f)$ is biharmonic, we now analyze the bitension field in more detail. To carry out this computation, we begin by expanding $\nabla_T^3 T$ using the Frenet equations. This gives

$$\nabla_T^3 T = 3\varepsilon_3 \kappa \kappa' T + \left(\varepsilon_2 \kappa'' - \kappa \left(\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2\right)\right) N - \varepsilon_1 \left(2\kappa' \tau + \kappa \tau'\right) B.$$

Next, we express the Frenet frame in terms of the pseudo-orthonormal frame $\{e_1, e_2, e_3\}$ defined by (2). We write the decomposition

$$T = T_1 e_1 + T_2 e_2 + T_3 e_3, N = N_1 e_1 + N_2 e_2 + N_3 e_3, B = B_1 e_1 + B_2 e_2 + B_3 e_3,$$

with real-valued smooth functions T_i, N_i, B_i.

With respect to the Frenet frame $\{T, N, B\}$, the curvature term $R(T, N)T$ can be expressed as

$$R(T, N)T = \varepsilon_2 g_f(R(T, N)T, N)N + \varepsilon_3 g_f(R(T, N)T, B)B.$$

Using the curvature components (4) together with the Lorentzian vector product identities (9), we compute the scalar terms of $R(T, N)T$ as follows

$$
\begin{aligned}
g_f(R(T, N)T, N) &= \frac{(f')^2 - 2ff''}{4f^2}(T_2 N_1 - T_1 N_2 + T_2 N_3 - T_3 N_2)^2 \\
&= \frac{(f')^2 - 2ff''}{4f^2}(B_1 + B_3)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
g_f(R(T, N)T, B) &= \frac{(f')^2 - 2ff''}{4f^2}(B_1 T_2 - B_2 T_1 + B_3 T_2 - B_2 T_3) \\
&\qquad\qquad\qquad \cdot (N_1 T_2 - N_2 T_1 + N_3 T_2 - N_2 T_3) \\
&= \frac{(f')^2 - 2ff''}{4f^2}(-\varepsilon_2 N_1 - \varepsilon_2 N_3)(\varepsilon_3 B_1 + \varepsilon_3 B_3) \\
&= \varepsilon_1 \frac{(f')^2 - 2ff''}{4f^2}(N_1 + N_3)(B_1 + B_3).
\end{aligned}
$$

Substituting into the biharmonic equation $\tau_2(\gamma) = 0$, we obtain

$$3\varepsilon_3 \kappa \kappa' T + \left(\varepsilon_2 \kappa'' - \kappa \left(\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2 \right) + \kappa \frac{(f')^2 - 2ff''}{4f^2}(B_1 + B_3)^2 \right) N$$
$$- \left(\varepsilon_1 \left(2\kappa' \tau + \kappa \tau' \right) + \kappa \frac{(f')^2 - 2ff''}{4f^2}(N_1 + N_3)(B_1 + B_3) \right) B = 0.$$

This leads to the following result

Proposition 1 *Let $\gamma : I \subset \mathbb{R} \to (\mathbb{R}^3, g_f)$ be a Frenet curve parametrized by arc-length. Then γ is a proper biharmonic curve if and only if*

$$\begin{cases} \kappa = constant \neq 0, \\ \varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2 = \dfrac{(f')^2 - 2ff''}{4f^2}(B_1 + B_3)^2, \\ \tau' = \varepsilon_1 \dfrac{2ff'' - (f')^2}{4f^2}(N_1 + N_3)(B_1 + B_3). \end{cases}$$

To investigate biharmonic helices in the three-dimensional Egorov space $(\mathbb{R}^3, g_f)$, we consider the case where the torsion τ is constant. Under this assumption, the third condition from the preceding proposition reduces to the following constraint:

$$\left(2ff'' - (f')^2 \right)(N_1 + N_3)(B_1 + B_3) = 0.$$

We now analyze the implications of this identity according to the causal character of the Frenet frame. Note that the sign of the product $\varepsilon_1 \varepsilon_3$ determines the causal nature of the normal vector N.

Case 1: If the normal vector is timelike, i.e., $\varepsilon_2 = -1$.

This corresponds to the situation where both the tangent vector γ' and the binormal vector B share the same causal character (both spacelike or both timelike), so that $\varepsilon_1 \varepsilon_3 = 1$.

We first assume that $2ff'' - (f')^2 = 0$. This condition characterizes the Egorov space $(\mathbb{R}^3, g_f)$ as flat. Substituting into the second condition of the proposition leads to both the curvature κ and the torsion τ vanishing identically. However, this contradicts the assumption that the curve is a proper biharmonic helix. A similar contradiction arises if we suppose $B_1 + B_3 = 0$.

Now, consider the case $N_1 + N_3 = 0$. Since

$$g_f(N, N) = N_1^2 + N_2^2 - N_3^2 = \varepsilon_2 = -1,$$

the condition $N_1 + N_3 = 0$ implies

$$g_f(N, N) = N_1^2 + N_2^2 - (N_1)^2 = N_2^2.$$

Consequently, $N_2^2 = -1$, which is impossible for a real-valued function N_2. Hence, the case $N_1 + N_3 = 0$ cannot occur. It follows that no proper biharmonic helices exist in $(\mathbb{R}^3, g_f)$ when the normal vector is timelike (i.e., when $\varepsilon_2 = -1$).

Case 2: If the normal vector is spacelike, i.e., $\varepsilon_2 = 1$.

In this case, the tangent vector γ' and the binormal vector B have opposite causal characters, which implies $\varepsilon_1 \varepsilon_3 = -1$. Hence, the second condition in the biharmonic characterization becomes

$$\kappa^2 - \tau^2 = \varepsilon_1 \frac{(f')^2 - 2ff''}{4f^2}(B_1 + B_3)^2. \tag{10}$$

Using the components of the Levi-Civita connection (3), we express $\nabla_T N$ and $\nabla_T B$ with respect to the pseudo-orthonormal basis (2) as follows:

$$\nabla_T N = \left(N_1' - \frac{f'}{2f}T_2 N_2\right) e_1 + \left(N_2' + \frac{f'}{2f}T_2 (N_1 + N_3)\right) e_2$$
$$+ \left(N_3' + \frac{f'}{2f}T_2 N_2\right) e_3,$$

$$\nabla_T B = \left(B_1' - \frac{f'}{2f}T_2 B_2\right) e_1 + \left(B_2' + \frac{f'}{2f}T_2 (B_1 + B_3)\right) e_2$$
$$+ \left(B_3' + \frac{f'}{2f}T_2 B_2\right) e_3.$$

Then, using the Frenet-Serret equations, we obtain

$$\begin{aligned} N_1' + N_3' &= -\varepsilon_1 \kappa (T_1 + T_3) + \varepsilon_3 \tau (B_1 + B_3), \\ B_1' + B_3' &= -\varepsilon_2 \tau (N_1 + N_3). \end{aligned} \tag{11}$$

Let us now explore the possible solutions.

If $B_1 + B_3 = 0$, then from equation (10), we have $\tau = \pm\kappa \neq 0$. Since $\tau \neq 0$, the second equation in (11) implies $N_1 + N_3 = 0$. Substituting $N_1 + N_3 = 0$ into the first equation of (11) yields $T_1 + T_3 = 0$. Thus, we obtain

$$T_1 + T_3 = N_1 + N_3 = 0.$$

This leads to

$$\begin{aligned} g_f(T, T) &= T_1^2 + T_2^2 - T_3^2 = T_2^2 = \varepsilon_1, \\ g_f(B, B) &= B_1^2 + B_2^2 - B_3^2 = B_2^2 = \varepsilon_3, \end{aligned}$$

which contradicts the assumption that $\varepsilon_1 \varepsilon_3 = -\varepsilon_2 = -1$. Therefore, this case cannot occur.

Now, if $N_1 + N_3 = 0$, then the first equation of (11) implies

$$T_1 + T_3 = -\frac{\tau}{\kappa}\left(B_1 + B_3\right). \tag{12}$$

On the other hand, from (9) we have, since $N_2^2 = \varepsilon_2 = 1$,

$$
\begin{aligned}
B_1 + B_3 &= \varepsilon_3\left(T_2 N_3 - T_3 N_2\right) + \varepsilon_3\left(T_2 N_1 - T_1 N_2\right), \\
&= -\varepsilon_3 N_2\left(T_1 + T_3\right), \\
&= \pm\varepsilon_3\left(T_1 + T_3\right).
\end{aligned}
$$

Combining this with equation (12), we obtain

$$\left(\tau \pm \varepsilon_3\kappa\right)\left(B_1 + B_3\right) = 0.$$

As we already excluded the case $B_1 + B_3 = 0$, we have $\tau = \pm\kappa$. Substituting this into equation (10), we get

$$\left((f')^2 - 2ff''\right)\left(B_1 + B_3\right) = 0.$$

Thus, $(f')^2 - 2ff'' = 0$, which again corresponds to the flat case of Egorov space.

Therefore, we conclude that in both causal configurations of the Frenet frame, proper biharmonic helices in $(\mathbb{R}^3, g_f)$ can exist only in the flat case of the Egorov space, and only when the normal vector is spacelike.

Theorem 1 *Let $\gamma : I \subset \mathbb{R} \to (\mathbb{R}^3, g_f)$ be a Frenet curve parametrized by arc-length in the three-dimensional Egorov space $(\mathbb{R}^3, g_f)$.*

If the normal vector is timelike, then there exists no proper biharmonic helices in $(\mathbb{R}^3, g_f)$.

If the normal vector is spacelike, then γ is a proper biharmonic helix if and only if $\tau = \pm\kappa = constant \neq 0$ and the defining function f satisfies

$$(f')^2 - 2ff'' = 0,$$

which characterizes the flat case of the Egorov space.

3.4   Biharmonic Helices in ε-Spaces $(\mathbb{R}^3, g_\varepsilon)$

We now investigate biharmonic helices in the setting of a 3-dimensional ε-space, $(\mathbb{R}^3, g_\varepsilon)$, where g_ε is the Lorentzian metric defined by (5). Our goal is to identify the conditions under which such curves are proper biharmonic (i.e., biharmonic but not geodesic).

Let $\gamma : I \subset \mathbb{R} \to (\mathbb{R}^3, g_\varepsilon)$ be a Frenet curve parametrized by arc-length and satisfying $g_\varepsilon(\gamma', \gamma') = \varepsilon_1 = \pm 1$. We consider along γ a Frenet frame

$\{T = \gamma', N, B\}$, with $g_\varepsilon(N, N) = \varepsilon_2 = \pm 1$ and $g_\varepsilon(B, B) = \varepsilon_3 = \pm 1$, and the corresponding geodesic curvature κ and torsion τ, respectively.

To analyze the biharmonic condition $\tau_2(\gamma) = 0$, we first express the Frenet frame in terms of the local pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ defined by (6). Using the same notation as in the previous section, we write $T = \sum_{i=1}^{3} T_i e_i$, $N = \sum_{i=1}^{3} N_i e_i$, and $B = \sum_{i=1}^{3} B_i e_i$, with real-valued smooth functions T_i, N_i, B_i. We then compute the third covariant derivative $\nabla_T^3 T$ along the curve γ, as well as the curvature term $R(T, N)T$, expressed in the Frenet frame. This leads to the following biharmonic equation

$$3\varepsilon_3 \kappa\kappa' T + \left(\varepsilon_2 \kappa'' - \kappa\left(\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2\right) - \varepsilon\kappa(B_1 + B_3)^2\right) N$$
$$+ \left(\varepsilon\kappa(N_1 + N_3)(B_1 + B_3) - \varepsilon_1\left(2\kappa'\tau + \kappa\tau'\right)\right) B = 0.$$

From this, we derive the following necessary and sufficient conditions for a curve to be proper biharmonic in $(\mathbb{R}^3, g_\varepsilon)$.

Proposition 2 *Let $\gamma : I \subset \mathbb{R} \to (\mathbb{R}^3, g_\varepsilon)$ be a Frenet curve parametrized by arc-length. Then γ is a proper biharmonic curve if and only if*

$$\begin{cases} \kappa = constant \neq 0, \\ \varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2 = -\varepsilon(B_1 + B_3)^2, \\ \tau' = \varepsilon\varepsilon_1(N_1 + N_3)(B_1 + B_3). \end{cases}$$

Proceeding as in the previous section, we examine the existence of proper biharmonic helices in the three-dimensional ε-space $(\mathbb{R}^3, g_\varepsilon)$. To this end, we assume that the torsion τ is constant. Under this assumption, the third condition in the above proposition reduces to

$$(N_1 + N_3)(B_1 + B_3) = 0.$$

Depending on the causal character of the Frenet frame, we distinguish two cases.

Case 1: The normal vector N is timelike, i.e., $\varepsilon_2 = -1$.

First, suppose $B_1 + B_3 = 0$. This forces both the curvature and torsion to vanish, which contradicts the requirement for proper biharmonic (non-geodesic) curves.

Next, assume $N_1 + N_3 = 0$. Then, using the condition $g_\varepsilon(N, N) = \varepsilon_2 = -1$, we find $N_2^2 = -1$, which is not possible.

Thus, both possibilities lead to contradictions.

Case 2: The normal vector N is spacelike, i.e., $\varepsilon_2 = 1$.

From the components of the Levi-Civita connection (7), we write down, in terms of the pseudo-orthonormal basis (6), $\nabla_T N$ obtaining

$$\nabla_T N = \left(N_1' + \varepsilon x_1 N_2(T_1 + T_3)\right) e_1 + \left(N_2' - \varepsilon x_1(T_1 + T_3)\left(N_1 + N_3\right)\right) e_2$$
$$+ \left(N_3' - \varepsilon x_1 N_2(T_1 + T_3)\right) e_3.$$

From the second Frenet-Serret equation, we get

$$N_1' + N_3' = -\varepsilon_1 \kappa \left(T_1 + T_3 \right) + \varepsilon_3 \tau \left(B_1 + B_3 \right). \tag{13}$$

Now suppose $N_1 + N_3 = 0$. Then

$$T_1 + T_3 = \frac{\varepsilon_1 \varepsilon_3 \tau}{\kappa} \left(B_1 + B_3 \right).$$

From the first equation in (9), and using the fact that $N_2^2 = \varepsilon_2 = 1$, we obtain

$$B_1 + B_3 = \pm \varepsilon_3 \left(T_1 + T_3 \right).$$

Combining the two identities, we get

$$\left(1 \pm \frac{\varepsilon_1 \tau}{\kappa} \right) \left(B_1 + B_3 \right) = 0.$$

Assuming $B_1 + B_3 \neq 0$, we obtain $\tau^2 = \kappa^2$. Substituting into the second biharmonic condition,

$$\left(\varepsilon_1 + \varepsilon_3 \right) \kappa^2 = -\varepsilon (B_1 + B_3)^2,$$

but since $\varepsilon_1 \varepsilon_3 = -1$, this implies $B_1 + B_3 = 0$, hence $T_1 + T_3 = 0$. This contradicts the causal character of the tangent and binormal vectors. Therefore, the assumption $N_1 + N_3 = 0$ must be false. We conclude that $B_1 + B_3 = 0$, leading to $\tau = \pm \kappa$.

Now, from the Levi-Civita connection (7), we compute

$$\nabla_T B = \left(B_1' + \varepsilon x_1 B_2 (T_1 + T_3) \right) e_1 + \left(B_2' - \varepsilon x_1 (T_1 + T_3) \left(B_1 + B_3 \right) \right) e_2 \\ + \left(B_3' - \varepsilon x_1 B_2 (T_1 + T_3) \right) e_3.$$

From the third Frenet-Serret equation, we obtain

$$B_1' + B_3' = -\varepsilon_2 \tau \left(N_1 + N_3 \right) = 0.$$

Since $\tau \neq 0$, this yields $N_1 + N_3 = 0$, which when substituted into equation (13), gives $T_1 + T_3 = 0$, once again leading to a contradiction with the causal character of the Frenet frame.

Therefore, neither case allows the existence of proper biharmonic helices in $(\mathbb{R}^3, g_\varepsilon)$. We conclude with the following result.

Theorem 2 *Let* $\gamma : I \subset \mathbb{R} \to (\mathbb{R}^3, g_\varepsilon)$ *be a Frenet curve parametrized by arc-length in the three-dimensional ε-space $(\mathbb{R}^3, g_\varepsilon)$. Then there exists no proper biharmonic helices in $(\mathbb{R}^3, g_\varepsilon)$.*

References

[1] W. Batat, G. Calvaruso and B. De Leo, Curvature properties of Lorentzian manifolds with large isometry groups. Math. Phys. Anal. Geom., **12** (2009), pp. 201–217. https://doi.org/10.1007/s11040-009-9060-4

[2] W. Batat, M. Brozos-Vázquez, E. García-Río and S. Gavino-Fernández, Ricci solitons on Lorentzian manifolds with large isometry groups. Bull. Lond. Math. Soc., **43** (2011), no. 6, pp. 1219–1227. https://doi.org/10.1112/blms/bdr057

[3] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds of S^3. Internat. J. Math, **12** (2001), pp. 867–876. https://doi.org/10.1142/S0129167X01001027

[4] R. Caddeo, S. Montaldo and P. Piu, Biharmonic curves on a surface. Rend. Mat. App., **21** (2001), pp. 143–157.

[5] R. Caddeo, C. Oniciuc and P. Piu, Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group. Rend. Semin. Mat. Univ. Politec. Torino, **62** (2004), no. 3, pp. 265–277.

[6] R. Caddeo, S. Montaldo, C. Oniciuc and P. Piu, The classification of biharmonic curves of Cartan-Vranceanu 3-dimensional spaces. In *Modern Trends in Geometry and Topology*, Cluj University Press: Cluj-Napoca, Romania, pp. 121–131, 2006.

[7] M. Cahen, J. Leroy, M. Parker, F. Tricerri and L. Vanhecke, Lorentz manifolds modelled on a Lorentz symmetric space. J. Geom. Phys., **7** (1990), no. 4, pp. 571–581. https://doi.org/10.1016/0393-0440(90)90007-P

[8] J. Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds. Amer. J. Math., **86** (1964), no. 1, pp. 109–160. https://doi.org/10.2307/2373037

[9] I.P. Egorov, Riemannian spaces of the first three lacunary types in the geometric sense (in Russian). Dokl. Akad. Nauk. SSSR, **150** (1963), pp. 730–732.

[10] D. Fetcu, Biharmonic curves in Cartan-Vranceanu (2n+1)-dimensional spaces. Bul. Acad. Ştiinţe Rep. Mold. Mat., **53** (2007), no. 1, pp. 59–65.

[11] D. Fetcu and C. Oniciuc, Biharmonic curves in 3-dimensional Heisenberg group. Math. Commun., **13** (2008), no. 2, pp. 311–320.

[12] J. Inoguchi, Biharmonic curves in Minkowski 3-space. Int. Electron. J. Geom., **1** (2008), no. 1, pp. 40–83.

[13] G.Y. Jiang, 2-harmonic maps and their first and second variational formulas. Chinese Ann. Math. Ser. A, **7** (1986), no. 4, pp. 389–402.

[14] J. Lee, Biharmonic spacelike curves in Lorentzian Heisenberg space. Commun. Korean Math. Soc., **33** (2018), no. 4, pp. 1309–1320. https://doi.org/10.4134/CKMS.c170376

[15] R. López, Differential geometry of curves and surfaces in Lorentz-Minkowski space. Int. Electron. J. Geom., **7** (2014), no. 1, pp. 44–107. https://doi.org/10.36890/iejg.594497

[16] V. Patrangenaru, Lorentz manifolds with the three largest degrees of symmetry. Geom. Dedicata, **102** (2003), pp. 25–33. https://doi.org/10.1023/B:GEOM.0000006588.95481.1c

[17] V. Patrangenaru, Locally homogeneous pseudo-Riemannian manifolds. J. Geom. Phys., **17** (1995), no. 1, pp. 59–72. https://doi.org/10.1016/0393-0440(94)00040-B

[18] J. Walrave, Curves and Surfaces in Minkowski Space. Doctoral Thesis, Katholieke Universiteit Leuven, Faculty of Science, Leuven, Belgium, 1995.

Fatima Z. Benzahra-Belkacem
Department of Preparatory Training in Science and Technology
of National Polytechnic School of Oran-Maurice Audin
B.P 1523 El M'naouar Oran 31000, Algeria.
fatima.benzahra-belkacem@enp-oran.dz

Noura Sidhoumi
Department of Preparatory Training in Science and Technology
of National Polytechnic School of Oran-Maurice Audin
B.P 1523 El M'naouar Oran 31000, Algeria.
noura.sidhoumi@enp-oran.dz