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Normal Automorphisms of Free Burnside Groups of Period 3

V. S. Atabekyan, H. T. Aslanyan and A. E. Grigoryan

Abstract. If any normal subgroup of a group G is ϕ-invariant for some automorphism ϕ of G, then ϕ is called a normal automorphism. Each inner automorphism of a group is normal, but the converse is not true in the general case. We prove that any normal automorphism of the free Burnside group $\mathbf{B}(m,3)$ of period 3 is inner for each rank $m \geq 3$.

Key Words: normal automorphism, inner automorphism, periodic group, free Burnside group, free group
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Introduction

For an arbitrary group G and for an automorphism $\phi \in \mathrm{Aut}(G)$ a subgroup H of G is called a *ϕ-invariant* subgroup if $\phi(H) = H$. If a normal subgroup of G is ϕ-invariant for some automorphism ϕ, then ϕ is called a *normal automorphism* of G.

Thus, for a given normal automorphism $\phi \in \mathrm{Aut}(G)$ every normal subgroup of G is ϕ-invariant. Clearly, if N is a ϕ-invariant normal subgroup of G, then ϕ induces an automorphism of the quotient group G/N.

Denote by $\mathrm{Aut}_n(G)$ the set of all normal automorphisms of G. It is obvious that

$$\mathrm{Inn}(G) \subset \mathrm{Aut}_n(G),$$

but the converse is not true (for example $\mathrm{Aut}(\mathbb{Z}) = \mathrm{Aut}_n(\mathbb{Z}) = \mathbb{Z}_2 \neq \mathrm{Inn}(\mathbb{Z}) = \{1\}$). A.Lubotzky proved in [1] that every normal automorphism of a non-cyclic absolutely free group F is inner, that is

$$\mathrm{Inn}(F) = \mathrm{Aut}_n(F).$$

The corresponding equation was subsequently proved for various interesting classes of groups (see References in [2]). In particular, Neshchadim [3] strengthened the result of [1] by proving that every normal automorphism of a free product of nontrivial groups is inner. Minasyan and Osin [4] showed that if G is a non-cyclic relatively hyperbolic group without non-trivial finite normal subgroups, then $\mathrm{Inn}(G) = \mathrm{Aut}_n(G)$. V.Atabekyan in [2] proved that for any odd number $n \geq 1003$ every normal automorphism of the free Burnside group $\mathbf{B}(m, n)$ of (finite or infinite) rank $m > 1$ is inner (see also [5]). Recall that a relatively free group of rank m in the variety of all groups satisfying the law $x^n = 1$ is denoted by $\mathbf{B}(m, n)$ and and is called a free periodic group or the free Burnside group of exponent n and rank m. The well-known theorem of S.Adian [6] (see also [7]) asserts that for all odd $n \geq 665$ and for $m > 1$ the group $\mathbf{B}(m, n)$ is infinite (the solution of the Burnside problem). In this paper we consider the groups $\mathbf{B}(m, 3)$ of period 3 which are finite groups. However, for these groups the equality $\mathrm{Inn}(G) = \mathrm{Aut}_n(G)$ again holds.

Theorem 1 *Any normal automorphism of a free Burnside group $\boldsymbol{B}(m, 3)$ of period 3 is inner for all finite ranks $m \geq 3$.*

Corollary 1 *$Inn(\boldsymbol{B}(m, 3)) = Aut_n(\boldsymbol{B}(m, 3))$ for all finite ranks $m \geq 3$.*

A survey on automorphisms of infinite free Burnside groups $\mathbf{B}(m, n)$ can be found in [8].

1 Preliminary lemmas

According to well known Magnus's theorem, if in some absolutely free group F the normal closures of $r \in F$ and $s \in F$ coincide, then r is conjugate to s or s^{-1}. We say that a group G possesses the *Magnus property*, if for any two elements r, s of G with the same normal closures we have that r is conjugate to s or s^{-1}. The following result is proved by authors in the paper [9].

Lemma 1 (see [9, Theorem 1]) A free Burnside group $\mathbf{B}(m, 3)$ of any rank m possesses the Magnus's property.

We will write $x \sim y$ if two elements x and y are conjugate in a given group.

Lemma 2 *An automorphism α of $\boldsymbol{B}(m, 3)$ is a normal automorphism if and only if for all $x \in \boldsymbol{B}(m, 3)$ we have $\alpha(x) \sim x$ or $\alpha(x) \sim x^{-1}$.*

Proof. Let α be a normal automorphism. Since the normal closure $\langle\langle x \rangle\rangle$ of element x is a normal subgroup and α is a normal automorphism, we have the equalities

$$\langle\langle x \rangle\rangle = \alpha(\langle\langle x \rangle\rangle) = \langle\langle \alpha(x) \rangle\rangle.$$

Consequently, by virtue of Lemma 1 we get that either $\alpha(x) \sim x$ or $\alpha(x) \sim x^{-1}$. The converse is trivial. $\square$

Lemma 3 *The identities* $(xy)^3 = 1$ *and*

$$yxy = x^{-1}y^{-1}x^{-1} \tag{1}$$

are equivalent.

Proof. The proof is obvious. □

Lemma 4 *Any element* $y \in \boldsymbol{B}(3)$ *permutes with any of its conjugates* $x^{-1}yx$.

Proof. Clearly, the identity $x^3 = 1$ implies the equality

$$(xy^{-1})^3 \cdot (yx^{-1}y^3xy^{-1}) \cdot (y(x^{-1}y^{-1})^3y^{-1}) \cdot y^3 = 1.$$

After reducing we get the equality $xy^{-1}xyx^{-1}y^{-1}x^{-1}y = 1$, which means that

$$x \cdot y^{-1}xy = y^{-1}xy \cdot x.$$

□

Lemma 5 *For any elements* $x, y \in \boldsymbol{B}(3)$ *and any* $k, l \in \mathbb{Z}$ *the equality*

$$x^k \cdot y^{-1}x^ly = y^{-1}x^ly \cdot x^k$$

holds.

Proof. Immediately follows from Lemma 4. □

Lemma 6 *Let* $\boldsymbol{B}(m, 3)$ *is the free Burnside group with the free generators* $X = \{x_1, x_2, ..., x_m\}$. *Then for any element* $u \in \boldsymbol{B}(m, 3)$ *and for any generator* $x_i \in X$ *there exists an element* $v \in Gp(X \setminus x_i)$ *such that the equality*

$$uxu^{-1} = vxv^{-1} \tag{2}$$

holds.

Proof. Consider an element $x_iwx_iw^{-1}x_i^{-1}$. By Lemma 5 we can permute wx_iw^{-1} and x_i^{-1} and get $x_iwx_iw^{-1}x_i^{-1} = wx_iw^{-1}$.

Therefore, if $u = u_1x_i^{\pm1}u_2x_i^{\pm1} \cdots x_i^{\pm1}u_k$, where $u_j \in Gp(X \setminus x_i)$ for all $j = 1, ..., k$, then

$$uxu^{-1} = u_1u_2 \cdots u_kx_i(u_1u_2 \cdots u_k)^{-1}.$$

So, $v = u_1u_2 \cdots u_k$. □

2 Proof of Theorem 1

Proof. Let α be a normal automorphism of $\mathbf{B}(m,3)$. By virtue of Lemma 4 for any element x we have $\alpha(x) \sim x$ or $\alpha(x) \sim x^{-1}$. We denote by $\bar{\alpha}$ the automorphism of the abelian group $\mathbf{B}(m,3)/[\mathbf{B}(m,3),\mathbf{B}(m,3)]$ induced by automorphism α and denote by $\bar{x}$ the image of x in $\mathbf{B}(m,3)/[\mathbf{B}(m,3),\mathbf{B}(m,3)]$, where $[\mathbf{B}(m,3),\mathbf{B}(m,3)]$ is the commutator subgroup of $\mathbf{B}(m,3)$.

Let x, y be some elements of $\mathbf{B}(m,3)$ that do not belong to the commutator subgroup $[\mathbf{B}(m,3),\mathbf{B}(m,3)]$. Suppose that $\alpha(x) \sim x$ and $\alpha(y) \sim y^{-1}$. Then $\bar{\alpha}(\bar{x}) = \bar{x}$ and $\bar{\alpha}(\bar{y}) = \bar{y}^{-1}$. Therefore, if $\alpha(xy) \sim xy$, then $\bar{\alpha}(\bar{x}\bar{y}) = \bar{x}\bar{y}$ and if $\alpha(xy) \sim (xy)^{-1}$, then $\bar{\alpha}(\bar{x}\bar{y}) = \bar{y}^{-1}\bar{x}^{-1}$.

In the case $\bar{\alpha}(\bar{x}\bar{y}) = \bar{x}\bar{y}$ we get that $\bar{x}\bar{y}^{-1} = \bar{x}\bar{y}$ and therefore, $\bar{y} = 1$.

In the second case $\bar{\alpha}(\bar{x}\bar{y}) = \bar{y}^{-1}\bar{x}^{-1}$ we get $\bar{x}\bar{y}^{-1} = \bar{y}^{-1}\bar{x}^{-1}$ and therefore, $\bar{x} = 1$. Hence, in both cases we obtain a contradiction with the condition $x, y \notin [\mathbf{B}(m,3),\mathbf{B}(m,3)]$.

This means that for all $x \in \mathbf{B}(m,3)$ which are not in $[\mathbf{B}(m,3),\mathbf{B}(m,3)]$ we have either

1) $\alpha(x) \sim x$

or

2) $\alpha(x) \sim x^{-1}$.

First consider Case 1). Let $x_1, ..., x_m$ be the free generators of $\mathbf{B}(m,3)$. It is obvious that $x_1, ..., x_m \notin [\mathbf{B}(m,3),\mathbf{B}(m,3)]$. Then by virtue of condition $\alpha(x) \sim x$ we have

$$\begin{aligned} \alpha(x_1) &= u_1 x_1 u_1^{-1} \\ \alpha(x_2) &= u_2 x_2 u_2^{-1} \\ &\cdots \\ \alpha(x_m) &= u_m x_m u_m^{-1}. \end{aligned} \tag{3}$$

Applying the inner automorphism $\beta_1 = i_{u_1^{-1}}$ to both sides of above equalities (3) we obtain

$$\begin{aligned} \beta_1(\alpha(x_1)) &= x_1 \\ \beta_1(\alpha(x_2)) &= v_2 x_2 v_2^{-1} \\ &\cdots \\ \beta_1(\alpha(x_m)) &= v_m x_m v_m^{-1}. \end{aligned} \tag{4}$$

According to Lemma 6 we can assume that any generator x_i does not occur in v_i for all $i = 2, ..., m$.

Now applying the automorphism $\beta_1 \circ \alpha$ to $x_1 x_2$ we get

$$(\beta_1 \circ \alpha)(x_1 x_2) = x_1 v_2 x_2 v_2^{-1}$$

The elements $x_1, x_1x_2, x_3, ..., x_m$ are free generators of $\mathbf{B}(m,3)$. Since the automorphism $\beta_1 \circ \alpha$ is normal, it induces an automorphism $\overline{\beta_1 \circ \alpha}$ of the quotient group $\mathbf{B}(m,3)/\langle\langle x_1x_2\rangle\rangle$. Note that $\mathbf{B}(m,3)/\langle\langle x_1x_2\rangle\rangle$ is canonically isomorphic to the free Burnside group $\mathbf{B}(m-1,3)$ with the free generators $x_1, x_3, ..., x_m$. Hence,

$$1 = (\overline{\beta_1 \circ \alpha})(x_1x_2) = x_1v_2x_2v_2^{-1}$$

in $\mathbf{B}(m-1,3)$.

Using the defining relation $x_1x_2 = 1$ of $\mathbf{B}(m,3)/\langle\langle x_1x_2\rangle\rangle$ from $1 = x_1v_2x_2v_2^{-1}$ we get

$$v_2x_1v_2^{-1} = x_1.$$

Since x_2 does not occur in v_2, we assume that x_1 and v_2 also permute in $\mathbf{B}(m,3)$.

Now consider the automorphism $\beta_2 = i_{v_2^{-1}}$. Applying it to both sides of the equalities (4) we obtain

$$\begin{aligned} \beta_2(\beta_1(\alpha(x_1))) &= x_1 \\ \beta_2(\beta_1(\alpha(x_2))) &= x_2 \\ \beta_2(\beta_1(\alpha(x_3))) &= w_3x_3w_3^{-1} \\ &\cdots \\ \beta_2(\beta_1(\alpha(x_m))) &= w_mx_mw_m^{-1}, \end{aligned} \tag{5}$$

Again using Lemma 6 we can assume that no generator x_i does not occur in w_i for all $i = 3, ..., m$. Further, considering the quotient groups $\mathbf{B}(m,3)/\langle\langle x_1x_3\rangle\rangle$ and $\mathbf{B}(m,3)/\langle\langle x_2x_3\rangle\rangle$ and repeating the above arguments we can assume that x_1 and x_2 permute with w_3 in $\mathbf{B}(m,3)$.

Now apply the inner automorphism $\beta_3 = i_{w_3^{-1}}$ to both sides of the equalities (5) and repeating above arguments for all generators x_j, $j \geq 4$ we finally find inner automorphisms $\beta_4, ..., \beta_m$ such that

$$\begin{aligned} \beta_m(...(\beta_1(\alpha(x_1)))...) &= x_1 \\ \beta_m(...(\beta_1(\alpha(x_2)))...) &= x_2 \\ &\cdots \\ \beta_m(...(\beta_1(\alpha(x_m)))...) &= x_m. \end{aligned}$$

This means that $\beta_m \circ ... \circ \beta_1 \circ \alpha$ is identical automorphism. Hence, $\alpha = (\beta_m \circ ... \circ \beta_1)^{-1}$. Because $\beta_1, ..., \beta_m$ are inner automorphisms, we get that α also is inner.

Case 2). By condition $\alpha(x) \sim x^{-1}$ for all $x \in \mathbf{B}(m,3)$ which are not in

$[\mathbf{B}(m,3), \mathbf{B}(m,3)]$ we have

$$\alpha(x_1) = u_1 x_1^{-1} u_1^{-1}$$
$$\alpha(x_2) = u_2 x_2^{-1} u_2^{-1}$$
$$\dots$$
$$\alpha(x_m) = u_m x_m^{-1} u_m^{-1}.$$

Reasoning in the same way as in Case 1) we obtain:

$$\beta_m(...(\beta_1(\alpha(x_1)))...) = x_1^{-1}$$
$$\beta_m(...(\beta_1(\alpha(x_2)))...) = x_2^{-1}$$
$$\dots$$
$$\beta_m(...(\beta_1(\alpha(x_m)))...) = x_m^{-1}.$$

Besides, $\beta_m(...(\beta_1(\alpha(x)))...) = x^{-1}$ for all $x \in \mathbf{B}(m,3) \setminus [\mathbf{B}(m,3), \mathbf{B}(m,3)]$. Denote $\gamma = \beta_m \circ ... \circ \beta_1 \circ \alpha$. Then

$$\gamma(x_1x_2x_3) = x_1^{-1}x_2^{-1}x_3^{-1} = (x_3x_2x_1)^{-1}. \tag{6}$$

On the other hand, we have

$$\beta_m(...(\beta_1(\alpha(x)))...) \sim x^{-1}$$

which is provided by the condition $\alpha(x) \sim x^{-1}$.

Consequently,

$$\gamma(x_1x_2x_3) \sim (x_1x_2x_3)^{-1} \tag{7}$$

It follows from (6) and (7) that $\langle\langle (x_3x_2x_1)^{-1} \rangle\rangle = \langle\langle (x_1x_2x_3)^{-1} \rangle\rangle$. So,

$$\langle\langle x_1x_2x_3 \rangle\rangle = \langle\langle x_3x_2x_1 \rangle\rangle. \tag{8}$$

Now consider the free generators $x_1x_2x_3, x_2, x_3$ of $\mathbf{B}(m,3)$. The normal automorphism γ induces an automorphism of the quotient group

$$\mathbf{B}(m,3)/\langle\langle x_1x_2x_3 \rangle\rangle \simeq \mathbf{B}(m-1,3).$$

The equalities (8) and $x_1x_2x_3 = 1$ in $\mathbf{B}(m,3)/\langle\langle x_1x_2x_3 \rangle\rangle$ imply $x_3x_2x_1 = 1$. Therefore, we get equality $x_2x_3 = x_3x_2$ in the free group $\mathbf{B}(m-1,3)$ with the free generators $\{x_2, x_3, ..., x_m\}$. This is contradiction, because for $m \geq 3$ the group $\mathbf{B}(m-1,3)$ is not Abelian. □

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V. S. Atabekyan
Yerevan State University
1 Alex Manoogian 0025 Yerevan, Armenia
avarujan@ysu.am

H. T. Aslanyan
Russian-Armenian University,
123 Hovsep Emin str. 0051 Yerevan, Armenia
haikaslanyan@gmail.com

A. E. Grigoryan
Russian-Armenian University,
123 Hovsep Emin str. 0051 Yerevan, Armenia
artgrigrau@gmail.com

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