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# Perfect 3-colorings of cubic graphs of order 8

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**Abstract.** Perfect coloring is a generalization of the notion of completely regular codes, given by Delsarte. A perfect $m$-coloring of a graph $G$ with $m$ colors is a partition of the vertex set of $G$ into m parts $A_1, \ldots, A_m$ such that, for all $i, j \in \{1, \cdots, m\}$, every vertex of $A_i$ is adjacent to the same number of vertices, namely, $a_{ij}$ vertices, of $A_j$ . The matrix $A = (a_{ij})_{i,j\in\{1,\cdots,m\}}$ is called the parameter matrix. We study the perfect 3-colorings (also known as the equitable partitions into three parts) of the cubic graphs of order 8. In particular, we classify all the realizable parameter matrices of perfect 3-colorings for the cubic graphs of order 8.

*Key Words:* perfect coloring, parameter matrices, Cubic graph, equitable partition
*Mathematics Subject Classification* 2010: 03E02, 05C15, 68R05

# Introduction

The concept of a perfect $m$-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see [11]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. We are looking for a positive answer to find the conjecture Delsarte for each cubic graphs of order 8. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6, 3)$, $J(7, 3)$, $J(8, 3)$, $J(8, 4)$, and $J(v, 3)$ ($v$ odd) (see [4, 5, 9]).

Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of $n$-dimensional hypercube $Q_n$ for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the $n$-dimensional cube with a given parameter matrix (see [6, 7, 8]).

In this paper all graphs are finite, undirected, simple and connected. Let $G = (V, E)$ be an undirected graph. Two vertices $u, v \in V(G)$ are adjacent if there exists an edge $e = \{u, v\} \in E(G)$ to which they are both incident. The adjacent will be shown $u \leftrightarrow v$.

A cubic graph is a 3-regular graph. In [12], it is shown that the number of connected cubic graphs with 8 vertices is 5. Each graph is described by a drawing as shown in Figure 1.

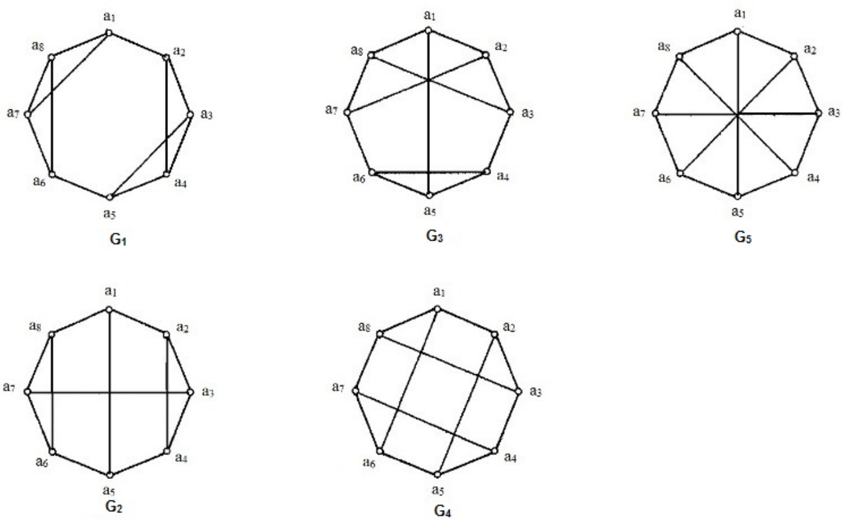


Figure 1: Connected cubic graphs of order 8

**Definition 1** *For a graph $G$ and an integer $m$, a mapping $T : V(G) \to \{1, \cdots, m\}$ is called a perfect $m$-coloring with matrix $A = (a_{ij})_{i,j \in \{1,\cdots,m\}}$ if it is surjective, and for all $i, j$, for every vertex of color $i$, the number of its neighbors of color $j$ is equal to $a_{ij}$. The matrix $A$ is called the parameter matrix of a perfect coloring. In the case $m = 3$, we use three colors: white, black and red. The sets of white, black and red vertices are denoted by W,B and R, respectively. In this paper, we generally show a parameter matrix by*

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

**Remark 1** *In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e. we identify the perfect 3-coloring with the matrices:*

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}, \begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix},$$

*obtained by switching the colors with the original coloring.*

# 1 Preliminaries and Analysis

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of connected graph of order 8 with a given parameter matrix $A$.
The simplest necessary condition for the existence of perfect 3-colorings of a cubic connected graph with the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is:

$$a + b + c = d + e + f = g + h + i = 3.$$

Also, it is clear that we cannot have $b = c = 0$, $d = f = 0$, or $g = h = 0$, since the graph is connected. In addition, $b = 0$, $c = 0$, $f = 0$ if $d = 0$, $g = 0$, $h = 0$, respectively.
The number $\theta$ is called an eigenvalue of a graph G, if $\theta$ is an eigenvalue of the adjacency matrix of this graph. The number $\lambda$ is called an eigenvalue of a perfect coloring T into three colors with the matrix A, if $\lambda$ is an eigenvalue of A. The following theorem demonstrates the connection between the introduced notions.

**Theorem 1 ([1])** *If $T$ is a perfect coloring of a graph $G$ in $m$ colors, then any eigenvalue of $T$ is an eigenvalue of $G$.*

The next theorem can be useful to find the eigenvalues of a parameter matrix.

**Theorem 2** *Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a parameter matrix of a $k$-regular graph. Then the eigenvalues of $A$ are*

$$\lambda_{1,2} = \frac{\operatorname{tr}(A) - k}{2} \pm \sqrt{\left(\frac{\operatorname{tr}(A) - k}{2}\right)^2 - \frac{\det(A)}{k}} \quad , \qquad \lambda_3 = k.$$

***Proof.*** By using the condition $a + b + c = d + e + f = g + h + i = k$, it is clear that one of the eigenvalues is $k$. Therefore $\det(A) = k\lambda_1\lambda_2$. From $\lambda_2 = \operatorname{tr}(A) - \lambda_1 - k$, we get

$$\det(A) = k\lambda_1(\operatorname{tr}(A) - \lambda_1 - k) = -k\lambda_1^2 + k(\operatorname{tr}(A) - k)\lambda_1.$$

By solving the equation $\lambda^2 + (k - \operatorname{tr}(A))\lambda + \dfrac{\det(A)}{k} = 0$, we obtain

$$\lambda_{1,2} = \frac{\operatorname{tr}(A) - k}{2} \pm \sqrt{\left(\frac{\operatorname{tr}(A) - k}{2}\right)^2 - \frac{\det(A)}{k}}.$$

□

The eigenvalues of the all cubic graphs of order 8 are stated in the next theorem.

**Theorem 3 ([12])** *The distinct eigenvalues of the graph $G_1$ are the numbers $3, \sqrt{5}, -1, -\sqrt{5}$. The distinct eigenvalues of the graph $G_2$ are the numbers $\sqrt{3}, 1, 1-\sqrt{2}, -1, -\sqrt{3}, -3+\sqrt{2}$. The distinct eigenvalues of the graph $G_3$ are the numbers $3, 1.5616, 0.618, 0, -1.618, -2.5616$. The distinct eigenvalues of the graph $G_4$ are the numbers $3, 1, -1, 3$.The distinct eigenvalues of the graph $G_5$ are the numbers $3, 1, 1-\sqrt{2}, -1, -2, -3+\sqrt{2}$.*

The next proposition gives a formula for calculating the number of white, black and red vertices, in a perfect 3-coloring.

**Proposition 1 ([3])** *Let $T$ be a perfect 3-coloring of a graph $G$ with the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.*

1. *If $b, c, f \neq 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

2. *If $b = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

3. *If $c = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.$$

4. *If $f = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.$$

In this section, without loss of generality, we may assume $|W| \leq |B| \leq |R|$.

**Lemma 1** *Let $G$ be a cubic connected graph of order* 8*. Then $G$ has no perfect 3-coloring $T$ with the matrix that $|W| = 1$.*

***Proof.*** Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a parameter matrix with $|W| = 1$. Consider the white vertex. It is clear that none of its adjacent vertices are white; i.e. $a = 0$. Therefore, we have two cases below.

(1) The adjacent vertices of the white vertex are the same color.
If they are black, then $b = 3$ and $c = 0$. From $c = 0$, we get $g = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

If the adjacent vertices of the white vertex are red, then $c = 3$, $b = 0$. From $b = 0$, we get $d = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Finally, by using Remark 1 and the fact that $|W| \leq |B| \leq |R|$, it is obvious that there are only six matrices in (1), as shown $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$.

$$A_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$
$$A_5 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

(2) The adjacent vertices of the white vertex are different colors. It immediately gives that $b, c \neq 0$. Also, it can be seen that $d = g = 1$. An easy computation, as in (1), shows that there are only five matrices that can be a parameter matrix in this case, as shown $A_7$, $A_8$, $A_9$, $A_{10}$,

$A_{11}$.

$$A_7 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, A_9 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, A_{10} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

By using the Proposition 1, it can be seen that no matrix can be a parameter.

□

We now peresent two lemmas which can be useful to reach our goal.

**Lemma 2** *Let $G$ be a cubic connected graph of order 8. If $T$ is a perfect 3-coloring with the matrix $A$, and $|W| = |B| = 2$, $|R| = 4$, then $A$ should be one of the following matrices:*

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

***Proof.*** First, suppose that $b, c \neq 0$. As $|W| = 2$, by Proposition 1, it follows that $\frac{b}{d} + \frac{c}{g} = 3$. From $b + c \leq 3$, we have $b = 2$, $c = g = d = 1$, or $c = 2$, $b = g = d = 1$. If $b = 2$, $c = g = d = 1$, we get a contradiction of $|B| = 2$. If $c = 2$, $b = d = g = 1$, then we conclude from $|B| = 2$ and $|R| = 4$ that $h = 1$, $f = 2$. Therefore $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.
Second, suppose that $b = 0$ and, in consequence, $d = 0$. As $|R| = 4$, by Proposition 1, it follows that $\frac{g}{c} + \frac{h}{f} = 1$. Therefore, $c = f = 2$, $g = h = 1$, or $c = f = 3$, $h = 2$, $g = 1$, or $c = f = 3$, $g = 2$, $h = 1$. If $c = f = 2$, $g = h = 1$, then $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$. In the other two cases, we get a contradiction of $|B| = 2$.
Third, suppose that $c = 0$ and, in consequence, $g = 0$. As $|B| = 2$, by Proposition 1, it follows that $\frac{d}{b} + \frac{f}{h} = 3$. Therefore $d = 2$, $b = f = h = 1$, or $f = 2$, $b = h = d = 1$. If $d = 2$, $b = f = h = 1$, then we get a contradiction of $|R| = 4$. If $f = 2$, $b = h = d = 1$, then $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Finally, note that the matrix $\begin{bmatrix}0&1&2\\1&2&0\\1&0&2\end{bmatrix}$ is the same as the matrix $\begin{bmatrix}2&1&0\\1&0&2\\0&1&2\end{bmatrix}$ up to renaming the colors, by Remark 1. □

**Lemma 3** *Let $G$ be a cubic connected graph of order 8. If $T$ is a perfect 3-coloring with the matrix $A$, and $|W| = 2$, $|B| = |R| = 3$, then $A$ should be the following matrix:*

$$\begin{bmatrix}0&3&0\\2&0&1\\0&1&2\end{bmatrix}.$$

***Proof.*** First, suppose that $b, c \neq 0$. As $|W| = 2$, by Proposition 1, it follows that $\frac{b}{d} + \frac{c}{g} = 3$. From $b + c \leq 3$, we get $b = 2$, $c = g = d = 1$, or $c = 2$, $b = g = d = 1$. If $b = 2$, $c = g = d = 1$, we get a contradiction of $|B| = 3$. If $c = 2$, $b = d = g = 1$, then from Proposition 1, we have $f = 2$, $h = 3$, which is a contradiction of $g + h \leq 3$.
Second, suppose that $b = 0$ and, in consequence, $d = 0$. As $|R| = 3$, by Proposition 1, it follows that $\frac{g}{c} + \frac{h}{f} = \frac{5}{3}$. Therefore, $c = 3$, $g = 2$, $h = f = 1$, or $f = 3$, $h = 2$, $c = g = 1$. If $c = 3$, $g = 2$, $h = f = 1$, then $A = \begin{bmatrix}0&0&3\\0&2&1\\2&1&0\end{bmatrix}$. In the other case, we get a contradiction of $|W| = 2$.
Third, suppose that $c = 0$ and, in consequence, $g = 0$. As $|B| = 3$, by Proposition 1, it follows that $\frac{d}{b} + \frac{f}{h} = \frac{5}{3}$. Therefore $h = 3$, $f = 2$, $b = d = 1$, or $b = 3$, $d = 2$, $f = h = 1$. If $h = 3$, $f = 2$, $b = d = 1$, then we get a contradiction of $|W| = 2$. If $b = 3$, $d = 2$, $f = h = 1$, then $A = \begin{bmatrix}0&3&0\\2&0&1\\0&1&2\end{bmatrix}$.
Finally, note that the matrix $\begin{bmatrix}0&0&3\\0&2&1\\2&1&0\end{bmatrix}$ is the same as the matrix $\begin{bmatrix}0&3&0\\2&0&1\\0&1&2\end{bmatrix}$ up to renaming the colors, by Remark 1. □

By using the Lemmas 1, 2 and 3, it can be seen that only the following matrices:

$$A_1 = \begin{bmatrix}0&1&2\\1&0&2\\1&1&1\end{bmatrix}, A_2 = \begin{bmatrix}1&0&2\\0&1&2\\1&1&1\end{bmatrix}, A_3 = \begin{bmatrix}0&3&0\\2&0&1\\0&1&2\end{bmatrix}, A_4 = \begin{bmatrix}2&1&0\\1&0&2\\0&1&2\end{bmatrix},$$

can be parameter ones.

# 2 Perfect 3-colorings of cubic graphs with 8 vertices

In this section we enumerate the parameter matrices of all perfect 3-colorings of cubic graphs with 8 vertices. As it has been shown in section 3, only matrices $A_1$ , $A_2$, $A_3$ and $A_4$ can be parameter matrices. With consideration of cubic graphs eigenvalues and using Theorems 1, 2 and 3, it can be seen that the connected cubic graphs with 8 vertices can have a perfect 3-coloring with the matrices $A_1$, $A_2$ and $A_3$ which is represented by table 1.

| graphs | matrix $A_1$ | matrix $A_2$ | matrix$A_3$ |
|---|---|---|---|
| $G_1$ | $\surd$ | $\surd$ | $\times$ |
| $G_2$ | $\surd$ | $\surd$ | $\times$ |
| $G_3$ | $\times$ | $\times$ | $\surd$ |
| $G_4$ | $\surd$ | $\surd$ | $\times$ |
| $G_5$ | $\surd$ | $\surd$ | $\times$ |

Table 1

**Theorem 4** *There are no perfect 3-colorings with the matrix $A_1$ for the graph $G_5$.*

***Proof.*** Countrary to our claim, suppose that $T$ is a perfect 3-coloring with the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ for the graph $G_5$. Without restriction of generality, suppose that $T(a_1) = 1$. Therefore, again without restriction of generality, suppose that $T(a_2) = T(a_8) = 3$ and $T(a_5) = 2$. From $T(a_5) = 2$, we can easily seen that $T(a_4) = T(a_6) = 3$. Therefore $T(a_4) = 2$, which is a contradiction with third row of the matrix $A_1$. $\square$

**Theorem 5** *The parameter matrices of cubic graphs of order* 8 *are listed in the following table.*

| *graphs* | *matrix $A_1$* | *matrix $A_2$* | *matrix$A_3$* |
|---|---|---|---|
| $G_1$ | $\surd$ | $\surd$ | $\times$ |
| $G_2$ | $\surd$ | $\surd$ | $\times$ |
| $G_3$ | $\times$ | $\times$ | $\surd$ |
| $G_4$ | $\surd$ | $\surd$ | $\times$ |
| $G_5$ | $\times$ | $\surd$ | $\times$ |

*Table* 2

***Proof.*** As it has been shown in the table 1, only the matrices $A_1$, $A_2$ and $A_3$ can be parameter matrices. Hence, from Theorem 4, it suffices to show that there are perfect 3-colorings with the matrices in the table 2. The graph $G_1$ has perfect 3-colorings with the matrices $A_1$ and $A_2$. Consider two mappings $T_1$ and $T_2$ as follows:

$$T_1(a_2) = T_1(a_6) = 1, T_1(a_4) = T_1(a_8) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_5) = T_1(a_7) = 3.$$
$$T_2(a_1) = T_2(a_2) = 1, T_2(a_5) = T_2(a_6) = 2,$$
$$T_2(a_3) = T_2(a_4) = T_2(a_7) = T_2(a_8) = 3.$$

It is clear that $T_1$ and $T_2$ are perfect 3-colorings with the matrices $A_1$ and $A_2$, respectively.

The graph $G_2$ has perfect 3-colorings with the matrices $A_1$ and $A_2$. Consider two mappings $T_1$ and $T_2$ as follows:

$$T_1(a_1) = T_1(a_3) = 1, T_1(a_5) = T_1(a_7) = 2,$$
$$T_1(a_2) = T_1(a_4) = T_1(a_6) = T_1(a_8) = 3,$$
$$T_2(a_1) = T_2(a_5) = 1, T_2(a_3) = T_2(a_7) = 2,$$
$$T_2(a_2) = T_2(a_4) = T_2(a_6) = T_2(a_8) = 3.$$

It is clear that $T_1$ and $T_2$ are perfect 3-coloring with the matrices $A_1$ and $A_2$, respectively.
The graph $G_3$ has perfect 3-colorings with the matrix $A_3$. Consider a mapping $T_1$ as follows:

$$T_1(a_2) = T_1(a_8) = 1, T_1(a_1) = T_1(a_3) = T_1(a_7) = 2,$$
$$T_1(a_4) = T_1(a_5) = T_1(a_6) = 3.$$

It is clear that $T_1$ is a perfect 3-colorings with the matrices $A_3$.
The graph $G_4$ has perfect 3-colorings with the matrices $A_1$ and $A_2$. Consider two mappings $T_1$ and $T_2$ as follows:

$$T_1(a_2) = T_1(a_7) = 1, T_1(a_5) = T_1(a_8) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_4) = T_1(a_6) = 3.$$
$$T_2(a_2) = T_2(a_5) = 1, T_2(a_7) = T_2(a_8) = 2,$$
$$T_2(a_1) = T_2(a_3) = T_2(a_4) = T_2(a_6) = 3.$$

It is clear that $T_1$ and $T_2$ are perfect 3-colorings with the matrices $A_1$ and $A_2$, respectively.
The graph $G_5$ has perfect 3-colorings with the matrix $A_3$. Consider a mapping $T_1$ as follows:

$$T_1(a_4) = T_1(a_8) = 1, T_1(a_2) = T_1(a_6) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_5) = T_1(a_7) = 3.$$

It is clear that $T_1$ is a perfect 3-colorings with the matrices $A_3$.
□

# References

[1] M. Alaeiyan and A. Abedi, *Perfect 2-colorings of Johnson graphs J(4, 3), J(4, 3), J(6, 3) and Petersen graph*, Ars Combinatoria,(to appear)

[2] M. Alaeiyan, H. Karami, *Perfect 2-colorings of the generalized Petersen graph,* Proceedings Mathematical Sciences, **126** (2016), pp. 1-6

[3] M. Alaeiyan and A. Mehrabani, *Perfect 3-colorings of cubic graphs of order* 10, Electronic Journal of Graph Theory and Applications (EJGTA), **5(2)** (2017), pp. 194-206.

[4] S. V. Avgustinovich, I. Yu. Mogilnykh, *Perfect 2-colorings of Johnson graphs J(6, 3) and J(7, 3),* Lecture Notes in Computer Science, **5228** (2008), pp. 11-19.

[5] S. V. Avgustinovich, I. Yu. Mogilnykh, *Perfect colorings of the Johnson graphs J(8, 3) and J(8, 4) with two colors*, Journal of Applied and Industrial Mathematics, **5** (2011), pp. 19-30.

[6] D. G. Fon-Der-Flaass, *A bound on correlation immunity,* Siberian Electronic Mathematical Reports Journal, **4** (2007), pp. 133-135.

[7] D. G. Fon-Der-Flaass, *Perfect 2-colorings of a hypercube,* Siberian Mathematical Journal, **4** (2007), pp. 923-930.

[8] D. G. Fon-der-Flaass, *Perfect 2-colorings of a 12-dimensional Cube that achieve a bound of correlation immunity,* Siberian Mathematical Journal, **4** (2007), pp. 292-295.

[9] A. L. Gavrilyuk and S.V. Goryainov, *On perfect 2-colorings of Johnson graphs J(v,3),* Journal of Combinatorial Designs, **21** (2013), pp. 232-252.

[10] Godsil, Chris, and Gordon Royle, Algebraic graph theory, volume 207 of Graduate Texts in Mathematics, (2001).

[11] C. Godsil, *Compact graphs and equitable partitions,*, Linear Algebra and Its Application, **255** (1997), pp. 259-266.

[12] F. C. Bussemaker, S. Cobeljic, D.M. Cvetkovic, J.J. Seidel, Computer investigation of cubic graphs, Technische Hogeschool Eindhoven Nederland Onderafedeling Der Wiskunde, January 1976.

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