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Geometric Properties of Operators Associated with Normalized Jackson and Hahn–Exton $q-$Bessel Functions and $q-$Extension of the Hohlov Integral Operator

M. K. Giri and K. Raghavendar

Abstract. In this paper, we examine the geometric properties of linear operators associated with normalized Jackson and Hahn–Exton $q-$Bessel functions that arise through suitable transformations and $q-$extension of the Hohlov integral operator. These operators are investigated in the framework of the function from the class $\mathcal{M}_{\xi,\varkappa}^{\varpi;\theta}(z;q)$. We derive coefficient bounds and sufficient conditions for functions in the class $\mathcal{M}_{\xi,\varkappa}^{\varpi;\theta}(z;q)$. Additionally, we explore inclusion properties by using the Taylor coefficients of $z_2\phi_1(a,b;c;q,z)$ and the normalized Jackson and Hahn–Exton $q-$Bessel functions. The primary objective is to derive sufficient conditions under which the convolution operators will be in different subclasses of $q-$starlike and $q-$convex functions.

Key Words: $q-$Convex Functions, $q-$Starlike Functions, $q-$Hypergeometric Function, $q-$Bessel Function
Mathematics Subject Classification 2020: 30C45, 33C10, 33D15

Introduction

The study of univalent analytic functions defined in the open unit disk $\mathbb{D}$ is fundamental to geometric function theory. The class of these functions is denoted by $\mathcal{S}$. The most studied subclasses of univalent functions are starlike and convex functions. These functions map $\mathbb{D}$ onto domains that are starlike and convex, respectively.

Let $\Psi(z)$ be a function of the form

$$\Psi(z) = z + \sum_{j=2}^{\infty} t_j z^j, \tag{1}$$

which is analytic in $\mathbb{D}$. The class of all such functions is denoted by $\mathcal{A}$. A function $\Psi(z) \in \mathcal{A}$ is said to be starlike of order σ (with $0 \leq \sigma < 1$) if it satisfies the condition

$$\Re\left\{\frac{z\Psi'(z)}{\Psi(z)}\right\} > \sigma.$$

Similarly, $\Psi(z)$ is termed convex of order σ if

$$\Re\left\{1 + \frac{z\Psi''(z)}{\Psi'(z)}\right\} > \sigma.$$

When $\sigma = 0$, these classes reduce to the standard starlike and convex function classes, denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. For more information about starlike and convex functions, we refer to Duren's book [16] and the references therein.

Recently, the study of $q-$calculus has attracted significant attention due to its wide applications in both mathematics and physics. The fundamental work in this area was carried out by Jackson [22, 23], who introduced and developed the concepts of the $q-$derivative and $q-$integral. Subsequent research, particularly in the study of quantum groups, has revealed geometric aspects of $q-$analysis. The development of $q-$calculus has made it possible to explore the $q-$analog of classical and special functions (see, for example, [4,6,15]). In [20], Ismail et al. established a relationship between the theory of $q-$calculus and starlike functions.

Many researchers have focused on $q-$analogues of starlike and convex functions because they have powerful capabilities for generalizing classical mathematical concepts. When $q \to 1$, these analogues typically reduce to their classical form. In this study, we explore the inclusion properties and coefficient estimates of subclasses of $q-$starlike and $q-$convex functions by utilizing the $q-$derivative operator and their connections with other $q-$analogs of analytic function classes.

We begin by introducing some fundamental concepts related to $q-$derivative, which will be essential for our analysis. For a real number q satisfying $0 < q < 1$, the $q-$derivative of a function $\Psi(z)$ is given by

$$D_q\Psi(z) = \begin{cases} \dfrac{\Psi(z) - \Psi(qz)}{(1-q)z} & \text{if} \quad z \neq 0, \\ \Psi'(0) & \text{if} \quad z = 0. \end{cases}$$

Note that if q approaches to 1, the $q-$derivative converges to the classical derivative, that is, $\lim_{q\to 1^-} D_q\Psi(z) = \Psi'(z)$.

Applying the $q-$derivative to the power series given in (1), we obtain

$$D_q\Psi(z) = 1 + \sum_{j=2}^{\infty}[j,q]t_j z^{j-1},$$

where

$$[j,q] = \frac{1-q^j}{1-q} = 1 + q + q^2 + \cdots + q^{j-1}.$$

The following notions were introduced in [31].

Definition 1 *Let $\Psi(z) \in \mathcal{A}$. The function is said to be $q-$starlike of order σ if*

$$\Re\left(\frac{zD_q\Psi}{\Psi}\right) > \sigma,$$

and said to be q-convex of order σ if

$$\Re\left(1 + \frac{zD_q^2\Psi}{D_q\Psi}\right) > \sigma.$$

The classes of $q-$starlike functions of order σ and $q-$convex functions of order σ are denoted by $\mathcal{S}_q^*(\sigma)$ and $\mathcal{C}_q(\sigma)$ respectively.

When $\sigma = 0$, these classes correspond to the classical $q-$starlike and $q-$convex functions, which are denoted by $\mathcal{S}_q^*$ and $\mathcal{C}_q$, respectively.

It is easy to see that the following lemma holds true.

Lemma 1 *A function $\Psi \in \mathcal{A}$ is in $\mathcal{S}_q^*(\sigma)$ if the following inequality holds:*

$$\sum_{j=2}^{\infty}[j,q]|t_j| \leq 1 + \sigma. \tag{2}$$

Similarly, Ψ is in $\mathcal{C}_q(\sigma)$ if

$$\sum_{j=2}^{\infty}[j,q][j-1,q]|t_j| \leq 1 + \sigma. \tag{3}$$

Definition 2 *For a fixed parameter $\lambda > 0$, the classes $\mathcal{S}_{q,\lambda}^*$ and $\mathcal{C}_{q,\lambda}$ are defined as*

$$\mathcal{S}_{q,\lambda}^* = \left\{\Psi \in \mathcal{A} : \left|\frac{zD_q\Psi}{\Psi} - 1\right| < \lambda\right\},$$

$$\mathcal{C}_{q,\lambda} = \left\{\Psi \in \mathcal{A} : \left|\frac{zD_q^2\Psi}{D_q\Psi}\right| < \lambda\right\}.$$

The following result is straightforward.

Lemma 2 *A function $\Psi \in \mathcal{A}$ is in $\mathcal{S}^*_{q,\lambda}$ if*

$$\sum_{j=2}^{\infty}([j,q]+\lambda-1)|t_j| \leq \lambda. \tag{4}$$

Similarly, a sufficient condition for $\Psi(z)$ to be in $\mathcal{C}_{q,\lambda}$ is given by

$$\sum_{j=2}^{\infty}[j,q]([j-1,q]+\lambda)|t_j| \leq \lambda. \tag{5}$$

Srivastava et al. [32] introduced two function classes $\mathcal{S}^*_q[V_1,V_2]$ and $\mathcal{C}_q[V_1,V_2]$.

Definition 3 *The function $\Psi \in \mathcal{A}$ is said to be in $\mathcal{S}^*_q[V_1,V_2]$ if*

$$\left| \frac{(V_2-1)\dfrac{zD_q\Psi}{\Psi}-(V_1-1)}{(V_2+1)\dfrac{zD_q\Psi}{\Psi}-(V_1+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Using the idea of Alexander's theorem, the class $\mathcal{C}_q[V_1,V_2]$ of q-convex functions is defined as follows.

Definition 4 *The function $\Psi \in \mathcal{A}$ is said to be in $\mathcal{C}_q[V_1,V_2]$ if $zD_q\Psi \in \mathcal{S}^*_q[V_1,V_2]$.*

We can see that $\lim_{q\to 1^-} \mathcal{S}^*_q[V_1,V_2] \equiv \mathcal{S}^*[V_1,V_2]$. The class $\mathcal{S}^*[V_1,V_2]$ was introduced and studied by Janowski (see [24]). The class $\mathcal{S}^*_q[V_1,V_2]$ generalizes other subclasses since for particular values of V_1 and V_2, one can obtain other classes. For example, if $V_1 = 1-2\alpha$ and $V_2 = -1$, $\mathcal{S}^*_q[1-2\alpha,-1] \equiv \mathcal{S}^*_q(\alpha)$ (see [2]); and if $V_1 = 1$ and $V_2 = -1$, then $\mathcal{S}^*_q[1,-1] \equiv \mathcal{S}^*_q$ (see [20]).

Lemma 3 *[32] The function $\Psi \in \mathcal{A}$ is in $\mathcal{S}^*_q[V_1,V_2]$ if the condition below holds:*

$$\sum_{j=2}^{\infty}(2q[j-1,q]+|(V_2+1)[j,q]-(V_1+1)|)|t_j| < |V_2-V_1|. \tag{6}$$

Similarly, Ψ is a member of the class $\mathcal{C}_q[V_1,V_2]$ provided that

$$\sum_{j=2}^{\infty}[j,q](2q[j-1,q]+|(V_2+1)[j,q]-(V_1+1)|)|t_j| < |V_2-V_1|. \tag{7}$$

Definition 5 *A function $\Psi \in \mathcal{A}$ belongs to class $k-\mathcal{UCV}_q[V_1, V_2]$ if it satisfies*

$$\Re\left(\frac{(V_2-1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1-1)}{(V_2+1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1+1)}\right)$$
$$> k\left|\frac{(V_2-1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1-1)}{(V_2+1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1+1)}-1\right|.$$

Srivastava et al. [33] introduced the $k-$uniformly Janowski $q-$starlike functions, represented by the class $k-\mathcal{ST}_q[V_1, V_2]$, which is defined below.

Definition 6 *A function $\Psi \in \mathcal{A}$ belongs to class $k-\mathcal{ST}_q[V_1, V_2]$ if*

$$\Re\left(\frac{(V_2-1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1-1)}{(V_2+1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1+1)}\right)$$
$$> k\left|\frac{(V_2-1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1-1)}{(V_2+1)\dfrac{D_q(zD_q\Psi)}{D_q\Psi}-(V_1+1)}-1\right|.$$

The following lemma provide sufficient conditions for functions to belong to classes defined above.

Lemma 4 *[33] The function $\Psi \in \mathcal{A}$ is in $k-\mathcal{UCV}_q[V_1, V_2]$ if the following inequality holds:*

$$\sum_{j=2}^{\infty}[j,q](2q(k+1)[j-1,q]+|(V_2+1)[j,q]+(V_1+1)|)|t_j| < |V_2-V_1|. \quad (8)$$

Similarly, Ψ lies in the class $k-\mathcal{ST}_q[V_1, V_2]$ if

$$\sum_{j=2}^{\infty}(2q(k+1)[j-1,q]+|(V_2+1)[j,q]+(V_1+1)|)|t_j| < |V_2-V_1|. \quad (9)$$

Various subclasses of normalized analytic functions were defined, and their geometric properties were studied using differential operators and parameterized conditions (see, for example [27,29]). Motivated by these developments, we propose a new subclass of analytic functions that incorporates first, second, and third-order derivatives.

Definition 7 *A function $\Psi(z) \in \mathcal{A}$ belongs to class $\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ if*

$$\left|\frac{D_q\Psi + \varkappa z D_q^2\Psi + \xi z^2 D_q^3\Psi - 1}{2\varpi(1-\theta) + D_q\Psi + \varkappa z D_q^2\Psi + \xi z^2 D_q^3\Psi - 1}\right| < 1, \tag{10}$$

where $\varpi \in \mathbb{C} \setminus 0$, $\theta < 1$, $0 \leq \xi < 1$ and $0 \leq \varkappa < 1$.

As $q \to 1$, the class $\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ tends toward class $\mathcal{M}_{\xi,\varkappa}^{\varpi}(\theta)$ introduced in [19]. Its inclusion and geometric properties, particularly under various integral operators, have been studied in [18, 28].

Heine (see [17,30]) defined the basic (or $q-$) hypergeometric series, commonly denoted by $\phi(a,b;c;q,z)$, as

$$\phi(a,b;c;q,z) = \sum_{j=0}^{\infty} \frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j} z^j,$$

where $q-$shifted factorial $(a;q)_j$ is given by

$$(a;q)_0 = 1, \quad (a;q)_j = \prod_{k=1}^{j}(1-aq^{k-1}), \quad (a;q)_\infty = \prod_{k\geq 1}(1-aq^{k-1}).$$

with the assumption that $c \neq q^{-t}$ for $t = 0, 1, 2, \ldots$ If $q \to 1$, then $\phi(a,b;c;q,z)$ approaches the classical Gauss hypergeometric series.

By multiplying $\phi(a,b;c;q,z)$ with z, we obtain

$$z\phi(a,b;c;q,z) = \sum_{j=1}^{\infty} A_j z^j,$$

where the coefficients A_j are given by

$$A_j = \frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}, \ j = 1, 2, 3, \ldots \tag{11}$$

Consider the following second-order differential equation, as discussed in the classical work of Watson [34]:

$$z^2\mathcal{V}''(z) + z\mathcal{V}'(z) + (z^2 - p^2)\mathcal{V}(z) = 0,$$

where the parameter $p \geq -1$. A well-known solution to this equation is the Bessel function of the first kind, denoted by $\mathcal{J}_p(z)$, which admits the power series representation

$$J_p(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(p+j+1)} \left(\frac{z}{2}\right)^{2j+p}, \ z \in \mathbb{C}.$$

Let us examine the Jackson and Hahn-Exton types of $q-$Bessel functions, which are given by the following explicit series representations:

$$J_p^{(2)}(z;q) = \frac{(q^{p+1};q)_\infty}{(q;q)_\infty} \sum_{j\geq 0} \frac{(-1)^j \left(\frac{z}{2}\right)^{2j+p}}{(q;q)_j (q^{p+1};q)_j} q^{j(j+p)}$$

and

$$J_p^{(3)}(z;q) = \frac{(q^{p+1};q)_\infty}{(q;q)_\infty} \sum_{j\geq 0} \frac{(-1)^j z^{2j+p}}{(q;q)_j (q^{p+1};q)_j} q^{\frac{1}{2}j(j+1)},$$

where $z \in \mathbb{C}$, $p > -1$, $q \in (0,1)$. These functions are recognized as $q-$analogues of the classical Bessel function of the first kind. In particular, for a fixed complex number z, one obtains the classical Bessel function in the limit as $q \nearrow 1$:

$$J_p^{(2)}((1-z);q) \to J_p(z) \quad \text{and} \quad J_p^{(3)}((1-z);q) \to J_p(2z).$$

A detailed study of the Bessel function of the first kind is described in the classic treatise by Watson [34]. Various properties of the Jackson and Hahn-Exton $q-$Bessel functions, which serve as $q-$extensions of the classical Bessel functions, are extensively studied in the literature (see, for example, [1, 5, 13, 14, 21, 25, 26] and the references therein). Investigations into the geometric properties of Bessel functions, such as their univalence, starlikeness, convexity, and close-to-convexity, are addressed in works including [7, 8, 10, 11].

In [3, 9], three analytic normalizations of the Jackson and Hahn-Exton $q-$Bessel functions were introduced, each defined on the open unit disk in the complex plane. Since the functions $J_p^{(2)}(z;q)$ and $J_p^{(3)}(z;q)$ are not in the class $\mathcal{A}$, suitable normalizations are considered to place them in this function class. For $p > -1$, the normalized versions are given by

$$\mathcal{L}_p^{(2)}(z;q) = 2^p S_p(q) z^{1-\frac{p}{2}} J_p^{(2)}(\sqrt{z};q),$$
$$\mathcal{L}_p^{(3)}(z;q) = S_p(q) z^{1-\frac{p}{2}} J_p^{(3)}(\sqrt{z};q),$$

where

$$S_p(q) = \frac{(q;q)_\infty}{(q^{p+1};q)_\infty}.$$

These transformations ensure that the resulting functions belong to the analytic function class $\mathcal{A}$.

Although infinitely many possible normalizations can be constructed for both Jackson and Hahn-Exton $q-$Bessel functions, the above choices are particularly meaningful because their classical limits have been previously studied in the literature, for example, in [12].

The series representation of $\mathcal{L}_p^{(2)}(z;q)$ is

$$\begin{aligned}\mathcal{L}_p^{(2)}(z;q) &= 2^p S_p(q) z^{1-\frac{p}{2}} J_p^{(2)}(\sqrt{z};q)\\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}} z^j = \sum_{j=1}^{\infty} L_j^{(2)} z^j,\end{aligned}$$

with the coefficients $L_j^{(2)}(z;q)$ given by

$$L_j^{(2)} = \frac{(-1)^{j-1} q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}}. \tag{12}$$

Similarly, the series expansion of $\mathcal{L}_p^{(3)}(z;q)$ is

$$\begin{aligned}\mathcal{L}_p^{(3)}(z;q) &= S_p(q) z^{1-\frac{p}{2}} J_p^{(3)}(\sqrt{z};q)\\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} z^j = \sum_{j=1}^{\infty} L_j^{(3)} z^j,\end{aligned}$$

where coefficients $L_j^{(3)}$ are

$$L_j^{(3)} = \frac{(-1)^{j-1} q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}. \tag{13}$$

Applying $q-$difference operator D_q to the functions $\mathcal{L}_p^{(2)}(z;q)$ and $\mathcal{L}_p^{(3)}(z;q)$, we obtain

$$\begin{aligned}D_q\mathcal{L}_p^{(2)}(z;q) &= \frac{\mathcal{L}_p^{(2)}(z;q) - \mathcal{L}_p^{(2)}(qz;q)}{(1-q)z}\\ &= 1 + \sum_{j=2}^{\infty} [j,q] \frac{(-1)^{j-1} q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}} z^{j-1}\end{aligned} \tag{14}$$

and

$$\begin{aligned}D_q\mathcal{L}_p^{(3)}(z;q) &= \frac{\mathcal{L}_p^{(3)}(z;q) - \mathcal{L}_p^{(3)}(qz;q)}{(1-q)z}\\ &= 1 + \sum_{j=2}^{\infty} [j,q] \frac{(-1)^{j-1} q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} z^{j-1}.\end{aligned} \tag{15}$$

To ensure that the considered functions are analytic in the open unit disk $\mathbb{D}$, appropriate normalization techniques are employed for Jackson and Hahn–Exton $q-$Bessel functions.

1 Main results

In this section, we study geometric properties of the introduced normalized $q-$Bessel functions.

Theorem 1 *Let $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$. Then coefficients t_j in its expansion satisfies the bound*

$$|t_j| \leq \frac{2(1-\theta)|\varpi|}{[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)}, \quad j=2,3,\ldots \tag{16}$$

The estimate is sharp and is achieved for the function

$$\Psi(z) = z + \frac{2(1-\theta)\varpi q^{k(j-1)}}{B} z^{k(j-1)+1}, \quad k=1,2,3,\ldots$$

where $B = [k(j-1)+1,q]\Big(1+\varkappa[k(j-1),q]+\xi[j-1,q][k(j-1)-1,q]\Big)$.

Proof. Let $\Upsilon(z;q)$ be an analytic function in the open unit disk $\mathbb{D}$ satisfying $\Upsilon(0;q)=0$ and $|\Upsilon(z;q)|<1$. Consider the following identity:

$$1+\frac{1}{\varpi}\left(D_q\Psi+\varkappa z D_q^2\Psi+\xi z^2 D_q^3\Psi-1\right) = \frac{1+(1-2\theta)\Upsilon(z;q)}{1-\Upsilon(z;q)}.$$

Expanding and simplifying, we get

$$\begin{aligned}&\frac{1}{\varpi}\left(D_q\Psi+\varkappa z D_q^2\Psi+\xi z^2 D_q^3\Psi-1\right)\\ &\quad= \left[\frac{1}{\varpi}\left(D_q\Psi+\varkappa z D_q^2\Psi+\xi z^2 D_q^3\Psi-1\right)+2(1-\theta)\right]\Upsilon(z;q).\end{aligned}$$

Now, let us express the expansion as a power series. We have

$$\left[2(1-\theta)+\frac{1}{\varpi}\sum_{j=2}^{\infty}\Delta_j t_j z^{j-1}\right]\Upsilon(z;q) = \frac{1}{\varpi}\sum_{j=2}^{\infty}\Delta_j t_j z^{j-1},$$

where $\Delta_j = [j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)$. Assume the power series for $\Upsilon(z;q)$ is given by $\Upsilon(z;q)=\sum_{j=1}^{\infty} u_j q^j z^j$. Then the equation becomes

$$\left[2(1-\theta)+\frac{1}{\varpi}\sum_{j=2}^{\infty}\Delta_j t_j z^{j-1}\right]\sum_{j=1}^{\infty}u_j q^j z^j = \frac{1}{\varpi}\sum_{j=2}^{\infty}\Delta_j t_j z^{j-1}.$$

Equating the coefficients shows that each t_j depends on $t_2,\ldots,t_{j-1}$. Let us extract the term of degree z^{k-1} on both sides:

$$\left[2(1-\theta)+\frac{1}{\varpi}\sum_{j=2}^{k-1}\Delta_j t_j z^{j-1}\right]\Upsilon(z;q) = \frac{1}{\varpi}\sum_{j=2}^{k}\Delta_j t_j z^{j-1}+\sum_{j=k+1}^{\infty} d_j z^j.$$

Using $|\Upsilon(z;q)| < 1$, we apply the maximum modulus principle to estimate

$$\left|2(1-\theta) + \frac{1}{\varpi}\sum_{j=2}^{k-1} \Delta_j t_j z^{j-1}\right| > \left|\frac{1}{\varpi}\sum_{j=2}^{k} \Delta_j t_j z^{j-1} + \sum_{j=k+1}^{\infty} d_j z^j\right|.$$

Squaring both sides and integrating over $|z| = t$, $0 < t < 1$, then letting $t \to 1$, we obtain

$$4(1-\theta)^2 \geq \frac{1}{|\varpi|^2}{\Delta_j}^2 |t_j|^2.$$

Solving this inequality for $|t_j|$, we obtain required bound.

To determine sharpness, consider the extremal function defined by

$$D_q\Psi + \varkappa z D_q^2\Psi + \xi z^2 D_q^3\Psi = 1 + \frac{2(1-\theta)\varpi q^{j-1} z^{j-1}}{1 - q^{j-1} z^{j-1}} \tag{17}$$

The left side of (17) expands as

$$D_q\Psi + \varkappa z D_q^2\Psi + \xi z^2 D_q^3\Psi = 1 + \sum_{j=2}^{\infty} \Delta_j t_j z^{j-1}.$$

The right side of (17) is a geometric series

$$1 + \frac{2(1-\theta)\varpi q^{j-1} z^{j-1}}{1 - q^{j-1} z^{j-1}} = 1 + 2(1-\theta)\varpi \sum_{k=1}^{\infty} q^{k(j-1)} z^{k(j-1)} = 1 + \sum_{n=1}^{\infty} B_n q^n z^n,$$

where

$$B_n = \begin{cases} 2(1-\theta)\varpi, & \text{if } n = k(j-1), \\ 0, & \text{otherwise.} \end{cases}$$

Shifting the index, this becomes $1 + \sum_{n=2}^{\infty} B_{n-1} q^{n-1} z^{n-1}$, which matches the left side of expansion. Equating the coefficients on both sides, we get

$$\Delta_j t_j = \begin{cases} 2(1-\theta)\varpi q^{n-1} & \text{if} \quad n-1 = k(j-1), \\ 0, & \text{otherwise.} \end{cases}$$

For $n = k(j-1)+1$,

$$t_j = \frac{2(1-\theta)\varpi q^{k(j-1)}}{B},$$

where $B = [k(j-1)+1, q]\Big(1 + \varkappa[k(j-1), q] + \xi[j-1, q][k(j-1)-1, q]\Big)$. For all other n, $t_j = 0$. Thus, we obtain

$$\Psi(z) = z + \frac{2(1-\theta)\varpi q^{k(j-1)}}{B} z^{k(j-1)+1}, \quad \text{for } k = 1, 2, 3, \ldots$$

□

Theorem 2 *A sufficient condition for a function Ψ to be a member of the class $\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ is given by*

$$\sum_{j=2}^{\infty}[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)|t_j| \leq (1-\theta)|\varpi|. \tag{18}$$

Proof. Using (1) and (18) it is easy to see that

$$\begin{aligned}
&\Re e^{i\phi}\left(D_q\Psi+\varkappa z D_q^2\Psi+\xi z^2 D_q^3\Psi-\theta\right)\\
&=(1-\theta)\cos\phi+\Re e^{i\phi}\sum_{j=2}^{\infty}[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)t_j z^{j-1}\\
&\geq(1-\theta)\cos\phi-\sum_{j=2}^{\infty}\left|[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)\right||t_j|\geq 0.
\end{aligned}$$

The resultant obtained above is equivalent to the analytic characterization of $\Psi(z)\in\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$, and the proof is complete. □

Theorem 3 *Let $c\neq 1$, $q\in\mathbb{C}\setminus\{0,1\}$, and suppose the parameters a, b, c, ξ, $\varkappa$, q, ϖ, θ satisfy the following condition:*

$$\begin{aligned}
&\frac{\xi q^3(a;q)_3(b;q)_3}{(1-q)^3(c;q)_3}\phi(aq^3,bq^3;cq^3;q,1)\\
&\quad+(\varkappa q^2+\xi[3,q])\frac{(a;q)_2(b;q)_2}{(1-q)^2(c;q)_2}\phi(aq^2,bq^2;cq^2;q,1)\\
&\quad+\frac{(q+\varkappa[2,q])(1-a)(1-b)}{(1-q)(1-c)}\phi(aq,bq;cq;q,1)+\phi(a,b;c;q,1)\\
&\leq 1+(1-\theta)|\varpi|.
\end{aligned}$$

Then $z\phi(a,b;c;q,z)\in\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$.

Proof. By substituting (11) into the sufficient condition for a function to be in the class $\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$, we obtain

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\\
&\quad\leq(1-\theta)|\varpi|.
\end{aligned}\tag{19}$$

Simplifying and rearranging the terms in the left-hand side of (19), we get

$$\begin{aligned}
&\sum_{j=1}^{\infty}\frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j}+\frac{q+\varkappa[2,q]}{(1-q)}\frac{(1-a)(1-b)}{(1-c)}\sum_{j=0}^{\infty}\frac{(aq;q)_j(bq;q)_j}{(cq;q)_j(q;q)_j}\\
&\quad+\frac{\varkappa q^2+\xi[3,q]}{(1-q)^2}\frac{(a;q)_2(b;q)_2}{(c;q)_2}\sum_{j=0}^{\infty}\frac{(aq^2;q)_j(bq^2;q)_j}{(cq^2;q)_j(q;q)_j}\\
&\quad+\frac{\xi q^3}{(1-q)^3}\frac{(a;q)_3(b;q)_3}{(c;q)_3}\sum_{j=0}^{\infty}\frac{(aq^3;q)_j(bq^3;q)_j}{(cq^3;q)_j(q;q)_j}\\
&\leq(1-\theta)|\varpi|. \qquad (20)
\end{aligned}$$

Equation (20) yields the required result. □

Theorem 4 *Let $\Psi\in\mathcal{A}$, $q\in(0,1)$, and suppose the parameters p, ξ, $\varkappa$, q, ϖ, θ satisfy the following condition:*

$$D_q\mathcal{L}_p^{(2)}(1;q)+\varkappa D_q^2\mathcal{L}_p^{(2)}(1;q)+\xi D_q^3\mathcal{L}_p^{(2)}(1;q)-1\leq(1-\theta)|\varpi|.$$

Then $\mathcal{L}_p^{(2)}(z;q)\in\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$.

Proof. Substituting (12) in (18), we can write

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)\frac{(-1)^{j-1}q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}}\\
&\quad\leq(1-\theta)|\varpi|.
\end{aligned}$$

Simplifying the obtained relation, we get

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q]\frac{(-1)^{j-1}q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}}\\
&\qquad+\varkappa\sum_{j=2}^{\infty}[j,q][j-1,q]\frac{(-1)^{j-1}q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}}\\
&\qquad+\xi\sum_{j=2}^{\infty}[j,q][j-1,q][j-2,q]\frac{(-1)^{j-1}q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}}\\
&\quad\leq(1-\theta)|\varpi|.
\end{aligned}$$

Hence,

$$D_q\mathcal{L}_p^{(2)}(1;q)-1+\varkappa D_q^2\mathcal{L}_p^{(2)}(1;q)+\xi D_q^3\mathcal{L}_p^{(2)}(1;q)\leq(1-\theta)|\varpi|.$$

□

Theorem 5 *Let $\Psi \in \mathcal{A}$, $q \in (0,1)$, and suppose the parameters p, ξ, $\varkappa$, q, ϖ, θ satisfy the following condition:*

$$D_q\mathcal{L}_p^{(3)}(1;q) + \varkappa D_q^2\mathcal{L}_p^{(3)}(1;q) + \xi D_q^3\mathcal{L}_p^{(3)}(1;q) - 1 \le (1-\theta)|\varpi|.$$

Then $\mathcal{L}_p^{(3)}(z;q) \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$.

Proof. Substituting (13) in (18), we can write

$$\sum_{j=2}^{\infty}[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right)\frac{(-1)^{j-1}q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} \le (1-\theta)|\varpi|,$$

or, equivalently,

$$\begin{aligned}&\sum_{j=2}^{\infty}[j,q]\frac{(-1)^{j-1}q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\\ &\quad+\varkappa\sum_{j=2}^{\infty}[j,q][j-1,q]\frac{(-1)^{j-1}q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\\ &\quad+\xi\sum_{j=2}^{\infty}[j,q][j-1,q][j-2,q]\frac{(-1)^{j-1}q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\\ &\le (1-\theta)|\varpi|.\end{aligned}$$

Thus,

$$D_q\mathcal{L}_p^{(3)}(1;q) - 1 + \varkappa D_q^2\mathcal{L}_p^{(3)}(1;q) + \xi D_q^3\mathcal{L}_p^{(3)}(1;q) \le (1-\theta)|\varpi|.$$

□

Let $\Psi \in \mathcal{A}$, and define the convolution operator $\mathcal{L}_{p,q}^{(2)}(\Psi,z)$ by

$$\mathcal{L}_{p,q}^{(2)}(\Psi,z) = \mathcal{L}_p^{(2)}(z;q) * \Psi(z) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}}t_jz^j = \sum_{j=1}^{\infty}X_jz^j,$$

where $*$ is the Hadamard product. The coefficients X_j are explicitly given by

$$X_j = \frac{(-1)^{j-1}q^{(j-1)(j+p-1)}}{4^{j-1}(q;q)_{j-1}(q^{p+1};q)_{j-1}}t_j. \tag{21}$$

Similarly, let $\mathcal{L}_{p,q}^{(3)}(\Psi,z)$ be defined as

$$\mathcal{L}_{p,q}^{(3)}(\Psi,z) = \mathcal{L}_p^{(3)}(z;q) * \Psi(z) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}t_jz^j = \sum_{j=1}^{\infty}Y_jz^j,$$

and the coefficients Y_j be

$$Y_j = \frac{(-1)^{j-1} q^{\frac{j}{2}(j-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} t_j. \tag{22}$$

Theorem 6 *Suppose that $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ satisfies the inequality*

$$\begin{aligned}
&(2k+V_2+3)D_q\mathcal{L}_p^{(2)}(1;q) + (V_1-2k-1)\mathcal{L}_p^{(2)}(1;q) \\
&\quad < V_2+V_1+2+|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.
\end{aligned}$$

Then $\mathcal{L}_{p,q}^{(2)}(\Psi,z) \in k-\mathcal{UCV}_q[V_1,V_2]$.

Proof. Substituting (21) in (8), we obtain

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q]((2k+V_2+3)[j,q]+V_1-2k-1)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \\
&\quad < |V_2-V_1|.
\end{aligned} \tag{23}$$

Due to the restrictions on the parameters, we have

$$[j,q]\left(1+\varkappa[j-1,q]+\xi[j-1,q][j-2,q]\right) \geq [j,q](1+\varkappa+\xi). \tag{24}$$

Substituting inequality (24) into (16), we obtain the upper bound for t_j as

$$|t_j| \leq \frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}, \quad j=2,3,\ldots \tag{25}$$

Replacing $|t_j|$ in (23) by the bound from (25), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q]((2k+V_2+3)[j,q]+V_1-2k-1)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \\
&\quad \leq \sum_{j=2}^{\infty}[j,q](2k+V_2+3)[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)} \\
&\qquad + \sum_{j=2}^{\infty}(V_1-2k-1)[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)} \\
&\quad = \sum_{j=2}^{\infty}(2k+V_2+3)[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)} \\
&\qquad + \sum_{j=2}^{\infty}(V_1-2k-1)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}.
\end{aligned}$$

Simplifying this summations and substituting the result into (23), we obtain

$$(2k+V_2+3)\sum_{j=2}^{\infty}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}$$
$$+(V_1-2k-1)\sum_{j=2}^{\infty}\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}$$
$$<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}. \tag{26}$$

Finally, using identity (14) in (26), we get

$$(2k+V_2+3)(D_q\mathcal{L}_p^{(2)}(1;q)-1)+(V_1-2k-1)(\mathcal{L}_p^{(2)}(1;q)-1)$$
$$<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}. \tag{27}$$

The hypothesis of the theorem together with the inequality in (27) confirms the required sufficient condition for $\mathcal{L}_{p,q}^{(2)}(\Psi,z)$ to belong to the class $k-\mathcal{UCV}_q[V_1,V_2]$. □

Theorem 7 *Suppose that* $\Psi\in\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$(2k+V_2+3)D_q\mathcal{L}_p^{(3)}(1;q)+(V_1-2k-1)\mathcal{L}_p^{(3)}(1;q)$$
$$<V_2+V_1+2+|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

Then $\mathcal{L}_{p,q}^{(3)}(\Psi,z)\in k-\mathcal{UCV}_q[V_1,V_2]$.

Proof. Substituting (22) in (8) and proceeding as in the previous proof, we obtain the desired result. □

Theorem 8 *Suppose that* $\Psi\in\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$(|V_2+1|+2)D_q\mathcal{L}_p^{(2)}(1;q)+(|V_1+1|-2)\mathcal{L}_p^{(2)}(1;q)-|V_2+1|-|V_1+1|$$
$$<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

Then $\mathcal{L}_{p,q}^{(2)}(\Psi,z)\in\mathcal{C}_q[V_1,V_2]$.

Proof. Substituting expression (21) for X_j into the condition (7) under which a function $\Psi(z)$ belongs to the class $\mathcal{C}_q[V_1,V_2]$, we obtain

$$\sum_{j=2}^{\infty}[j,q](2q[j-1,q]+|(V_2+1)[j,q]-(V_1+1)|)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j|$$
$$<|V_2-V_1|. \tag{28}$$

Applying (25) to the above gives

$$\sum_{j=2}^{\infty}(|V_2+1|+2)[j,q]^2\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}$$
$$+\sum_{j=2}^{\infty}(|V_1+1|-2)[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}$$
$$=(|V_2+1|+2)\sum_{j=2}^{\infty}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}$$
$$+(|V_1+1|-2)\sum_{j=2}^{\infty}\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}. \tag{29}$$

Simplifying the summations and substituting the result from (29) into (28), we obtain

$$(|V_2+1|+2)\sum_{j=2}^{\infty}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}$$
$$+(|V_1+1|-2)\sum_{j=2}^{\infty}\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}$$
$$<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}. \tag{30}$$

Using identity (14) in (30), we get

$$(|V_2+1|+2)(D_q\mathcal{L}_p^{(2)}(1;q)-1)+(|V_1+1|-2)(\mathcal{L}_p^{(2)}(1;q)-1)$$
$$<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}. \tag{31}$$

The hypothesis of the theorem together with the inequality in (31) confirms the required sufficient condition for $\mathcal{L}_{p,q}^{(2)}(\Psi,z)$ to belong to the class $\mathcal{C}_q[V_1,V_2]$. □

Theorem 9 *Suppose that* $\Psi\in\mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$(|V_2+1|+2)D_q\mathcal{L}_p^{(3)}(1;q)+(|V_1+1|-2)\mathcal{L}_p^{(3)}(1;q)-|V_2+1|-|V_1+1|$$
$$<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

Then $\mathcal{L}_{p,q}^{(3)}(\Psi,z)\in\mathcal{C}_q[V_1,V_2]$.

Proof. Substituting the expression (22) for Y_j into the condition (8) and proceeding as in the previous proof, we obtain the desired result. □

Theorem 10 *Suppose that* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$D_q\mathcal{L}_p^{(2)}(1;q) + (\lambda q - 1)\mathcal{L}_p^{(2)}(1;q) \leq \lambda q \left(1 + \frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|}\right).$$

Then $\mathcal{L}_{p,q}^{(2)}(\Psi, z) \in \mathcal{C}_{q,\lambda}$.

Proof. Substituting expression (21) for X_j into condition (5) under which $\Psi(z)$ belongs to the class $\mathcal{C}_{q,\lambda}$, we obtain

$$\sum_{j=2}^{\infty}[j,q]([j-1,q]+\lambda)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \leq \lambda. \tag{32}$$

Applying (25) to the left side of (32), we get

$$\begin{aligned}&\sum_{j=2}^{\infty}[j,q]([j-1,q]+\lambda)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \\ &\leq \sum_{j=2}^{\infty}[j,q]\left(\frac{1}{q}[j,q]+\lambda-\frac{1}{q}\right)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}.\end{aligned} \tag{33}$$

Simplifying the summations and substituting the result from (33) into (32), we obtain

$$\begin{aligned}&\sum_{j=2}^{\infty}\frac{1}{q}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} + \sum_{j=2}^{\infty}\left(\lambda-\frac{1}{q}\right)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} \\ &\quad\leq \lambda\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.\end{aligned} \tag{34}$$

Using identity (14) in (34) gives

$$\frac{1}{q}(D_q\mathcal{L}_p^{(2)}(1;q)-1) + \left(\lambda-\frac{1}{q}\right)(\mathcal{L}_p^{(2)}(1;q)-1) \leq \lambda\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

Hence, the result follows. □

Theorem 11 *Suppose that* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$D_q\mathcal{L}_p^{(3)}(1;q) + (\lambda q - 1)\mathcal{L}_p^{(3)}(1;q) \leq \lambda q \left(1 + \frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|}\right).$$

Then $\mathcal{L}_{p,q}^{(3)}(\Psi, z) \in \mathcal{C}_{q,\lambda}$.

Proof. Substituting the expression (22) for Y_j into the condition (5) and proceeding as in the previous proof, we obtain the desired result. □

Theorem 12 *Suppose that $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ satisfies the inequality*

$$\mathcal{L}_p^{(2)}(1;q) - 1 \leq (1+\sigma)\frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|}.$$

Then $\mathcal{L}_{p,q}^{(2)}(\Psi, z) \in \mathcal{S}_q^*(\sigma)$.

Proof. Substituting expression (21) for X_j into and condition (2) for a function $\Psi(z)$ to belong to the class $\mathcal{S}_q^*(\sigma)$, we obtain

$$\sum_{j=2}^{\infty}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \leq 1+\sigma. \tag{35}$$

Applying (25) to the left side of (35), we get

$$\begin{aligned}&\sum_{j=2}^{\infty}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \\ &\quad \leq \sum_{j=2}^{\infty}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}.\end{aligned} \tag{36}$$

Simplifying the summations and substituting the result from (36) into (35), we obtain

$$\sum_{j=2}^{\infty}\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} \leq (1+\sigma)\frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|}. \tag{37}$$

Using identity (14) in (37) gives

$$\mathcal{L}_p^{(2)}(1;q) - 1 \leq (1+\sigma)\frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|},$$

and the result follows. □

Theorem 13 *Suppose that $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ satisfies the inequality*

$$\mathcal{L}_p^{(3)}(1;q) - 1 \leq (1+\sigma)\frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|}.$$

Then $\mathcal{L}_{p,q}^{(3)}(\Psi, z) \in \mathcal{S}_q^*(\sigma)$.

Proof. Substituting the expression (22) for Y_j into the condition (2) and proceeding as in the previous proof, we obtain the desired result. □

Theorem 14 *Suppose that* $\Psi \in \mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$ *satisfies the inequality*

$$D_q\mathcal{L}^{(2)}_p(1;q) - \mathcal{L}^{(2)}_p(1;q) \leq (1+\sigma)q\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

Then $\mathcal{L}^{(2)}_{p,q}(\Psi, z) \in \mathcal{C}_q(\sigma)$.

Proof. Substituting expression (21) for X_j into condition (3) under which a function $\Psi(z)$ belongs to the class $\mathcal{C}_q(\sigma)$, we obtain

$$\sum_{j=2}^{\infty}[j,q][j-1,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \leq 1+\sigma. \tag{38}$$

Applying (25) to the left side of (38), we get

$$\begin{aligned}&\sum_{j=2}^{\infty}[j,q][j-1,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}|t_j| \\ &\quad \leq \sum_{j=2}^{\infty}\frac{1}{q}[j,q]([j,q]-1)\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}.\end{aligned} \tag{39}$$

Simplifying the summations and substituting the result from (39) into (38), we obtain

$$\begin{aligned}&\frac{1}{q}\sum_{j=2}^{\infty}[j,q]\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} - \frac{1}{q}\sum_{j=2}^{\infty}\frac{(-1/4)^{j-1}q^{(j-1)(j+p-1)}}{(q;q)_{j-1}(q^{p+1};q)_{j-1}} \\ &\quad \leq (1+\sigma)\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.\end{aligned} \tag{40}$$

Using identity (14) in (40) gives

$$\frac{1}{q}(D_q\mathcal{L}^{(2)}_p(1;q)-1) - \frac{1}{q}(\mathcal{L}^{(2)}_p(1;q)-1) \leq (1+\sigma)\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}, \tag{41}$$

and the proof is complete. □

Theorem 15 *Suppose that* $\Psi \in \mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$ *satisfies the inequality*

$$D_q\mathcal{L}^{(3)}_p(1;q) - \mathcal{L}^{(3)}_p(1;q) \leq (1+\sigma)q\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

Then $\mathcal{L}^{(3)}_{p,q}(\Psi, z) \in \mathcal{C}_q(\sigma)$.

Proof. Substituting the expression (22) for Y_j into the condition (3) and proceeding as in the previous proof, we obtain the desired result. □

The following corollaries are immediate consequences of Theorems 12, 13, 14, and 15, respectively.

Corollary 1 *If* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$\mathcal{L}_p^{(2)}(1;q) - 1 \leq \frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|},$$

then $\mathcal{L}_{p,q}^{(2)}(\Psi, z) \in \mathcal{S}_q^*$.

Corollary 2 *If* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$\mathcal{L}_p^{(3)}(1;q) - 1 \leq \frac{1+\varkappa+\xi}{2(1-\theta)|\varpi|},$$

then $\mathcal{L}_{p,q}^{(3)}(\Psi, z) \in \mathcal{S}_q^*$.

Corollary 3 *If* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(q,z)$ *satisfies the inequality*

$$D_q\mathcal{L}_p^{(2)}(1;q) - \mathcal{L}_p^{(2)}(z;q) \leq q\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},$$

then $\mathcal{L}_{p,q}^{(2)}(\Psi, z) \in \mathcal{C}_q$.

Corollary 4 *If* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$D_q\mathcal{L}_p^{(3)}(1;q) - \mathcal{L}_p^{(3)}(1;q) \leq q\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},$$

then $\mathcal{L}_{p,q}^{(3)}(\Psi, z) \in \mathcal{C}_q$.

Let $\Psi \in \mathcal{A}$. The $q-$extension of Hohlov operator, denoted by $\mathcal{H}_{a,b,c,q}(\Psi, z)$, is defined as

$$\mathcal{H}_{a,b,c,q}(\Psi, z) = z\phi(a,b;c;q,z) * \Psi(z) = \sum_{j=1}^{\infty} \frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}} t_j z^j$$
$$= \sum_{j=1}^{\infty} B_j z^j,$$

and coefficients B_j are given by

$$B_j = \frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}} t_j, \;\; j = 1, 2, \ldots \tag{42}$$

We now derive sufficient conditions under which the operator $\mathcal{H}_{a,b,c,q}(\Psi, z)$ belongs to the classes $k-\mathcal{UCV}_q[V_1, V_2]$, $k-\mathcal{ST}_q[V_1, V_2]$, $\mathcal{C}_q[V_1, V_2]$, $\mathcal{S}_q^*[V_1, V_2]$, $\mathcal{C}_{q,\lambda}$, $\mathcal{S}_{q,\lambda}^*$, $\mathcal{S}_q^*(\sigma)$, $\mathcal{C}_q(\sigma)$, $\mathcal{S}_q^*$ and $\mathcal{C}_q$.

Theorem 16 *Suppose* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$\begin{aligned}\frac{(2k+V_2+3)q(1-a)(1-b)}{(1-q)(1-c)}&\phi(aq,bq;cq;q,1)\\ &+(V_2+V_1+2)\phi(a,b;c;q,1)-(V_2+V_1+2)\\ &<\frac{(|V_2-V_1|)(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},\end{aligned}$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z) \in k-\mathcal{UCV}_q[V_1,V_2]$.

Proof. Substituting expression (42) for B_j into condition (8) under which a function $\Psi(z)$ belongs to the class $k-\mathcal{UCV}_q[V_1,V_2]$, we obtain

$$\begin{aligned}&\sum_{j=2}^{\infty}[j,q](2q(k+1)[j-1,q]+|(V_2+1)[j,q]+(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|\\ &\quad<|V_2-V_1|. \end{aligned} \tag{43}$$

Applying (25) to the left side of (43), we get

$$\begin{aligned}&\sum_{j=2}^{\infty}[j,q](2q(k+1)[j-1,q]+|(V_2+1)[j,q]+(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|\\ &\quad\leq\sum_{j=2}^{\infty}[j,q]q[j-1,q](2k+V_2+3)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}\\ &\qquad+\sum_{j=2}^{\infty}[j,q](V_2+V_1+2)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}\\ &\quad=\sum_{j=2}^{\infty}q[j-1,q](2k+V_2+3)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\\ &\qquad+\sum_{j=2}^{\infty}(V_2+V_1+2)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}.\end{aligned}$$

Simplifying the summations and substituting this result into (43), we obtain

$$\begin{aligned}(2k+V_2+3)q\frac{(1-a)(1-b)}{(1-q)(1-c)}&\sum_{j=0}^{\infty}\frac{(aq;q)_j(bq;q)_j}{(q;q)_j(cq;q)_j}\\ &+(V_2+V_1+2)\sum_{j=1}^{\infty}\frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j}\\ &<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.\end{aligned} \tag{44}$$

Using the hypothesis of the theorem together with (31), we obtain the desired result. □

Theorem 17 *Assume that* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$\begin{aligned}&\frac{(V_1-1-2k)q(q-c)}{(1-q)(q-a)(q-b)}\phi\left(\frac{a}{q},\frac{a}{q};\frac{a}{q};q,1\right)+(2k+V_2+3)\phi(a,b;c;q,1)\\&\quad-\frac{(V_1-1-2k)q(q-c)}{(1-q)(q-a)(q-b)}-\frac{(V_1-1-2k)}{(1-q)^2}-(2k+V_2+3)\\&\quad<\frac{|V_2-V_1|(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},\end{aligned}$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z)\in k-\mathcal{ST}_q[V_1,V_2]$.

Proof. Substituting the expression (42) for B_j into condition (9) under which $\Psi(z)$ belongs to the class $k-\mathcal{ST}_q[V_1,V_2]$, we obtain

$$\begin{aligned}&\sum_{j=2}^{\infty}(2q(k+1)[j-1,q]+|(V_2+1)[j,q]+(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|\\&\quad<|V_2-V_1|.\end{aligned}\tag{45}$$

Applying (25) to the left side of (45), we get

$$\begin{aligned}&\sum_{j=2}^{\infty}(2q(k+1)[j-1,q]+|(V_2+1)[j,q]+(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|\\&\le\sum_{j=2}^{\infty}\left((2k+V_2+3)[j,q]+V_1-1-2k\right)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}\\&=\sum_{j=1}^{\infty}(2k+V_2+3)\frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\\&\quad+\sum_{j=2}^{\infty}\frac{(V_1-1-2k)q(q-c)}{(1-q)(q-a)(q-b)}\frac{\left(\frac{a}{q};q\right)_j\left(\frac{b}{q};q\right)_j}{\left(\frac{c}{q};q\right)_j(q;q)_j}\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\\&=\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\left((2k+V_2+3)\sum_{j=1}^{\infty}\frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j}\right.\\&\quad\left.+\frac{(V_1-1-2k)q(q-c)}{(1-q)(q-a)(q-b)}\sum_{j=2}^{\infty}\frac{\left(\frac{a}{q};q\right)_j\left(\frac{b}{q};q\right)_j}{\left(\frac{c}{q};q\right)_j(q;q)_j}\right).\end{aligned}$$

Simplifying the summations and substituting this result into (45), we obtain

$$\begin{aligned}&\frac{(V_1-1-2k)q(q-c)}{(1-q)(q-a)(q-b)}\left(\phi\left(\frac{a}{q},\frac{a}{q};\frac{a}{q};q,1\right)-1-\frac{(q-a)(q-b)}{(q-c)q(1-q)}\right)\\&\quad+(2k+V_2+3)\left(\phi(a,b;c;q,1)-1\right)<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},\end{aligned}$$

and the desired result follows. □

Theorem 18 *Suppose* $\Psi \in \mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$ *satisfies the inequality*

$$\frac{(2+|V_2+1|)q(1-a)(1-b)}{(|V_2-V_1|)(1-q)(1-c)}\phi(aq,bq;cq;q,1)+\phi(a,b;c;q,1) < 1+\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z) \in \mathcal{C}_q[V_1,V_2]$.

Proof. Substituting expression (42) for B_j into condition (7) for a function $\Psi(z)$ to belong to the class $\mathcal{C}_q[V_1,V_2]$, we obtain

$$\sum_{j=2}^{\infty}[j,q](2q[j-1,q]+|(V_2+1)[j,q]-(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| < |V_2-V_1|. \tag{46}$$

Applying (25) to the left side of (46), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q](2q[j-1,q]+|(V_2+1)[j,q]-(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| \\
&\leq \sum_{j=2}^{\infty}(2+|V_2+1|)q[j-1,q][j,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)} \\
&\quad + \sum_{j=2}^{\infty}|V_2-V_1|[j,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)} \\
&= \frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}\left(q(2+|V_2+1|)\frac{(1-a)(1-b)}{(1-q)(1-c)}\sum_{j=2}^{\infty}\frac{(aq;q)_{j-2}(bq;q)_{j-2}}{(q;q)_{j-2}(cq;q)_{j-2}}\right. \\
&\quad \left. +|V_2-V_1|\sum_{j=2}^{\infty}\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\right) \\
&= \frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}\left(q(2+|V_2+1|)\frac{(1-a)(1-b)}{(1-q)(1-c)}\sum_{j=0}^{\infty}\frac{(aq;q)_{j}(bq;q)_{j}}{(q;q)_{j}(cq;q)_{j}}\right. \\
&\quad \left. +|V_2-V_1|\sum_{j=1}^{\infty}\frac{(a;q)_{j}(b;q)_{j}}{(q;q)_{j}(c;q)_{j}}\right).
\end{aligned} \tag{47}$$

Simplifying the summations and substituting this result into (46), we obtain

$$q(2+|V_2+1|)\frac{(1-a)(1-b)}{(1-q)(1-c)}\phi(aq,bq;cq;q,1)+|V_2-V_1|(\phi(a,b;c;q,1)-1) < |V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},$$

which completes the proof. □

Theorem 19 *Suppose that* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *and satisfies the inequality*

$$\begin{aligned}&\frac{(|V_1+1|-2)q(q-c)}{(1-q)(q-a)(q-b)}\phi\left(\frac{a}{q},\frac{a}{q};\frac{a}{q};q,1\right)+(2+|V_2+1|)\phi(a,b;c;q,1)\\&\qquad-\frac{(|V_1+1|-2)q(q-c)}{(1-q)(q-a)(q-b)}-\frac{(|V_1+1|-2)}{(1-q)^2}-(2+|V_2+1|)\\&\qquad<\frac{|V_2-V_1|(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},\end{aligned}$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z)\in\mathcal{S}_q^*[V_1,V_2]$.

Proof. Substituting expression (42) for B_j into condition (6) under which $\Psi(z)$ belongs to the class $\mathcal{S}_q^*[V_1,V_2]$, we obtain

$$\sum_{j=2}^{\infty}(2q[j-1,q]+|(V_2+1)[j,q]-(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|<|V_2-V_1|. \tag{48}$$

Applying (25) to the left side of (48), we get

$$\begin{aligned}&\sum_{j=2}^{\infty}(2q[j-1,q]+|(V_2+1)[j,q]-(V_1+1)|)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|\\&\leq\sum_{j=2}^{\infty}\left((2+|V_2+1|)[j,q]+|V_1+1|-2\right)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}\\&=\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\left((2+|V_2+1|)\sum_{j=1}^{\infty}\frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j}\right.\\&\qquad\left.+\frac{(|V_1+1|-2)q(q-c)}{(1-q)(q-a)(q-b)}\sum_{j=2}^{\infty}\frac{\left(\frac{a}{q};q\right)_j\left(\frac{b}{q};q\right)_j}{\left(\frac{c}{q};q\right)_j(q;q)_j}\right)\end{aligned}$$

Simplifying the summations and substituting this result into (46), we obtain

$$\begin{aligned}&\frac{(|V_1+1|-2)q(q-c)}{(1-q)(q-a)(q-b)}\left(\phi\left(\frac{a}{q},\frac{a}{q};\frac{a}{q};q,1\right)-1-\frac{(q-a)(q-b)}{(q-c)q(1-q)}\right)\\&\qquad+(2+|V_2+1|)\left(\phi(a,b;c;q,1)-1\right)<|V_2-V_1|\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.\end{aligned}$$

Hence, we obtain the desired result. □

Theorem 20 *Suppose that* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *and satisfies the inequality*

$$\begin{aligned}&\frac{q(q-c)(\lambda-1)}{(1-q)(q-a)(q-b)}\phi\left(\frac{a}{q},\frac{b}{q};\frac{c}{q};q,1\right)+\phi(a,b;c;q,1)-\frac{\lambda(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}\\&\qquad\leq 1+(\lambda-1)\frac{(q-a)(q-b)+q(1-q)(q-c)}{(1-q)^2(q-a)(q-b)},\end{aligned}$$

then $\mathcal{H}_{a,b,c,q}(\Psi, z) \in \mathcal{S}^*_{q,\lambda}$.

Proof. Substituting expression (42) for B_j into condition (4) under which a function $\Psi(z)$ belongs to the class $\mathcal{S}^*_{q,\lambda}$, we obtain

$$\sum_{j=2}^{\infty}([j,q]+\lambda-1)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| \leq \lambda. \tag{49}$$

Applying (25) to the left side of (49), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty}([j,q]+\lambda-1)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| \\
&\quad \leq \sum_{j=2}^{\infty}([j,q]+\lambda-1)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)} \\
&\quad = \frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\left(\sum_{j=2}^{\infty}\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}} + \sum_{j=2}^{\infty}(\lambda-1)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(c;q)_{j-1}(q;q)_{j-1}[j,q]}\right) \\
&\quad = \frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\left(\sum_{j=1}^{\infty}\frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j}\right. \\
&\qquad \left. + \frac{(\lambda-1)}{(1-q)}\frac{(1-\frac{c}{q})}{(1-\frac{a}{q})(1-\frac{b}{q})}\sum_{j=2}^{\infty}\frac{(\frac{a}{q};q)_j(\frac{b}{q};q)_j}{(\frac{c}{q};q)_j(q;q)_j}\right).
\end{aligned}$$

Substituting this into (49), we obtain

$$\begin{aligned}
&\phi(a,b;c;q,1)-1 \\
&\qquad + \frac{(\lambda-1)}{(1-q)}\frac{(1-\frac{c}{q})}{(1-\frac{a}{q})(1-\frac{b}{q})}\left(\phi\left(\frac{a}{q},\frac{b}{q};\frac{c}{q};q,1\right)-1-\frac{(1-\frac{a}{q})(1-\frac{b}{q})}{(1-\frac{c}{q})(1-q)}\right) \\
&\quad \leq \frac{\lambda(1+\varkappa+\xi)}{2(1-\theta)|\varpi|},
\end{aligned}$$

which is what was required. □

Theorem 21 *Suppose* $\Psi \in \mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$ *satisfies the inequality*

$$\frac{(1-a)(1-b)}{(1-q)(1-c)\lambda}\phi(aq,bq;cq;q,1)+\phi(a,b;c;q,1)-\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|} \leq 1,$$

then $\mathcal{H}_{a,b,c,q}(\Psi, z) \in \mathcal{C}_{q,\lambda}$.

Proof. Substituting expression (42) for B_j into condition (5) under which $\Psi(z)$ belongs to the class $\mathcal{C}_{q,\lambda}$, we obtain

$$\sum_{j=2}^{\infty}[j,q]([j-1,q]+\lambda)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| \leq \lambda. \tag{50}$$

Applying (25) to the left side of (50), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q]([j-1,q]+\lambda)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| \\
&\quad\leq \sum_{j=2}^{\infty}[j,q]([j-1,q]+\lambda)\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)} \\
&\quad= \frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\left(\sum_{j=2}^{\infty}[j-1,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}} + \sum_{j=2}^{\infty}\lambda\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\right) \\
&\quad= \frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\left(\frac{(1-a)(1-b)}{(1-q)(1-c)}\sum_{j=0}^{\infty}\frac{(aq;q)_j(bq;q)_j}{(q;q)_j(cq;q)_j} + \lambda\sum_{j=1}^{\infty}\frac{(a;q)_j(b;q)_j}{(q;q)_j(c;q)_j}\right)
\end{aligned}$$

Simplifying the summations and substituting this result into (49), we obtain

$$\frac{(1-a)(1-b)}{(1-q)(1-c)}\phi(aq,bq;cq;q,1) + \lambda(\phi(a,b;c;q,1)-1) \leq \lambda\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

Thus we obtain the desired result. □

Theorem 22 *Suppose* $\Psi \in \mathcal{M}_{\xi,\varkappa}^{\varpi,\theta}(z;q)$ *satisfies the inequality*

$$\phi(a,b;c;q,1) \leq \frac{(1+\sigma)(1+\varkappa+\xi)}{2(1-\theta)|\varpi|} + 1,$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z) \in \mathcal{S}_q^*(\sigma)$.

Proof. Substituting expression (42) for B_j into condition (2) for a function $\Psi(z)$ to belong to the class $\mathcal{S}_q^*(\sigma)$, we obtain

$$\sum_{j=2}^{\infty}[j,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| \leq 1+\sigma. \tag{51}$$

Applying (25) to the left side of (51), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty}[j,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j| \\
&\quad\leq \sum_{j=2}^{\infty}[j,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)} \\
&\quad= \frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\frac{(1-q)(1-\frac{c}{q})}{(1-\frac{a}{q})(1-\frac{b}{q})}\sum_{j=2}^{\infty}\frac{(\frac{a}{q};q)_j(\frac{b}{q};q)_j}{(\frac{a}{q};q)_j(q;q)_j},
\end{aligned}$$

and substituting this inequality into (51), we obtain

$$\phi\left(\frac{a}{q},\frac{b}{q};\frac{c}{q};q,1\right)-1-\frac{(1-\frac{a}{q})(1-\frac{b}{q})}{(1-\frac{c}{q})}\le(1+\sigma)\frac{(1-\frac{a}{q})(1-\frac{b}{q})}{(1-q)(1-\frac{c}{q})}\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}.$$

□

Example 1 *Let $\Psi_4(z)$ be a function defined by the series*

$$\Psi_4(z)=z+\sum_{j=2}^{\infty}\frac{z^j}{j^2}=z+\frac{z^2}{4}+\frac{z^3}{9}+\dots, \tag{52}$$

it is not difficult to see that $\Psi_4\in\mathcal{A}$. Consider the parameter values $q=0.6$, $a=0.3$, $b=0.4$, $c=0.6$, $\varkappa=0.3$, $\xi=0.4$, $\theta=0.2$, $\varpi=1$, and $\sigma=0.5$. It can be verified that $\Psi(z)\in\mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$, and that

$$\phi(a,b;c;q,1)\approx1.34\le\frac{(1+\sigma)(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}+1=2.59375.$$

Thus, the conditions of Theorem 22 are satisfied, and it follows that

$$\mathcal{H}_{a,b,c,q}(\Psi,z)\in\mathcal{S}^*_q(\sigma).$$

Theorem 23 *Suppose $\Psi\in\mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$ satisfies the inequality*

$$\phi\left(\frac{a}{q},\frac{b}{q};\frac{c}{q};q,1\right)\le(1+\sigma)\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}\frac{(1-q)(q-a)(q-b)}{q(q-c)},$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z)\in\mathcal{C}_q(\sigma)$.

Proof. Substituting the expression (42) for B_j into condition (3) under which a function $\Psi(z)$ belongs to the class $\mathcal{C}_q(\sigma)$, we obtain

$$\sum_{j=2}^{\infty}[j,q][j-1,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|\le1+\sigma. \tag{53}$$

Applying (25) to the left side of (53), we get

$$\begin{aligned}&\sum_{j=2}^{\infty}[j,q][j-1,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}|t_j|\\&\le\sum_{j=2}^{\infty}[j,q][j-1,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\frac{2(1-\theta)|\varpi|}{[j,q](1+\varkappa+\xi)}\\&=\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\sum_{j=2}^{\infty}[j-1,q]\frac{(a;q)_{j-1}(b;q)_{j-1}}{(q;q)_{j-1}(c;q)_{j-1}}\\&=\frac{2(1-\theta)|\varpi|}{(1+\varkappa+\xi)}\frac{q(q-c)}{(1-q)(q-a)(q-b)}\sum_{j=0}^{\infty}\frac{(\frac{a}{q};q)_j(\frac{a}{q};q)_j}{(q;q)_j(\frac{a}{q};q)_j}.\end{aligned}$$

Substituting this into (53), we obtain

$$\frac{q(q-c)}{(1-q)(q-a)(q-b)}\phi\left(\frac{a}{q},\frac{b}{q};\frac{c}{q};q,1\right)\leq(1+\sigma)\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}. \tag{54}$$

□

Remark 1 *In Example 1, if the parameters are chosen as* $q=0.6$, $a=0.3$, $b=0.4$, $c=0.5$, $\varkappa=0.4$, $\xi=0.3$, $\theta=0.2$, $\varpi=1$ *and* $\sigma=0.5$, *the function* $\Psi_4(z)$ *does not satisfy the conditions stated in Theorem 23. However, if the parameters are chosen as* $q=0.6$, $a=0.1$, $b=0.2$, $c=0.4$, *with the other values unchanged, the function* $\Psi_4(z)$ *satisfies the conditions of Theorem 23 and it follows that* $\mathcal{H}_{a,b,c,q}(\Psi_4,z)\in\mathcal{C}_q(\sigma)$.

Corollary 5 *Suppose* $\Psi\in\mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$ *satisfies the inequality*

$$\phi(a,b;c;q,1)\leq\frac{(1+\varkappa+\xi)}{2(1-\theta)|\varpi|}+1,$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z)\in\mathcal{S}^*_q$.

Corollary 6 *Suppose* $\Psi\in\mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$ *satisfies the inequality*

$$\phi\left(\frac{a}{q},\frac{b}{q};\frac{c}{q};q,1\right)\leq\frac{(1-q)(q-a)(q-b)(1+\varkappa+\xi)}{q(q-c)2(1-\theta)|\varpi|},$$

then $\mathcal{H}_{a,b,c,q}(\Psi,z)\in\mathcal{C}_q$.

Conclusion

In this work, we have studied the normalized forms of Jackson and Hahn–Exton $q-$Bessel functions, focusing on their convolution operators. Our findings provide sufficient conditions for operators that are associated with normalized Jackson and Hahn–Exton $q-$Bessel functions and $q-$extension of the Hohlov integral operator. We derive coefficient bounds and sufficient conditions for a function to belong to the class $\mathcal{M}^{\varpi,\theta}_{\xi,\varkappa}(z;q)$. Furthermore, by using the Taylor coefficients of $z_2\phi_1(a,b;c;q,z)$ and the normalized Jackson and Hahn–Exton $q-$Bessel functions, we explored the inclusion properties and geometric properties of related convolution operators.

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Manas Kumar Giri
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology,
Vellore, 632014, Tamil Nadu, India.
mgiri3029@gmail.com; manaskumar.giri2020@vitstudent.ac.in

Kondooru Raghavendar
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology,
Vellore, 632014, Tamil Nadu, India.
raghavendar248@gmail.com; raghavendar.k@vit.ac.in

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