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Convergence acceleration of quasi-periodic and quasi-periodic-rational interpolations by polynomial corrections

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Abstract

The paper considers convergence acceleration of the quasi-periodic and the quasi-periodic-rational interpolations by application of polynomial corrections. We investigate convergence of the resultant quasi-periodic-polynomial and quasi-periodic-rational-polynomial interpolations and derive exact constants of the main terms of asymptotic errors in the regions away from the endpoints. Results of numerical experiments clarify behavior of the corresponding interpolations.

Key Words: Trigonometric interpolation, Quasi-periodic interpolation, Convergence acceleration, Polynomial correction, Rational correction
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1 Introduction

The quasi-periodic (QP) trigonometric interpolation $I_{N,m}(f,x)$, $m \geq 0$ (m is integer), $x \in [-1,1]$ interpolates function f on equidistant grid

$$x_k = \frac{k}{N}, \ |k| \leq N \tag{1}$$

and is exact for the following set of quasi-periodic functions

$$e^{i\pi n\sigma x}, \ |n| \leq N, \ \sigma = \frac{2N}{2N+m+1} \tag{2}$$

with the period $2/\sigma$ (which tends to 2 as $N \to \infty$).

The idea of the QP-interpolation is introduced in [3] where it is investigated based on the results of numerical experiments. Explicit representation of the interpolation is derived in [8] and [9] (see also [6] and [7]). There, the convergence of the interpolation is investigated in the framework of the $L_2(-1,1)$-norm and at the endpoints $x = \pm 1$ in terms of the limit function. Pointwise convergence in the interval $(-1,1)$ is explored in [10] where the exact constant of the main term of asymptotic error is obtained.

Paper [11] considers convergence acceleration of the QP-interpolation by rational corrections in terms of $e^{i\pi\sigma x}$ which leads to the quasi-periodic-rational (QPR) interpolation. There, the pointwise convergence in the interval $(-1,1)$ is investigated and the exact constant of the main term of asymptotic error is obtained.

In the current paper, we continue investigations concerning the problem of convergence acceleration of interpolations and consider the polynomial corrections of the corresponding errors. Such approach leads to quasi-periodic-polynomial (QPP) and quasi-periodic-rational-polynomial (QPRP) interpolations.

Polynomial corrections ([13]) are linear combination of some polynomials with derivatives of f at the points $x = \pm 1$ in the coefficients. We derive exact constants of the asymptotic errors in the case when the exact values of these derivatives are known, otherwise, we consider the problem of their approximation by means of the discrete Fourier coefficients by solution of some systems of linear equations. Some results concerning the $L_2(-1,1)$-convergence of QPP-interpolation are presented in [13].

Numerical experiments show the behavior of the corresponding interpolations for moderate number of interpolation points for specific functions.

Some results of this research are reported in [12].

2 The Quasi-Periodic Interpolation

The QP-interpolation can be realized by the following formula

$$I_{N,m}(f,x) = \sum_{n=-N}^{N} F_{n,m} e^{i\pi n\sigma x},\ m \in Z,\ m \geq 0, \tag{3}$$

with the error

$$R_{N,m}(f,x) = f(x) - I_{N,m}(f,x), \tag{4}$$

where

$$F_{n,m} = \check{f}_{n,m} - \sum_{\ell=1}^{m} \theta_{n,\ell} \check{f}_{\ell+N,m}, \tag{5}$$

$$\check{f}_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-\frac{2i\pi nk}{2N+m+1}} \tag{6}$$

and

$$\theta_{n,\ell} = e^{\frac{2i\pi(\ell+N-n)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi n(s-1)}{2N+m+1}}. \tag{7}$$

Here, $v_{\ell,s}^{-1}$ are the elements of the inverse of the following Vandermonde matrix

$$v_{s,\ell} = \alpha_\ell^{s-1}, \ \alpha_\ell = e^{\frac{2i\pi(\ell+N)}{2N+m+1}}. \tag{8}$$

Next theorems describe the behavior of the QP-interpolation in the interval $(-1,1)$.

Let

$$A_{sk}(f) = f^{(k)}(1) - (-1)^{k+s} f^{(k)}(-1), \ k = 0, \ldots, q. \tag{9}$$

Theorem 1 *[10] Let $f^{(q+1)} \in AC[-1,1]$ for some $q \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \ldots, q-1. \tag{10}$$

Then, the following estimate holds for $|x| < 1$

$$\begin{aligned} R_{N,0}(f,x) = A_{0q}(f) \frac{(-1)^N}{(2N)^{q+1}} \frac{\sin(\pi N x)}{\cos \frac{\pi x}{2}} \sum_{k=0}^{[q/2]} \frac{(-1)^k 2^{2k}}{(q-2k)! \pi^{2k+1}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{2k+1}} \\ + o(N^{-q-1}), \ N \to \infty. \end{aligned} \tag{11}$$

Theorem 2 *[10] Let $f^{(q+2m)} \in AC[-1,1]$ for some $m \geq 1$, $q \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \ldots, q-1. \tag{12}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$\begin{aligned} R_{N,m}(f,x) = C_{q,m}(f) \frac{(-1)^N}{N^{q+m+1}} \Bigg[\sin\left(\pi(N+1)\sigma x\right) \sum_{k=0}^{\left[\frac{m}{2}\right]} \binom{m-k}{k} \frac{(-1)^k}{2^{2k+1} \cos^{2k+2} \frac{\pi x}{2}} \\ - \sin\left(\pi N \sigma x\right) \sum_{k=0}^{\left[\frac{m}{2}\right]-1} \binom{m-k-2}{k} \frac{(-1)^k}{2^{2k+3} \cos^{2k+4} \frac{\pi x}{2}} \Bigg] + o(N^{-q-m-1}), \end{aligned} \tag{13}$$

where

$$C_{q,m}(f) = \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k+1} i^{k-1} \pi^{k-m+1} (q-k)!} \Phi_{k,m}^{(m)}(-1), \tag{14}$$

and

$$\Phi_{k,m}(e^{i\pi x}) = e^{\frac{i\pi}{2}(m-1)x} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}}. \tag{15}$$

3 The Quasi-Periodic-Rational Interpolation

Consider a finite sequence $\lambda = \{\lambda_k\}_{|k|=1}^{p}$. By $\delta_n^k(\lambda, c_n)$, we denote generalized finite differences defined by the following recurrent relations

$$\delta_n^0(\lambda, c_n) = c_n, \tag{16}$$

and

$$\delta_n^k(\lambda, c_n) = \delta_n^{k-1}(\lambda, c_n) + \lambda_{-k}\delta_{n-1}^{k-1}(\lambda, c_n) + \lambda_k \left(\delta_{n+1}^{k-1}(\lambda, c_n) + \lambda_{-k}\delta_n^{k-1}(\lambda, c_n)\right) \tag{17}$$

for some sequence c_n.

The QPR-interpolation can be realized as follows

$$\begin{aligned} I_{N,m}^p(f,x) &= I_{N,m}(f,x) \\ &+ \sum_{k=1}^{p} \lambda_{-k} \frac{\delta_{-N-1}^{k-1}(\lambda, F_{n,m})\, e^{-i\pi\sigma N x} - \delta_N^{k-1}(\lambda, F_{n,m})\, e^{i\pi\sigma(N+1)x}}{\prod_{s=1}^{k}(1+\lambda_{-s}e^{i\pi\sigma x})(1+\lambda_s e^{-i\pi\sigma x})} \\ &+ \sum_{k=1}^{p} \lambda_k \frac{\delta_{N+1}^{k-1}(\lambda, F_{n,m})\, e^{i\pi\sigma N x} - \delta_{-N}^{k-1}(\lambda, F_{n,m})\, e^{-i\pi\sigma(N+1)x}}{\prod_{s=1}^{k}(1+\lambda_{-s}e^{i\pi\sigma x})(1+\lambda_s e^{-i\pi\sigma x})} \end{aligned} \tag{18}$$

with the error

$$R_{N,m}^p(f,x) = f(x) - I_{N,m}^p(f,x). \tag{19}$$

Different methods are known for determination of parameters λ_k (see [11]). Throughout the paper, we assume that

$$\lambda_{-k} = \lambda_k = 1 - \frac{\tau_k}{N},\ \ k = 1, \dots, p, \tag{20}$$

where τ_k are some new parameters independent of N. In the numerical experiments below we put $\tau_k = k$.

Let $\gamma_k(\tau)$ be the coefficients of polynomial

$$\prod_{s=1}^{p}(1+\tau_s x) = \sum_{k=0}^{p}\gamma_k(\tau)x^k, \tag{21}$$

where $\tau = \{\tau_1, \dots, \tau_p\}$.

Let

$$\psi_{p,m,j}^{\pm}(\tau) = \sum_{s=0}^{p}(-1)^s\gamma_s(\tau)\sum_{k=0}^{p}\gamma_k(\tau)(2p-k-s+j)!\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r\pm 1)^{2p-k-s+j+1}}. \tag{22}$$

Theorem 3 *[11] Let $f^{(q+2p+1)} \in AC[-1,1]$ for some $q \geq 0$, $p \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0,\ \ k = 0, \dots, q-1. \tag{23}$$

Let parameters λ_k be chosen as in (20)*. Then, the following estimate holds for* $|x| < 1$

$$R^p_{N,0}(f,x) = \frac{(-1)^{N+p}}{(2N)^{q+2p+1}} \frac{\sin(\pi N x)}{\cos^{2p+1}\left(\frac{\pi x}{2}\right)} \sum_{k=0}^{q} \frac{A_{kq}(f) 2^k}{i^k \pi^{k+1} (q-k)! k!} \psi^+_{p,0,k}(\tau) + o\left(N^{-q-2p-1}\right), \ N \to \infty. \tag{24}$$

Theorem 4 *[11] Let* $f^{(q+2p+m)} \in AC[-1,1]$ *for some* $q \geq 0$*,* $p, m \geq 1$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \ldots, q-1. \tag{25}$$

Let parameters λ_k be chosen as in (20)*. Then, the following estimate holds for* $|x| < 1$

$$R^p_{N,m}(f,x) = \frac{(-1)^N}{(2N)^{q+2p+1}} \frac{\sin\left(\pi\sigma(N+\frac{1}{2})x - \frac{\pi m}{2}\right)}{\cos^{2p+1}\frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k} 2^k}{(q-k)! i^{k-m} \pi^{k+1}} \times \left(\frac{(-1)^p}{k!} \psi^+_{p,m,k}(\tau) - \sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu - m + 1, \tau) \right) + o(N^{-q-2p-1}), \ N \to \infty, \tag{26}$$

where

$$h_p(\beta,\tau) = \left(\frac{\pi\beta}{2}\right)^{2p} \sum_{s=0}^{p} \gamma_s(\tau) \sum_{k=0}^{p} (-1)^k \gamma_k(\tau) \left(\frac{2}{i\pi\beta}\right)^{k+s}. \tag{27}$$

Let us consider some results of numerical experiments. Let

$$f_1(x) = (1-x^2)^4 \sin(x-1). \tag{28}$$

In view of smoothness of f_1 Theorems 1-4 are valid with $q = 4$.

Figure 1 shows the behavior of $|R_{256,m}(f,x)|$ for different values of m on $[-0.7, 0.7]$. According to Theorems 1 and 2 the rate of convergence is $O(N^{-m-5})$.

Figure 2 shows the behavior of $|R^p_{256,m}(f,x)|$ for different values of m and p on the same interval with $\tau_k = k$. According to Theorems 3 and 4 the rate of convergence is $O(N^{-2p-5})$.

The first columns of Figures 1 and 2 correspond to $m = 0$. In this case, the QP-interpolation has convergence rate $O(N^{-5})$ and the QPR-interpolations have rates $O(N^{-7})$, $O(N^{-11})$ for $p = 1, 3$, respectively, and, we see that in both cases the QPR-interpolations provide with better accuracy.

The second columns correspond to $m = 2$ and the QP-interpolation has convergence rate $O(N^{-7})$ which coincides with the convergence rate of the QPR-interpolation for $p = 1$ and in this case the accuracies are similar. For $p = 3$ the QPR-interpolations is more accurate.

The third columns correspond to $m = 4$ when the QP-interpolation is $O(N^{-9})$. The accuracies of the QPR-interpolations are worse for $p = 1$ and better for $p = 3$.

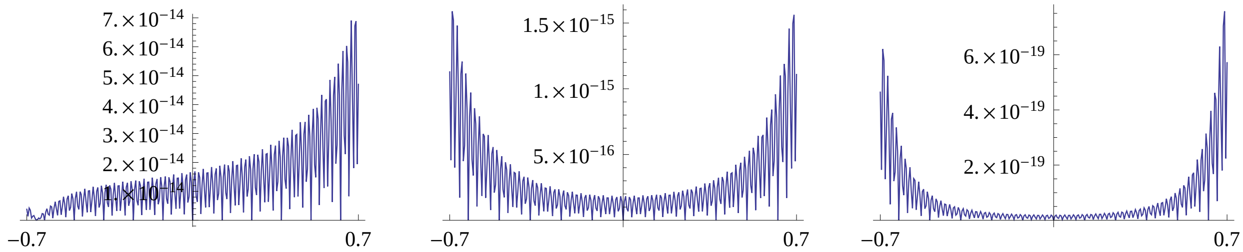

Figure 1: The graphs of the absolute errors $|R_{256,m}(f,x)|$ on $[-0.7,0.7]$ for $m=0,2,4$ (from left to right) while interpolating f_1 by the QP-interpolation.

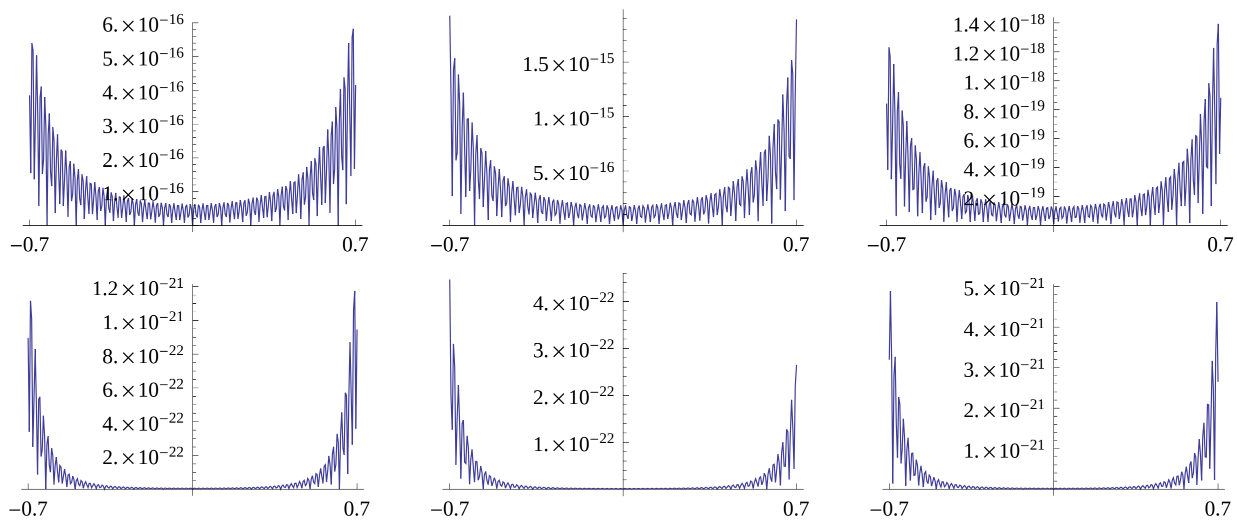

Figure 2: The graphs of the absolute errors $|R^p_{256,m}(f,x)|$ on $[-0.7,0.7]$ for $m=0,2,4$ (from left to right) and $p=1,3$ (from top to bottom) while interpolating f_1 by the QPR-interpolation with $\tau_k=k$.

4 Convergence acceleration by polynomial corrections

In this section, we consider convergence acceleration of the QP and QPR-interpolations by polynomial corrections ([13]). We see that in Theorems 1-4 the convergence rates of the corresponding interpolations essentially depend on the property

$$f^{(k)}(-1)=f^{(k)}(1)=0,\ \ k=0,\dots,q-1. \tag{29}$$

We will show how the same rates can be derived for functions without property (29) by application of polynomial corrections.

We need construction of some polynomials. First, we consider polynomials $B_k(x)$, $k=0,\dots,q-1$ with the property

$$B_k^{(s)}(1)-B_k^{(s)}(-1)=\delta_{k,s}, \tag{30}$$

where $\delta_{k,s}$ is the Kronecker symbol. These are well-known 2-periodic Bernoulli polynomials ([2]) defined by the recurrence relations

$$B_0(x) = \frac{x}{2},\ B_k(x) = \int B_{k-1}(x)dx,\ x \in [-1,1],\ \int_{-1}^{1} B_k(x)dx = 0. \tag{31}$$

Here are some of the Bernoulli polynomials

$$B_1(x) = \frac{x^2}{4} - \frac{1}{12},\ B_2(x) = \frac{x^3}{12} - \frac{x}{12},\ B_3(x) = \frac{x^4}{48} - \frac{x^2}{24} + \frac{7}{720}. \tag{32}$$

Knowledge of these polynomials leads to the Lanczos representation ([2])

$$f(x) = F^{-}(x) + \sum_{k=0}^{q-1} A_k^{-}(f)B_k(x), \tag{33}$$

where

$$A_k^{-}(f) = f^{(k)}(1) - f^{(k)}(-1) \tag{34}$$

and F is a 2-periodic and relatively smooth function on the real line $F^{-} \in C^{q-1}(\mathbb{R})$ if $f \in C^{q-1}[-1,1]$. If the exact values of $A_k^{-}(f)$ are unknown then their approximations can be derived from the Lanczos representation (33). By calculation of the discrete Fourier coefficients

$$\check{f}_n = \frac{1}{2N} \sum_{k=-N}^{N-1} f\left(\frac{k}{N}\right) e^{-i\pi n \frac{k}{N}} \tag{35}$$

of functions in the both sides of (33), we get

$$\check{f}_n = \check{F}_n^{-} + \sum_{k=0}^{q-1} A_k^{-}(f)\check{B}_n(k) \tag{36}$$

and taking into account the fast decay of $\check{F}_n^{-}$, $n \sim N$ as $N \to \infty$ compared to other coefficients, we get the following system for determination of approximations $\widetilde{A}_k^{-}(f)$ to $A_k^{-}(f)$

$$\check{f}_n = \sum_{k=0}^{q-1} \tilde{A}_k^{-}(f)\check{B}_n(k),\ n = n_1, \dots, n_q. \tag{37}$$

Investigation of the error $\widetilde{A}_k^{-}(f) - A_k^{-}(f)$ can be performed as in [4], where similar system is considered.

Now, we consider polynomials $E_k(x)$, $k = 0, \dots, q-1$ with the property

$$E_k^{(s)}(1) + E_k^{(s)}(-1) = \delta_{k,s}. \tag{38}$$

These polynomials can be constructed by the following recurrence relations

$$E_0(x) = \frac{1}{2},\ E_k(x) = \int E_{k-1}(x)dx,\ x \in [-1,1],\ E_k(1) + E_k(-1) = 0. \tag{39}$$

Here are some of them

$$E_1(x) = \frac{x}{2},\ E_2(x) = \frac{x^2}{4} - \frac{1}{4},\ E_3(x) = \frac{x^3}{12} - \frac{x}{4},\ E_4(x) = \frac{x^4}{48} - \frac{x^2}{8} + \frac{5}{48}. \tag{40}$$

Similar to the Lanczos representation, we consider the following one

$$f(x) = F^+(x) + \sum_{k=0}^{q-1} A_k^+(f) E_k(x), \tag{41}$$

where

$$A_k^+(f) = f^{(k)}(1) + f^{(k)}(-1). \tag{42}$$

If the exact values of $A_k^+(f)$ are unknown then their approximations can be found from (41). By multiplication of the both sides of (41) by $e^{i\pi x/2}$ and taking into account that $F^+(x)e^{i\pi x/2}$ has the same properties as $F^-(x)$, we calculate the discrete Fourier coefficients of the both sides, disregard the coefficients of F^+ and get the following system for determination of approximate values $\widetilde{A}_k^+(f)$ of $A_k^+(f)$

$$\check{f}_n^\dagger = \sum_{k=0}^{q-1} \widetilde{A}_k^+(f) \check{E}_n^\dagger(k),\ n = n_1, \dots, n_q, \tag{43}$$

where

$$\check{f}_n^\dagger = \frac{1}{2N} \sum_{k=-N}^{N-1} f\left(\frac{k}{N}\right) e^{-i\pi\left(n+\frac{1}{2}\right)\frac{k}{N}}. \tag{44}$$

Having approximate values of $\widetilde{A}_k^+(f)$ and $\widetilde{A}_k^-(f)$, we can calculate approximations to $f^{(k)}(1)$ and $f^{(k)}(-1)$ if needed.

Finally, we need to construct two different sets of polynomials $\mu_{k,q}(x)$ and $\nu_{k,q}(x)$, $k = 0, \dots, q-1$ with the properties

$$\mu_{k,q}^{(s)}(1) - \mu_{k,q}^{(s)}(-1) = \delta_{k,s},\ \mu_{k,q}^{(s)}(1) + \mu_{k,q}^{(s)}(-1) = 0,\ k, s = 0, \dots, q-1 \tag{45}$$

and

$$\nu_{k,q}^{(s)}(1) + \nu_{k,q}^{(s)}(-1) = \delta_{k,s},\ \nu_{k,q}^{(s)}(1) - \nu_{k,q}^{(s)}(-1) = 0,\ k, s = 0, \dots, q-1. \tag{46}$$

We put

$$\mu_{q-1,q}(x) = \frac{x^2(x^2-1)^{q-1}}{2^q(q-1)!},\ \nu_{q-1,q}(x) = \frac{x(x^2-1)^{q-1}}{2^q(q-1)!} \tag{47}$$

if q is even, and

$$\mu_{q-1,q}(x) = \frac{x(x^2-1)^{q-1}}{2^q(q-1)!},\ \nu_{q-1,q}(x) = \frac{x^2(x^2-1)^{q-1}}{2^q(q-1)!} \tag{48}$$

if q is odd.

Then

$$\mu_{k,q}(x) = \mu_{k,q-1}(x) - \left(\mu_{k,q-1}^{(q-1)}(1) + \mu_{k,q-1}^{(q-1)}(-1)\right) \nu_{q-1,q}(x), \tag{49}$$

$$\nu_{k,q}(x) = \nu_{k,q-1}(x) - \left(\nu_{k,q-1}^{(q-1)}(1) - \nu_{k,q-1}^{(q-1)}(-1)\right) \mu_{q-1,q}(x) \tag{50}$$

if $q - k$ is even, and

$$\mu_{k,q}(x) = \mu_{k,q-1}(x) - \left(\mu_{k,q-1}^{(q-1)}(1) - \mu_{k,q-1}^{(q-1)}(-1)\right) \mu_{q-1,q}(x), \tag{51}$$

$$\nu_{k,q}(x) = \nu_{k,q-1}(x) - \left(\nu_{k,q-1}^{(q-1)}(1) + \nu_{k,q-1}^{(q-1)}(-1)\right) \nu_{q-1,q}(x) \tag{52}$$

if $q - k$ is odd.

Let us show some of these polynomials. When $q = 1$

$$\mu_{0,1}(x) = \frac{x}{2},\ \nu_{0,1}(x) = \frac{x^2}{2}. \tag{53}$$

When $q = 2$

$$\mu_{0,2}(x) = -\frac{1}{4}x\left(-3 + x^2\right),\ \mu_{1,2}(x) = \frac{1}{4}x^2\left(-1 + x^2\right), \tag{54}$$

$$\nu_{0,2}(x) = \frac{x^2}{2} - \frac{1}{2}x^2\left(-1 + x^2\right),\ \nu_{1,2}(x) = \frac{1}{4}x\left(-1 + x^2\right). \tag{55}$$

When $q = 3$

$$\begin{gathered}\mu_{0,3}(x) = \frac{1}{16}x\left(15 - 10x^2 + 3x^4\right),\ \mu_{1,3}(x) = \frac{1}{16}\left(-9x^2 + 14x^4 - 5x^6\right), \\ \mu_{2,3}(x) = \frac{1}{16}x\left(-1 + x^2\right)^2,\end{gathered} \tag{56}$$

and

$$\begin{gathered}\nu_{0,3}(x) = \frac{1}{2}x^2\left(3 - 3x^2 + x^4\right),\ \nu_{1,3}(x) = \frac{1}{16}\left(-7x + 10x^3 - 3x^5\right), \\ \nu_{2,3}(x) = \frac{1}{16}x^2\left(-1 + x^2\right)^2.\end{gathered} \tag{57}$$

Now, let us consider the main representation of f:

$$f(x) = G(x) + \sum_{k=0}^{q-1} A_k^-(f)\mu_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\nu_{k,q}(x). \tag{58}$$

Taking into account the properties of functions $\mu_{k,q}$ and $\nu_{k,q}$, we see that

$$G^{(k)}(1) = G^{(k)}(-1) = 0,\ k = 0, \dots, q-1. \tag{59}$$

Approximation of G, in (58), by the QP-interpolation, leads to the following quasi-periodic-polynomial (QPP) interpolation

$$I_{N,m,q}(f,x) = I_{N,m}(G,x) + \sum_{k=0}^{q-1} A_k^-(f)\mu_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\nu_{k,q}(x), \tag{60}$$

with the error

$$R_{N,m,q}(f,x) = f(x) - I_{N,m,q}(f,x), \tag{61}$$

where the discrete Fourier coefficients of G can be calculated based on (58) if the exact values of A_k^+ and A_k^- are known. Otherwise, their approximations can be derived from systems (37) and (43) and the corresponding QPP-interpolation we denote by $\widetilde{I}_{N,m,q}(f,x)$ with the corresponding error $\widetilde{R}_{N,m,q}(f,x)$.

Then, we can reformulate Theorems 1 and 2 for the QPP-interpolation.

Theorem 5 *Let $f^{(q+1)} \in AC[-1,1]$ for some $q \geq 1$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$R_{N,0,q}(f,x) = A_{0q}(f)\frac{(-1)^N}{2^{q+1}N^{q+1}}\frac{\sin(\pi N x)}{\cos\frac{\pi x}{2}}\sum_{k=0}^{[q/2]}(-1)^k\frac{2^{2k}}{(q-2k)!\pi^{2k+1}} \times \sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2k+1}} + o(N^{-q-1}),\ N \to \infty. \tag{62}$$

Theorem 6 *Let $f^{(q+2m)} \in AC[-1,1]$ for some $m, q \geq 1$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$R_{N,m,q}(f,x) = C_{q,m}(f)\frac{(-1)^N}{N^{q+m+1}}\left[\sin\left(\pi(N+1)\sigma x\right)\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{m-k}{k}\frac{(-1)^k}{2^{2k+1}\cos^{2k+2}\frac{\pi x}{2}} - \sin\left(\pi N \sigma x\right)\sum_{k=0}^{\left[\frac{m}{2}\right]-1}\binom{m-k-2}{k}\frac{(-1)^k}{2^{2k+3}\cos^{2k+4}\frac{\pi x}{2}}\right] + o(N^{-q-m-1}), \tag{63}$$

where $C_{q,m}(f)$ and $\Phi_{k,m}(x)$ are defined in Theorem 2.

Let us show some numerical results. Let

$$f_2(x) = \sin(x-1). \tag{64}$$

Figure 3 shows the behavior of $R_{256,m}(f,x)$ for different values of m on $[-0.7, 0.7]$. The QP-interpolation has convergence rate $O(N^{-m-1})$ for (64) ($q = 0$).

Figures 4 and 5 show the behaviors of $R_{256,m,q}(f,x)$ and $\widetilde{R}_{256,m,q}(f,x)$, respectively, for different values of q and m.

Approximation of G, in (58), by the QPR-interpolation, leads to the following quasi-periodic-rational-polynomail (QPRP) interpolation

$$I_{N,m,q}^p(f,x) = I_{N,m}^p(G,x) + \sum_{k=0}^{q-1}A_k^-(f)\mu_{k,q}(x) + \sum_{k=0}^{q-1}A_k^+(f)\nu_{k,q}(x), \tag{65}$$

with the error

$$R_{N,m,q}^p(f,x) = f(x) - I_{N,m,q}^p(f,x), \tag{66}$$

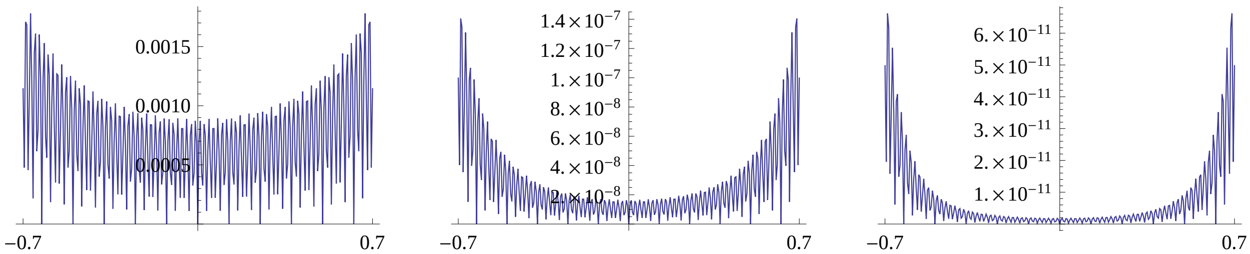

Figure 3: The graphs of the absolute errors $|R_{256,m}(f,x)|$ on interval $[-0.7,0.7]$ for $m=0,2,4$ (from left to right) while interpolating (64) by the QP-interpolation.

where again the discrete Fourier coefficients of G can be calculated based on (58) if the exact values of A_k^+ and A_k^- are known. If approximated values of jumps are used in the QPRP-interpolation then the corresponding one we denote by $\widetilde{I}^p_{N,m,q}(f,x)$ with the corresponding error $\widetilde{R}^p_{N,m,q}(f,x)$.

Now, we reformulate Theorems 3 and 4 for the QPRP-interpolation.

Theorem 7 *Let $f^{(q+2p+1)} \in AC[-1,1]$ for some $q,p \geq 1$. Let parameters λ_k be defined as in (20) and $m=0$. Then, the following estimate holds for $|x|<1$ as $N \to \infty$*

$$\begin{aligned} R^p_{N,0,q}(f,x) = {} & \frac{(-1)^{N+p}}{(2N)^{q+2p+1}} \frac{\sin(\pi N x)}{\cos^{2p+1}\left(\frac{\pi x}{2}\right)} \sum_{k=0}^{q} \frac{A_{kq}(f) 2^k}{i^k \pi^{k+1}(q-k)!k!} \psi^+_{p,0,k}(\tau) \\ & + o\left(N^{-q-2p-1}\right), \end{aligned} \tag{67}$$

where $\psi^+_{p,0,k}(\tau)$ is defined by (22).

Theorem 8 *Let $f^{(q+2p+m)} \in AC[-1,1]$ for some $q,p,m \geq 1$. Let parameters λ_k be defined as in (20). Then, the following estimate holds for $|x|<1$*

$$\begin{aligned} R^p_{N,m,q}(f,x) = {} & \frac{(-1)^N}{(2N)^{q+2p+1}} \frac{\sin\left(\pi\sigma(N+\frac{1}{2})x - \frac{\pi m}{2}\right)}{\cos^{2p+1}\frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k} 2^k}{(q-k)! i^{k-m} \pi^{k+1}} \\ & \times \left(\frac{(-1)^p}{k!} \psi^+_{p,m,k}(\tau) - \sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu-m+1,\tau) \right) \\ & + o(N^{-q-2p-1}), \ N \to \infty, \end{aligned} \tag{68}$$

where $\psi^+_{p,m,k}$ is defined by (22), $\Phi_{k,m}$ by (15), and h_p by (27).

Figures 6 and 7 show the behaviors of $R^p_{256,m,q}(f,x)$ and $\widetilde{R}^p_{256,m,q}(f,x)$, respectively, for different values of p and m when $q=3$ and $\tau_k = k$ while interpolating (64). We see that, in general, utilization of approximate values of jumps does not degrade the quality of interpolations.

Numerical experiments are performed by Wolfram's Mathematica package with high accuracy option.

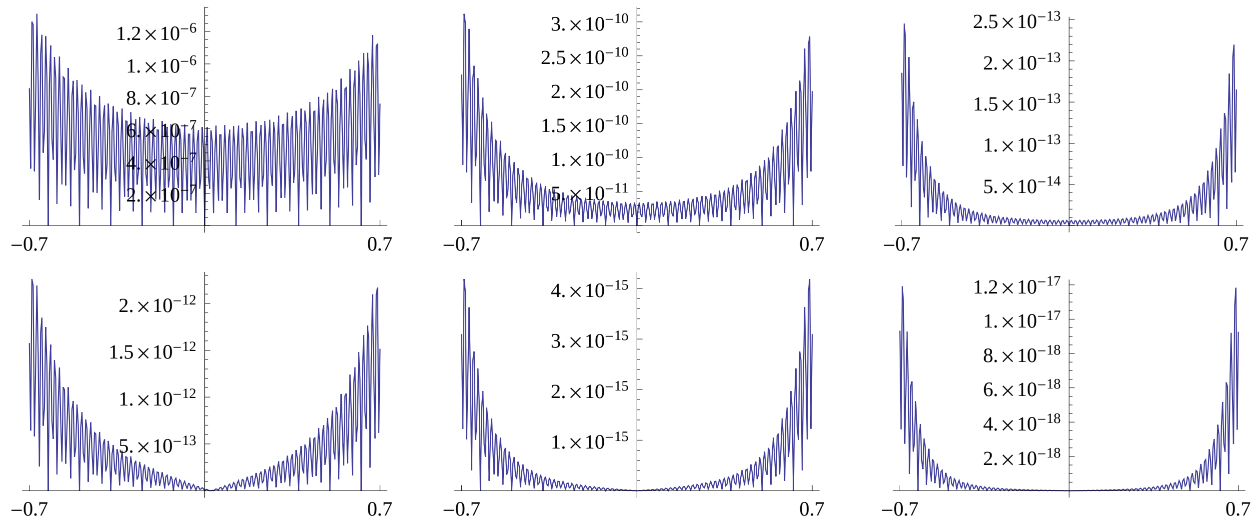

Figure 4: The graphs of the absolute errors $|R_{256,m,q}(f,x)|$ on interval $[-0.7,0.7]$ for $m=0,2,4$ (from left to right) and $q=1,3$ (from top to bottom) while interpolating f_2 by the QPP-interpolation with the exact values of A_k^+ and A_k^-.

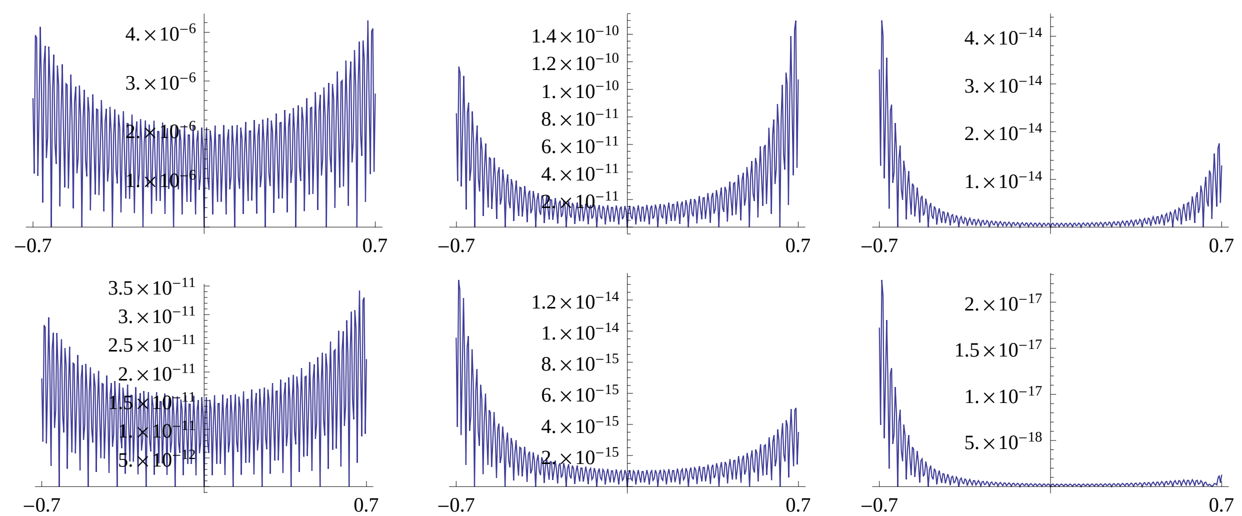

Figure 5: The graphs of the absolute errors $|\widetilde{R}_{256,m,q}(f,x)|$ on interval $[-0.7,0.7]$ for $m=0,2,4$ (from left to right) and $q=1,3$ (from top to bottom) while interpolating f_2 by the QPP-interpolation with the approximate values of A_k^+ and A_k^- derived from systems (37) and (43) with $n_1=N-1$, $n_2=-N$, $n_3=N-2$, $n_4=-N+1,\cdots$.

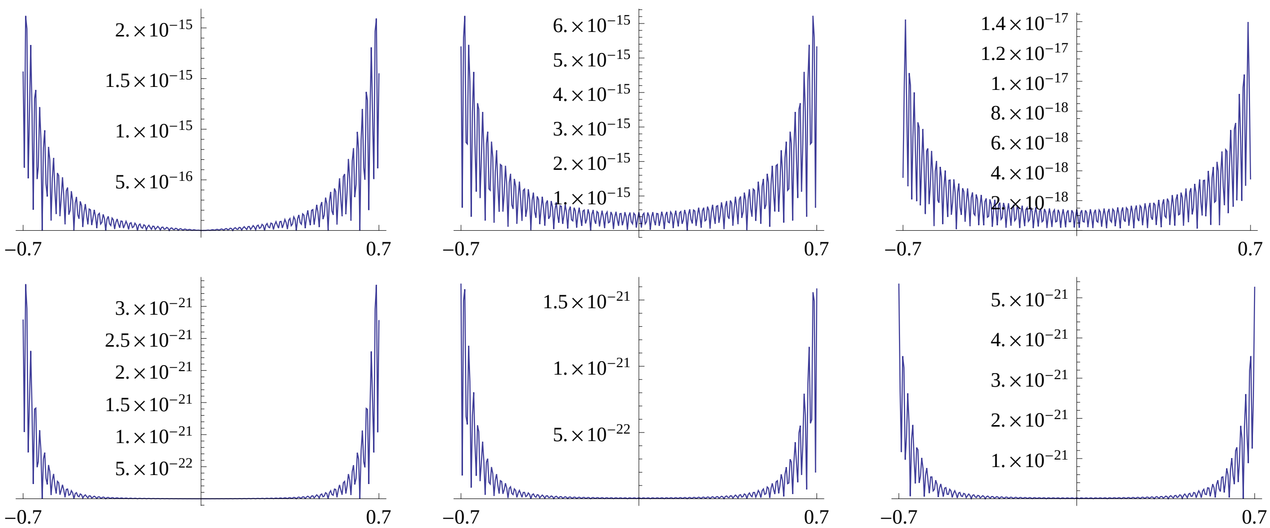

Figure 6: The graphs of the absolute errors $|R^{p}_{256,m,q}(f,x)|$ on interval $[-0.7, 0.7]$ for $q = 3$, $m = 0, 2, 4$ (from left to right) and $p = 1, 3$ (from top to bottom) while interpolating f_2 by the QPRP-interpolation with the exact values of A^{+}_{k} and A^{-}_{k}. We put $\tau_k = k$ in the rational corrections.

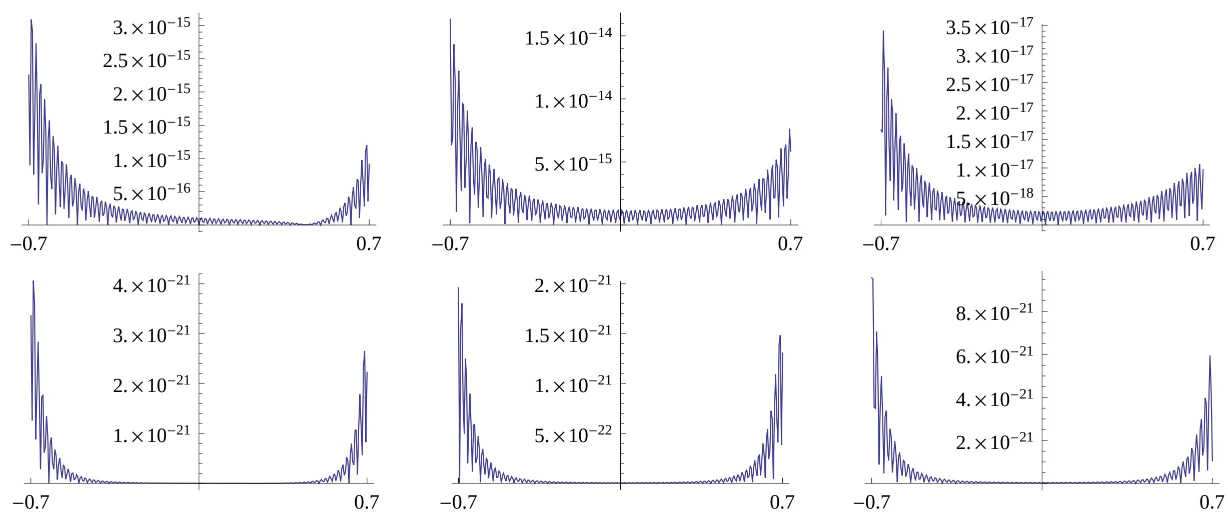

Figure 7: The graphs of the absolute errors $|\widetilde{R}^{p}_{256,m,q}(f,x)|$ on interval $[-0.7, 0.7]$ for $q = 3$, $m = 0, 2, 4$ (from left to right) and $p = 1, 3$ (from top to bottom) while interpolating f_2 by the QPRP-interpolation with the approximate values of A^{+}_{k} and A^{-}_{k} derived from systems (37) and (43) with $n_1 = N - 1$, $n_2 = -N$, $n_3 = N - 2$, $n_4 = -N + 1, \cdots$. We put $\tau_k = k$ in the rational corrections.

5 Conclusion

In this paper, the convergence of QP, QPR, QPP and QPRP interpolations are compared in terms of the pointwise convergence in $(-1,1)$ where interpolated function is smooth.

The QP interpolation $I_{N,m}(f,x)$ interpolates function f on equidistant grid (1) and is exact for the quasi-periodic functions (2). It can be realized by (3) where m indicates the size of the Vandermonde matrix (8) which must be inverted. Although the Vandermonde matrix is bad conditioned its inverse can be efficiently calculated by the well-known Bjorck-Pereyra algorithm (see [1]) however for moderate values of m. According to Theorems 1 and 2, the QP interpolation has convergence rate $O(N^{-q-m-1})$ for smooth functions with the property (29). The same accuracy in $(-1,1)$ can be achieved by the classical trigonometric interpolation by utilization of $q+m$ Bernoulli polynomials as error corrections (see [4]) which is known as the Krylov-Lanczos interpolation. However, for functions with rapidly growing derivatives at the points $x=\pm 1$, the QP interpolation is much more accurate compared to the Krylov-Lanczos approach as the main term of the error of the QP interpolation (see Theorems 1 and 2) depends on A_{0q} and A_{1q} while for the Krylov-Lanczos case we will have $f^{(q+m)}(1)-f^{(q+m)}(-1)$.

The QPR interpolation $I_{N,m}^{p}(f,x)$ performs convergence acceleration of the QP interpolation by rational corrections in terms of $e^{i\pi\sigma x}$. Theorems 3 and 4 show that the accuracy of the QPR interpolation is $O(N^{-q-2p-1})$ for smooth functions with the property (29). We see that for $p>m/2$, the accuracy of the QPR interpolation is higher than the QP provides with the same m. From practical point of view, it is more reasonable to fix the value of parameter m and achieve more accurate interpolation by applying rational corrections as inversion of large size Vandermone matrices, even with the Bjorck-Pereyra algorithm, will eventually lead to round-off errors.

The QPP and QPRP interpolations, $I_{N,m,q}(f,x)$ and $I_{N,m,q}^{p}(f,x)$, respectively, perform convergence acceleration of the QP and QPR interpolations by polynomial correction functions constructed as linear combinations of some polynomials with derivatives of f at the points $x=\pm 1$ in the coefficients. Such corrections allow to apply the QP and QPR interpolations to functions without the property (29). We show, how the corresponding polynomials can be constructed and, how the approximate values of jumps can be calculated by solution of some systems of linear equations. Theorems 5 and 6 show that the rate of convergence of the QPP interpolation is $O(N^{-q-m-1})$ and Theorems 7 and 8 state that the convergence rate of the QPRP interpolation is $O(N^{-q-2p-1})$. Numerical experiments show that approximation of jumps doesn't degrade the convergence quality of the corresponding interpolations. Practically, we recommend to fix the values of parameters m and q and additional accuracy achieve by application of rational corrections. This is important especially for functions with rapidly growing derivatives at the points $x=\pm 1$ as the main terms of asymptotic erros depend on A_{0q} and A_{1q} (see the corresponding theorems).

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