Armenian Journal of Mathematics
Volume 17, Number 13, 2025, 1–20
https://doi.org/10.52737/18291163-2025.17.13-1-20

Gibbsian Description of Gaussian Random Fields

L.A. Khachatryan

Abstract. This paper provides the solution to the problem of Gibbsian description of Gaussian random fields based on the Gibbs scheme currently being developed in the theory of lattice random fields.

Key Words: Gibbs Random Field, Gaussian Random Field, Conditional Densities, Gibbs Form of Conditional Densities, Transition Energy, Hamiltonian, Markov Random Field
Mathematics Subject Classification 2020: 60G60, 60G15, 60J99, 82B03

Introduction

At the beginning of the creation of mathematical statistical physics, the problem of Gibbsian description of various classes of random fields was quite relevant and was considered by many authors (see, for example, [1,2,13,18]). The main attention was paid to random fields with a state space of finite measure, and the case of a non-compact state space has not been studied enough. As Dobrushin [7] noted, due to the difficulties that arise when transferring known results to the non-compact case, it is natural first consider the problem of Gibbsian description for a class of well-studied random fields, namely, Gaussian random fields.

This problem for homogeneous Gaussian random fields was considered by Sinai [17] and Dobrushin [7]; Georgi [10] provides a solution to the same problem for inhomogeneous Gaussian random fields as well. Let us also note the work of Künsch [16], in which Dobrushin's results [7] were applied to some problems of thermodynamics and statistics.

All mentioned works were based on the DLR–definition of a Gibbs random field (the only definition present at that time) according to which a random filed P is Gibbsian if its conditional distributions coincide (for P-a.e. boundary conditions) with a Gibbsian specification constructed by some uniformly convergent potential (see [6]). The DLR–definition is strongly based on the theorem of existence of a random field corresponding to a given specification. This was the reason why the solution to the problem of Gibbsian description was carried out not individually for each given Gaussian random field, but for a set of random fields corresponding to a suitable (quadratic) potential (see, for example, Theorem 4.1 in [7]). Note also that in the case of a non-compact state space, corresponding Hamiltonian (potential) can be defined only for some subset of the set of infinite boundary conditions (see, for example, the remarks after formula (13.11) in [10]), and after that one has to prove that this set has P-measure one. That is, DLR–definition of a Gibbs random field with a finite state space (or a state space of a finite measure) has to be modified if one wants to apply it to Gaussian random fields.

In [4], a purely probabilistic definition of a Gibbs random field with finite state space was introduced. This definition does not use physical concept of potential, and within its framework, the problem of Gibbsianess of a random field is solved directly for each random field through the behavior of its finite-conditional distributions. However, this definition has some limitations in application to random fields with non-compact state space.

In [12], a purely probabilistic definition (or Pr–definition) of a Gibbs random field (with a finite state space) was improved, on the base of which the theory of Gibbs random fields can be constructed according to a scheme typical for the theory of random processes. These results can be extended to the case of positive random fields with Polish state space whose finite-dimensional distributions have densities. This allows us to give a solution to the problem of Gibbsian description of Gaussian random fields within the framework of Gibbs scheme developed in [12]. Namely, we obtain conditions on the elements of matrices, inverse to finite-volume covariance matrices, under which the corresponding Gaussian random field is Gibbsian, and give a Gibbs form of its conditional densities in terms of transition energies. It should be emphasized that in our solution, we do not use any objects external to the random field, but obtain the result through direct computations and the proposed definition of a Gibbs random field.

Finally, let us note that Dobrushin [7] and Georgii [10] observed that to solve the problem of whether a Gaussian random field is Gibbsian, it is sufficient to consider one-point conditional densities. Since the definition of a Gibbs random field they used is essentially based on solving the (inverse) problem of the existence of a random field with a given (Gibbsian) specification, they still had to deal with the entire specification, albeit reconstructed from its one-point subsystem. This was due to the absence at that time of explicit consistency conditions that would allow one to define a one-point specification as an independent object, from which a specification can be constructed. This problem (Dobrushin's problem) was solved in [3] (see also the paper by Fernández and Maillard [8] where sufficient conditions were provided). In the present paper, we also consider one-point conditional densities only. However, we do not need to solve the inverse problem since we use Pr–definition of a Gibbs random field that is formulated in terms of finite-conditional densities but not in terms of specification.

The remainder of the paper is organized as follows. In the next section, we give some preliminary notations and definitions. Section 2 shows how the main results of [12] are transferred to the case of random fields with Polish state spaces whose finite-dimensional distributions have densities. Section 3 presents the main part of the work. We obtain Gibbs form for conditional densities of Gaussian random fields in terms of transition energies and present conditions under which the corresponding Gaussian random field is Gibbsian. Particularly, we consider the case of Gaussian Markov random fields which are widely used in various applications. The paper ends with Appendix which contains proof of auxiliary results about inverse matrices.

1 Preliminaries

In this section, we present some general notations and definitions used in the following.

Let $\mathbb{Z}^d$ be a d-dimensional integer lattice, i.e., a set of d-dimensional vectors with integer components, $d \geq 1$. For $S \subset \mathbb{Z}^d$, denote by $W(S) = \{V \subset S, 0 < |V| < \infty\}$ the set of all non-empty finite subsets of S. In the case $S = \mathbb{Z}^d$, we will use a simpler notation W. To denote the complement of the set S, we will write S^c. When denoting one-point sets (singletons) $\{t\}$, $t \in \mathbb{Z}^d$, the brackets will be usually omitted.

Let $(X, \mathscr{B}, \mu)$ be a complete separable space (Polish space), where X is a non-empty set (state space), $\mathscr{B}$ is a σ-algebra generated by open subsets of X, and μ is a not identically equal to zero σ-additive measure. In particular, we will consider state space $X = \mathbb{R}$ with Borel σ-algebra $\mathscr{B}$ of subsets of $\mathbb{R}$ and Lebesgue measure μ on $\mathbb{R}$. Further, let each point $t \in \mathbb{Z}^d$ be associated with the space $(X^t, \mathscr{B}^t, \mu_t)$ which is a copy of $(X, \mathscr{B}, \mu)$.

Denote by X^S the set of all configurations on S, $S \subset \mathbb{Z}^d$, that is, the set $X^S = \{(x_t, t \in S) : x_t \in X^t, t \in S\}$, of all functions defined on S and tacking values in X. For $S = \emptyset$, we assume that $X^\emptyset = \{\boldsymbol{\emptyset}\}$ where $\boldsymbol{\emptyset}$ is an empty configuration. For any disjoint $S, T \subset \mathbb{Z}^d$ and any $x \in X^S$, $y \in X^T$, denote by xy the concatenation of x and y, that is, the configuration on $S \cup T$ equal to x on S and to y on T. When $T \subset S$, we denote by x_T the restriction of configuration $x \in X^S$ on T, i.e., $x_T = (x_t, t \in T)$.

For each $S \subset \mathbb{Z}^d$, the set X^S is endowed with the σ-algebra $\mathscr{B}^S$, which is the product of the σ-algebras $\mathscr{B}^t$, $t \in S$.

A *random field* on $\mathbb{Z}^d$ with state space X is a probability measure P on $(X^{\mathbb{Z}^d}, \mathscr{B}^{\mathbb{Z}^d})$. By P_S we denote the restriction of P on S, $S \subset \mathbb{Z}^d$, that is, a probability distribution on $(X^S, \mathscr{B}^S)$ such that

$$P_S(A) = (P)_S(A) = P(AX^{\mathbb{Z}^d \setminus S}), \qquad A \in \mathscr{B}^S.$$

For $S = \emptyset$, we assume $P_\emptyset(\boldsymbol{\emptyset}) = 1$. By $\mathscr{P}_P = \{P_V, V \in W\}$ we denote the system of finite-dimensional distributions of the random field P.

In this paper, we consider random fields whose finite-dimensional distributions $\{P_V, V \in W\}$ have densities $\{p_V, V \in W\}$ with respect to the measure μ:

$$P_V(A) = \int\limits_A p_V(x)\mu_V(dx), \qquad A \in \mathscr{B}^V, V \in W.$$

By the Radon–Nikodym theorem, for each $V \in W$, function p_V is defined on X^V, and there exist other densities coinciding with p_V on a set of μ_V-measure 1. In what follows, for each random field P, we choose a version of its densities $\{p_V, V \in W\}$ and associate the random field with this set by writing $P = \{p_V, V \in W\}$.

A random field $P = \{p_V, V \in W\}$ is called positive if for any $V \in W$, one has $p_V(x) > 0$ for all $x \in X^V$.

For a given random field $P = \{p_V, V \in W\}$ and any $t, s \in \mathbb{Z}^d$, denote

$$m_t = \int\limits_{X^t} x p_t(x)\mu_t(dx),$$

$$c_{ts} = \int\limits_{X^{\{t,s\}}} (x_t - m_t)(x_s - m_s)p_{\{t,s\}}(x)\mu_{\{t,s\}}(dx),$$

Then $m = \{m_t, t \in \mathbb{Z}^d\}$ is the vector of mean values of the random field P and $C = \{c_{ts}, t, s \in \mathbb{Z}^d\}$ is its covariance matrix.

For any $S \subset \mathbb{Z}^d$ and any function $h : W(S) \to \mathbb{R}$, the notation $\lim\limits_{\Lambda \uparrow S} h(\Lambda) = a$ means that for any $\varepsilon > 0$ there exists $\Lambda_\varepsilon \in W(S)$ such that for any $\Lambda \in W(S)$, $\Lambda \supset \Lambda_\varepsilon$, it holds $|h(\Lambda) - a| < \varepsilon$. We will also consider limits $\lim\limits_{n\to\infty} h(\Lambda_n)$ with respect to some increasing sequence $\{\Lambda_n\}_{n\geq 1}$ of finite sets converging to S, that is, $\Lambda_n \subset \Lambda_{n+1} \in W(S)$, $n \geq 1$, and $\bigcup\limits_{n=1}^{\infty} \Lambda_n = S$. It is clear that if $\lim\limits_{\Lambda \uparrow S} h(\Lambda) = a$, then for any increasing sequence $\{\Lambda_n\}_{n\geq 1}$ of finite sets converging to S, we have $\lim\limits_{n\to\infty} h(\Lambda_n) = a$. At the same time, if $\lim\limits_{n\to\infty} h(\Lambda_n) = a$ for any increasing sequence $\{\Lambda_n\}_{n\geq 1}$, then $\lim\limits_{\Lambda \uparrow S} h(\Lambda) = a$.

Any increasing sequence $\mathscr{F} = \{\mathscr{F}_n\}_{n\geq 1}$ of σ-subalgebras of $\mathscr{B}^{\mathbb{Z}^d}$, that is, $\mathscr{F}_n \subset \mathscr{F}_{n+1} \subset \mathscr{B}^{\mathbb{Z}^d}$, $n \geq 1$, is called a filtration on $(X^{\mathbb{Z}^d}, \mathscr{B}^{\mathbb{Z}^d})$. In particular, $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ forms a filtration corresponding to increasing sequence

$\{\Lambda_n\}_{n\geq 1}$ of finite sets converging to $\mathbb{Z}^d$. In what follows, we will consider only such filtrations, usually without explicitly specifying the corresponding sequence $\{\Lambda_n\}_{n\geq 1}$.

A function $f : X^S \to \mathbb{R}$, $S \subset \mathbb{Z}^d$, is called *local* if there exists $\Lambda \in W(S)$ such that for $x \in X^S$, the value $f(x)$ depends only on the restriction x_Λ of configuration x on Λ, that is, $f(uy) = f(uz)$ for any $u \in X^\Lambda$ and $y, z \in X^{S\setminus\Lambda}$.

A function $f : X^S \to \mathbb{R}$, $S \subset \mathbb{Z}^d$, is called *quasilocal* if it is the uniformly convergent limit of some sequence of local functions. Equivalently, f is quasilocal if

$$\lim_{\Lambda\uparrow S} \sup_{x,y\in X^S\,:\,x_\Lambda=y_\Lambda} \big|f(x) - f(y)\big| = 0.$$

We will say that f is quasilocal on $\mathscr{Y} \subset X^S$ if the relation above holds when supremum is taken over configurations $x, y \in \mathscr{Y}$ coinciding on Λ.

2 Gibbs random fields and Gibbs form of conditional densities

In this section, we show how the main results of [12] can be transferred to the case of positive random fields with Polish state space whose finite-dimensional distributions have densities. In what follows, by a random field we mean such a random field only.

We start with the definition of a Gibbs random field in terms of its one-point conditional densities. Further, we show that conditional densities of any random field admit a Gibbs form in terms of corresponding transition energies and present conditions on the system of transition energies under which the corresponding random field is Gibbsian. All results presented in this section can be obtained under appropriate (technical) restrictions in a manner similar to that used in [12], and therefore, their proofs will be omitted.

Let $P = \{p_V, V \in W\}$ be a random field and let

$$g_V^z(x) = \frac{p_{V\cup\Lambda}(xz)}{p_\Lambda(z)}, \qquad x \in X^V,$$

be its conditional density on X^V under condition $z \in X^\Lambda$, $\Lambda \in W(V^c)$, $V \in W$. Such densities will be called *finite-conditional densities.*

Consider filtration $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ where $\{\Lambda_n\}_{n\geq 1}$ is some increasing sequence of finite sets converging to $\mathbb{Z}^d$. For any fixed $V \in W$ and $x \in X^V$, let us consider a sequence of random variables $\xi_n : X^{V^c} \to \mathbb{R}$ defined by $\xi_n(\bar{x}) = q_V^{\bar{x}_{\Lambda_n\setminus V}}(x)$, $\bar{x} \in X^{V^c}$, $n \geq 1$. Then the sequence $(\xi_n, \mathscr{B}^{\Lambda_n})$ forms a martingale, and since $\sup\limits_{n\geq 1} E|\xi_n| = p_V(x) < \infty$, by the martingale convergence theorem, the limit

$$g_V^{\bar{x}}(x) = \lim_{n\to\infty} g_V^{\bar{x}_{\Lambda_n \setminus V}}(x), \tag{1}$$

exists for P-almost all $\bar{x} \in X^{V^c}$.

Limits (1) will be called *conditional densities* of P with respect to $\mathscr{F}$. Following the terminology accepted in mathematical statistical physics, any configuration $\bar{x}$ will be called a boundary condition.

For the filtration $\mathscr{F}$ and any $V \in W$, $x \in X^V$, denote by $\mathscr{X}_{V,x}(\mathscr{F})$ the set of all boundary conditions $\bar{x} \in X^{V^c}$ for which the limit in (1) exists, and let

$$\mathscr{X}_V(\mathscr{F}) = \bigcup_{x \in X^V} \mathscr{X}_{V,x}(\mathscr{F}).$$

We will call the elements of $\mathscr{X}_V(\mathscr{F})$, $V \in W$, *admissible boundary conditions* for the random field P with respect to filtration $\mathscr{F}$. It should be noted that, unlike the case of finite state space, the set $\mathscr{X}_V$ may not have P-measure 1.

The set $G(P,\mathscr{F}) = \{g_V^{\bar{x}}, \bar{x} \in \mathscr{X}_V(\mathscr{F}), V \in W\}$ will be called a *system of conditional densities of the random field P with respect to filtration $\mathscr{F}$*. The system $G(P,\mathscr{F})$ will be called (*strictly*) *positive* if its elements are positive, and will be called *quasilocal* if its elements are quasilocal as functions on the set of admissible boundary conditions, that is, if for any $V \in W$ and $x \in X^V$, it holds

$$\lim_{\Lambda \uparrow V^c} \sup_{\bar{x},\bar{y} \in \mathscr{X}_V(\mathscr{F}) : \bar{x}_\Lambda = \bar{y}_\Lambda} \left| g_V^{\bar{x}}(x) - g_V^{\bar{y}}(x) \right| = 0. \tag{2}$$

The system $G_1(P,\mathscr{F}) = \{g_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ of one-point conditional densities of the random field P (with respect to filtration $\mathscr{F}$) is of special interest. It can be shown (see [11] and [3]) that the positive system $G_1(P,\mathscr{F})$ restores $G(P,\mathscr{F})$, and $G(P,\mathscr{F})$ inherits such properties of $G_1(P,\mathscr{F})$ as positivity, homogeneity, quasilocality and Markov property. Therefore, in what follows, we will consider one-point conditional densities only.

The following statement holds true.

Proposition 1 *Let $G_1(P,\mathscr{F})$ be a system of one-point conditional densities of the random field P with respect to filtration $\mathscr{F}$. Then for any $t, s \in \mathbb{Z}^d$, $x, u \in X^t$, $y, v \in X^s$ and $\bar{x} \in \mathscr{X}_{\{t,s\}}(\mathscr{F})$, it holds*

$$g_t^{\bar{x}y}(x) g_s^{\bar{x}x}(v) g_t^{\bar{x}v}(u) g_s^{\bar{x}u}(y) = g_t^{\bar{x}y}(u) g_s^{\bar{x}u}(v) g_t^{\bar{x}v}(x) g_s^{\bar{x}x}(y).$$

2.1 Gibbs random fields

Following the approach developed in [12], we present the purely probabilistic definition (or Pr–definition) of a Gibbs random field under the assumptions of the present paper.

A random filed $P = \{p_V, V \in W\}$ will be called a *Gibbs random field* if there exists a filtration $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ and sets of admissible boundary conditions $\mathscr{Y}_t(\mathscr{F}) \subset \mathscr{X}_t(\mathscr{F})$, $P(\mathscr{Y}_t(\mathscr{F})) = 1$, $t \in \mathbb{Z}^d$, such that for all $t \in \mathbb{Z}^d$ and any $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$, the limits of finite-conditional densities

$$\lim_{n\to\infty} g_t^{\bar{x}_{\Lambda_n\setminus t}}(x) = \lim_{n\to\infty} \frac{p_{\Lambda_n}(x\bar{x}_{\Lambda_n\setminus t})}{p_{\Lambda_n\setminus t}(\bar{x}_{\Lambda_n\setminus t})} = g_t^{\bar{x}}(x), \qquad x \in X^t,$$

are positive and convergence with respect to $\bar{x}$ is uniform. Any such filtration $\mathscr{F}$ is called a *determining filtration*. Note that any Gibbs random field may have several determining filtrations.

Thus, for a random field P to be Gibbsian, it must have a version of its finite-dimensional densities $\{p_V, V \in W\}$ that satisfies the above conditions.

The following statements are analogous of the corresponding statements in [12] (Theorem 3 and Theorem 4, respectively).

Theorem 1 *Let P be a Gibbs random field, $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ be its determining filtration and $\mathscr{Y}_t(\mathscr{F})$, $t \in \mathbb{Z}^d$, be corresponding sets of admissible boundary conditions. Then for any filtration $\mathscr{F}' = \{\mathscr{B}^{\Lambda'_n}\}_{n\geq 1}$, the sequence of finite-conditional densities $\left\{g_t^{\bar{x}_{\Lambda'_n\setminus t}}(x)\right\}_{n\geq 1}$ converge (when $n \to \infty$) uniformly with respect to $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$ for all $x \in X^t$ and $t \in \mathbb{Z}^d$.*

Theorem 2 *A random field $P = \{p_V, V \in W\}$ is a Gibbs random field if there exists a filtration $\mathscr{F}$ such that for each $t \in \mathbb{Z}^d$, there is a set of admissible boundary conditions $\mathscr{Y}_t(\mathscr{F}) \in \mathscr{X}_t(\mathscr{F})$, $P(\mathscr{Y}_t(\mathscr{F})) = 1$, on which one-point conditional densities of P with respect to $\mathscr{F}$ are quasilocal, and for each $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$, one has $g_t^{\bar{x}}(x) > 0$, $x \in X^t$.*

Conversely, if $P = \{p_V, V \in W\}$ is a Gibbs random field, then there exist sets of boundary conditions $\mathscr{Y}_t \subset X^{t^c}$, $P(\mathscr{Y}_t) = 1$, $t \in \mathbb{Z}^d$, such that for any filtration $\mathscr{F}$ one-point conditional densities $G_1(P, \mathscr{F})$ are quasilocal on $\mathscr{Y}_t$, $t \in \mathbb{Z}^d$, and for each $\bar{x} \in \mathscr{Y}_t$, one has $g_t^{\bar{x}}(x) > 0$, $x \in X^t$.

2.2 Gibbs form of conditional densities

The concepts of a system of transition energies and a Hamiltonian of a random field with finite state space were considered in [12], while their autonomous (without appeal to the notion of a random field) analogues were introduced in [5]. The definitions of these objects can be easily transferred on the case considered in the present paper. Here we will consider one-point systems only.

Let $P = \{p_V, V \in W\}$ be a random field having a positive system of one-point conditional densities $G_1(P, \mathscr{F}) = \{g_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ with respect to some filtration $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$. Denote by $\mathbf{\Delta}_1(P,\mathscr{F}) = \{\Delta_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ the set of functions defined by

$$\Delta_t^{\bar{x}}(x,u) = \ln \frac{g_t^{\bar{x}}(x)}{g_t^{\bar{x}}(u)} = \lim_{n\to\infty} \ln \frac{p_{\Lambda_n}(x\bar{x}_{\Lambda_n\setminus t})}{p_{\Lambda_n}(u\bar{x}_{\Lambda_n\setminus t})}, \qquad x,u \in X^t.$$

It is not difficult to see that for all $t \in \mathbb{Z}^d$ and $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, the following equalities take place:

$$\Delta_t^{\bar{x}}(x,u) = \Delta_t^{\bar{x}}(x,y) + \Delta_t^{\bar{x}}(y,u), \qquad x,u,y \in X^t. \tag{3}$$

The set $\mathbf{\Delta}_1(P,\mathscr{F})$ will be called a *system of (one-point) transition energies* of the random field P with respect to filtration $\mathscr{F}$.

Let us also mention the following property of the system $\mathbf{\Delta}_1(P,\mathscr{F})$: for all $t,s \in \mathbb{Z}^d$ and $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, $x,u \in X^t$, $y,v \in X^s$, it holds

$$\Delta_t^{\bar{x}y}(x,u) + \Delta_s^{\bar{x}u}(y,v) = \Delta_s^{\bar{x}x}(y,v) + \Delta_t^{\bar{x}v}(x,u). \tag{4}$$

Both properties (3) and (4) can be interpreted as the energy conservation law.

The following statement directly follows from the definition of $\mathbf{\Delta}_1(P,\mathscr{F})$.

Theorem 3 *Let $P = \{p_V, V \in W\}$ be a random field with a positive system $G_1(P,\mathscr{F}) = \{g_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ and let $\mathbf{\Delta}_1(P,\mathscr{F}) = \{\Delta_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ be its system of transition energies. Then the elements of $G_1(P,\mathscr{F})$ necessarily have a Gibbs form*

$$g_t^{\bar{x}}(x) = \frac{\exp\{\Delta_t^{\bar{x}}(x,u)\}}{\int\limits_{\alpha\in X^t} \exp\{\Delta_t^{\bar{x}}(\alpha,u)\}\mu_t(d\alpha)}, \qquad x \in X^t, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d,$$

where $u \in X^t$.

Let $\mathbf{\Delta}_1(P,\mathscr{F}) = \{\Delta_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ be a system of transition energies of the random field P. Since the elements of $\mathbf{\Delta}_1(P,\mathscr{F})$ satisfy (3), for all $t \in \mathbb{Z}^d$ and $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, they can be presented in the form

$$\Delta_t^{\bar{x}}(x,u) = H_t^{\bar{x}}(u) - H_t^{\bar{x}}(x), \qquad x,u \in X^t.$$

The set of functions $H_1(P,\mathscr{F}) = \{H_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ forms a *one-point Hamiltonian* corresponding to the random field P (with respect to filtration $\mathscr{F}$). Due to (4), the elements of $H_1(P,\mathscr{F})$ satisfy the following consistency conditions: for all $t,s \in \mathbb{Z}^d$ and $\bar{x} \in \mathscr{X}_{\{t,s\}}(\mathscr{F})$, $x,u \in X^t$, $y,v \in X^s$, it holds

$$H_t^{\bar{x}y}(x) + H_s^{\bar{x}x}(v) + H_t^{\bar{x}v}(u) + H_s^{\bar{x}u}(y) = \\ = H_t^{\bar{x}y}(u) + H_s^{\bar{x}u}(v) + H_t^{\bar{x}v}(x) + H_s^{\bar{x}x}(y).$$

A direct corollary of Theorem 3 is the following result.

Corollary 1 *Let conditional densities* $G_1(P,\mathscr{F})$ *of the random field* $P = \{p_V, V \in W\}$ *be positive. Then they have a Gibbs form*

$$g_t^{\bar{x}}(x) = \frac{\exp\{-H_t^{\bar{x}}(x)\}}{\int\limits_{\alpha\in X^t} \exp\{-H_t^{\bar{x}}(\alpha)\}\mu_t(d\alpha)}, \qquad x \in X^t, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d,$$

where $H_1(P,\mathscr{F}) = \{H_t^{\bar{x}}, \bar{x} \in \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ *is a Hamiltonian corresponding to the random field* P*.*

Let us emphasize that Theorem 3 and Corollary 1 hold true for any random field with positive conditional densities. At the same time, according to the following proposition, quasilocality of transition energies as functions on boundary conditions is a sufficient condition for a random field to be Gibbsian.

Proposition 2 *A random field* $P = \{p_V, V \in W\}$ *is a Gibbs random field if there exists a filtration* $\mathscr{F}$ *such that for all* $t \in \mathbb{Z}^d$*, there is a set of admissible boundary conditions* $\mathscr{Y}_t(\mathscr{F})$*,* $P(\mathscr{Y}_t(\mathscr{F})) = 1$*, on which the elements of* $\mathbf{\Delta}_1(P,\mathscr{F})$ *are quasilocal, that is,*

$$\lim_{\Lambda\uparrow t^c} \sup_{\bar{x},\bar{y}\in\mathscr{Y}_t(\mathscr{F}):\bar{x}_\Lambda=\bar{y}_\Lambda} \left|\Delta_t^{\bar{x}}(x,u) - \Delta_t^{\bar{y}}(x,u)\right| = 0. \qquad x, u \in X^t,$$

3 Gibbsian description of Gaussian random fields

In this section, we apply the approach presented above to Gaussian random fields. First, we compute corresponding systems of one-point conditional densities and one-point transition energies. Further, we present conditions on the covariances under which the corresponding Gaussian random field is Gibbsian. Finally, we consider the case of Gaussian Markov random fields.

3.1 Conditional densities and transition energies of a Gaussian random field

Let P be a Gaussian random field with a vector of mean values $m = \{m_t, t \in \mathbb{Z}^d\}$ and covariance matrix $C = \{c_{ts}, t, s \in \mathbb{Z}^d\}$. We consider non-degenerate Gaussian random fields only, that is, such random fields that $c_{tt} > 0$ for any $t \in \mathbb{Z}^d$.

For each $\Lambda \in W$, denote $C_\Lambda = \{c_{ts}, t, s \in \Lambda\}$, and let $A_\Lambda = \{a_{ts}^\Lambda, t, s \in \Lambda\} = C_\Lambda^{-1}$. The matrices C_Λ and A_Λ are symmetric and positive definite. In particular, it follows that $\det C_\Lambda = (\det A_\Lambda)^{-1} > 0$, and all diagonal elements c_{ss} and a_{ss}^Λ, $s \in \Lambda$, are positive.

According to the next proposition, the elements of inverse matrices do not depend on the order of adding points to Λ (the proof is given in Appendix).

Proposition 3 *For any $t, s \in \mathbb{Z}^d$ and $\Lambda \in W(\{t,s\}^c)$, matrices $A_{t\cup s\cup\Lambda}$ and $A_{s\cup t\cup\Lambda}$ differ only by the order of first two rows and columns, that is,*

$$a_{ij}^{t\cup s\cup\Lambda} = a_{ij}^{s\cup t\cup\Lambda}, \qquad i,j \in \{t,s\}\cup\Lambda.$$

The following proposition establishes the recurrent relations between the elements of inverse matrices $A_{t\cup\Lambda}$ and A_Λ (see Appendix for the proof).

Proposition 4 *Let $A_\Lambda = \{a_{sk}^\Lambda, s,k\in\Lambda\}$ be the inverse of the matrix $C_\Lambda = \{c_{sk}, s,k \in \Lambda\}$, $\Lambda \in W$. Then for any $t \in \Lambda^c$, the elements of $A_{t\cup\Lambda} = \{a_{sk}^{t\cup\Lambda}, s,k \in t\cup\Lambda\} = C_{t\cup\Lambda}^{-1}$ have the following form:*

$$a_{tt}^{t\cup\Lambda} = \frac{\det C_\Lambda}{\det C_{t\cup\Lambda}} = \Big(c_{tt} - \sum_{i,j\in\Lambda} a_{ij}^\Lambda c_{ti} c_{tj}\Big)^{-1}, \tag{5}$$

$$a_{ts}^{t\cup\Lambda} = -\frac{\det C_\Lambda}{\det C_{t\cup\Lambda}} \sum_{j\in\Lambda} a_{sj}^\Lambda c_{tj}, \qquad s\in\Lambda, \tag{6}$$

$$a_{sk}^{t\cup\Lambda} = a_{sk}^\Lambda + \frac{\det C_\Lambda}{\det C_{t\cup\Lambda}} \sum_{i,j\in\Lambda} a_{si}^\Lambda a_{jk}^\Lambda c_{ti} c_{tj}, \qquad s,k\in\Lambda. \tag{7}$$

Finite-dimensional densities of Gaussian random field P have the following form:

$$p_\Lambda(x) = \sqrt{\frac{\det A_\Lambda}{(2\pi)^{|\Lambda|}}} \exp\Big\{-\frac{1}{2}\sum_{t,s\in\Lambda} a_{ts}^\Lambda (x_t - m_t)(x_s - m_s)\Big\}, \quad x \in \mathbb{R}^\Lambda, \Lambda \in W.$$

Let us note that using relations from Proposition 4, one can directly check their consistency in Kolmogorov's sense without appealing to characteristic functions or other technics.

Suppose for a Gaussian random field P, there exists an increasing sequence $\{\Lambda_n\}_{n\geq 1}$ of finite sets converging to $\mathbb{Z}^d$ such that for all $t\in\mathbb{Z}^d$, the following limit exists:

$$a_{tt} = \lim_{n\to\infty} a_{tt}^{\Lambda_n}. \tag{8}$$

Let $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ be the filtration generated by the sequence $\{\Lambda_n\}_{n\geq 1}$, and let $G_1(P,\mathscr{F})$ be the corresponding system of one-point conditional densities.

Proposition 5 *The elements of the system $G_1(P,\mathscr{F}) = \{g_t^{\bar{x}}, \mathscr{X}_t(\mathscr{F}), t \in \mathbb{Z}^d\}$ have the following form:*

$$g_t^{\bar{x}}(x) = \sqrt{\frac{a_{tt}}{2\pi}} \exp\Big\{-\frac{a_{tt}}{2}\Big((x-m_t) + \frac{1}{a_{tt}} \lim_{n\to\infty} \sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s)\Big)^2\Big\},$$

$x \in \mathbb{R}^t$, and for each $t \in \mathbb{Z}^d$, the set of admissible boundary conditions $\mathscr{X}_t(\mathscr{F})$ is the set of those configurations $\bar{x} \in \mathbb{R}^{t^c}$ for which there exist limits

$$\lim_{n\to\infty} \sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s). \tag{9}$$

If $a_{tt} > 0$ for all $t \in \mathbb{Z}^d$, then system $G_1(P, \mathscr{F})$ is positive.

Proof. For all $t \in \mathbb{Z}^d$, $\Lambda \in W(t^c)$ and $x \in \mathbb{R}^t$, $z \in \mathbb{R}^\Lambda$, we have

$$\begin{aligned} g_t^z(x) &= \frac{p_{t\cup\Lambda}(xz)}{p_\Lambda(z)} = \frac{p_{t\cup\Lambda}(xz)}{\int\limits_{\mathbb{R}} p_{t\cup\Lambda}(uz)du} = \\ &= \sqrt{\frac{a_{tt}^{t\cup\Lambda}}{2\pi}} \exp\Big\{ -\frac{a_{tt}^{t\cup\Lambda}}{2}(x-m_t)^2 - (x-m_t)\sum_{s\in\Lambda} a_{ts}^{t\cup\Lambda}(z_s - m_s) - \\ &\qquad -\frac{1}{2a_{tt}^{t\cup\Lambda}}\Big(\sum_{s\in\Lambda} a_{ts}^{t\cup\Lambda}(z_s-m_s)\Big)^2\Big\} = \\ &= \sqrt{\frac{a_{tt}^{t\cup\Lambda}}{2\pi}} \exp\Big\{ -\frac{a_{tt}^{t\cup\Lambda}}{2}\Big((x-m_t) + \frac{1}{a_{tt}^{t\cup\Lambda}}\sum_{s\in\Lambda} a_{ts}^{t\cup\Lambda}(z_s-m_s)\Big)^2\Big\}. \end{aligned}$$

It remains to note that by the martingale convergence theorem, for all $t \in \mathbb{Z}^d$, $x \in \mathbb{R}^t$ and P-a.e. $\bar{x} \in \mathbb{R}^{t^c}$, there exist conditional densities

$$\begin{aligned} g_t^{\bar{x}}(x) &= \lim_{n\to\infty} g_t^{\bar{x}_{\Lambda_n\setminus t}}(x) = \\ &= \sqrt{\frac{a_{tt}}{2\pi}} \exp\Big\{ -\frac{a_{tt}}{2}\Big((x-m_t) + \frac{1}{a_{tt}}\lim_{n\to\infty}\sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s-m_s)\Big)^2\Big\}. \end{aligned}$$

In particular, from here it follows the existence of limits (9). On the other hand, the existence of limits (9) on a set of probability 1 guarantees the convergence of corresponding finite-conditional densities. Thus, the sets of admissible boundary conditions are $\mathscr{X}_t(\mathscr{F})$, $t \in \mathbb{Z}^d$. Further, from the existence of limits (9), we conclude that there are positive numbers N_t, $t \in \mathbb{Z}^d$, such that for all $n \geq 1$,

$$\Big|\sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s)\Big| < N_t, \qquad \bar{x} \in \mathscr{X}_t(\mathscr{F}).$$

If $a_{tt} > 0$, $t \in \mathbb{Z}^d$, then for all $x \in \mathbb{R}^t$ and $\mathscr{X}_t(\mathscr{F})$, we have

$$g_t^{\bar{x}}(x) > \sqrt{\frac{a_{tt}}{2\pi}} \exp\Big\{ -\frac{a_{tt}}{2}\left(|x-m_t| + \frac{N_t}{a_{tt}}\right)^2\Big\} > 0.$$

□

The following statement presents the corresponding transition energies.

Proposition 6 *Let Gaussian random field P be such that $a_{tt} > 0$, $t \in \mathbb{Z}^d$. Then the elements of its system $\mathbf{\Delta}_1(P, \mathscr{F})$ have the following form:*

$$\Delta_t^{\bar{x}}(x,u) = \\ = \frac{a_{tt}}{2}\left((u-m_t)^2 - (x-m_t)^2\right) + (u-x)\lim_{n\to\infty}\sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s), \quad (10)$$

$x, u \in \mathbb{R}^t$, $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, $t \in \mathbb{Z}^d$.

Proof. For each $t \in \mathbb{Z}^d$, $\Lambda \in W(t^c)$ and $x, u \in \mathbb{R}^t$, $z \in \mathbb{R}^\Lambda$, we have

$$\ln\frac{p_{t\cup\Lambda}(xz)}{p_{t\cup\Lambda}(uz)} = \frac{a_{tt}^{t\cup\Lambda}}{2}\left((u-m_t)^2 - (x-m_t)^2\right) + (u-x)\sum_{s\in\Lambda} a_{ts}^{t\cup\Lambda}(z_s - m_s).$$

Hence, for all $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, we obtain

$$\Delta_t^{\bar{x}}(x,u) = \lim_{n\to\infty}\ln\frac{p_{t\cup\Lambda_n}(x\bar{x}_{\Lambda_n})}{p_{t\cup\Lambda_n}(u\bar{x}_{\Lambda_n})} = \\ = \frac{a_{tt}}{2}\left((u-m_t)^2 - (x-m_t)^2\right) + (u-x)\lim_{n\to\infty}\sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s).$$

□

From (10), it follows that a Hamiltonian $H_1(P, \mathscr{F})$ of the Gaussain random field can be defined in the following way:

$$\begin{aligned} H_t^{\bar{x}}(x) &= \frac{a_{tt}}{2}(x-m_t)^2 + (x-m_t)\lim_{n\to\infty}\sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s) = \\ &= \frac{1}{2}\lim_{n\to\infty}\sum_{s\in\Lambda_n} a_{ts}^{\Lambda_n}(x-m_t)(\alpha_s - m_s), \end{aligned}$$

where $\alpha_t = x$, $\alpha_s = \bar{x}_s$, $s \in t^c$, and $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, $x \in \mathbb{R}^t$, $t \in \mathbb{Z}^d$.

3.2 Gibbsianess of Gaussian random fields

Now we can apply the suggested definition of a Gibbs random field to the case of Gaussian random fields.

Theorem 4 *Let P be a Gaussian random field with covariance matrix C and a vector of mean values m. Suppose there is an increasing sequence $\{\Lambda_n\}_{n\geq 1}$ of finite sets converging to $\mathbb{Z}^d$ such that limits*

$$a_{tt} = \lim_{n\to\infty}\frac{\det C_{\Lambda_n}}{\det C_{t\cup\Lambda_n}}, \qquad t \in \mathbb{Z}^d,$$

exist and are positive. If there exist sets of admissible boundary conditions $\mathscr{Y}_t(\mathscr{F}) \subset \mathscr{X}_t(\mathscr{F})$, $P(\mathscr{Y}_t(\mathscr{F})) = 1$, $t \in \mathbb{Z}^d$, such that for all $t \in \mathbb{Z}^d$, the convergence in limits

$$\lim_{n\to\infty} \sum_{s\in\Lambda_n\setminus t} a_{ts}^{\Lambda_n}(\bar{x}_s - m_s),$$

where $a_{ts}^{\Lambda_n} = (C_{\Lambda_n}^{-1})_{ts}$, is uniform with respect to $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$, then P is a Gibbs random field.

Proof. Let $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ be the filtration generated by the sequence $\{\Lambda_n\}_{n\geq 1}$. Since $a_{tt} > 0$, $t \in \mathbb{Z}^d$, by Proposition 5, the corresponding system $G_1(P, \mathscr{F})$ is positive, and by Proposition 6, the elements of $\mathbf{\Delta}_1(P, \mathscr{F})$ have the form (10). Denote

$$\Delta_t^{\bar{x}\Lambda_n}(x,u) = \frac{a_{tt}^{t\cup\Lambda_n}}{2}\left((u-m_t)^2 - (x-m_t)^2\right) + (u-x)\sum_{s\in\Lambda_n} a_{ts}^{t\cup\Lambda_n}(z_s - m_s),$$

$x, u \in \mathbb{R}^t$, $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$, $t \in \mathbb{Z}^d$, $n \geq 1$. Due to the uniform (in $\bar{x}$) convergence of limits (9), each function $\Delta_t^{\bar{x}\Lambda_n}$ converges to $\Delta_t^{\bar{x}}$ uniformly in $\bar{x} \in \mathscr{Y}_t(\mathscr{F})$. Therefore, $\mathbf{\Delta}_1(P, \mathscr{F})$ is quasilocal, and thus, according to Proposition (2), P is Gibbsian. $\square$

Let us briefly recall the previous results before comparing them with the ones obtained.

Sinai [17] considered a stationary Gaussian random field P whose spectral density satisfies certain conditions and showed that such random fields admit a Gibbs representation with a Hamiltonian written in terms of the Fourier coefficients of the function on the spectral density of P.

Dobrushin [7] studied the same problem in more detail. First, he obtained the form of conditional densities of a stationary Gaussian random field P by considering the corresponding characteristic functional. He then showed that for a centered Gaussian random field, these densities are representable in Gibbs form with a Hamiltonian corresponding to a pair quadratic potential with parameters depending on the Fourier coefficients $U = \{U(t), t \in \mathbb{Z}^d\}$ of the spectral density of P. In this case, certain conditions for the convergence of such potential are naturally imposed. Dobrushin's main result (Theorem 4.1 in [7]) consists of describing the set of Gibbsian random fields corresponding to a certain Hamiltonian depending on U and some real number. Particularly, it follows that a centered stationary Gaussian random field is Gibbsian (see Remark 1 in [7]).

Georgii [10] considered Gaussian random fields with arbitrary mean vectors in both the Markov and general cases. For Markov Gaussian random fields, he shows that their conditional densities are representable in Gibbs form with a Hamiltonian whose parameters can be expressed in terms of the conditional covariances and means of the random fields. In the case of an arbitrary Gaussian random field, conditional densities with admissible boundary conditions are representable in Gibbs form by some vector $h = (h_t, t \in \mathbb{Z}^d)$ and a symmetric positive definite matrix $J = \{J_{ts}, t, s \in \mathbb{Z}^d\}$. According to Georgii, for each $\Lambda \in W$, the restriction J_Λ of matrix J on Λ can be considered as the inverse of the covariance matrix C_Λ of the Gaussian random field, and the vector h is defined as the solution to some equation. Thus, Georgii [10] generalizes Dobrushin's result to the case of non-stationary random fields, namely, he shows that centered Gaussian random fields admit a Gibbsian representation of their conditional densities.

In all mentioned works, a set of admissible boundary conditions must be specified (in terms of Fourier coefficients U in [7] and [17], or through infinite matrix J in [10]). Within the framework of the DLR–definition, one has to modify the definition of a Gibbs random field correspondingly, namely, to define potential on the set of admissible boundary conditions only (see, for example, the remarks after formula (13.11) in [10]). At the same time, the Pr–definition turns out to be more appropriate: we obtained a similar result (Theorem 4) as a direct corollary of the suggested Pr–definition of Gibbs random fields.

As for Hamiltonian, in all works it has the same quadratic form, differing in the meaning of the coefficients. In previous works, the Hamiltonian was defined ambiguously and depended on the additional parameter h, which is the solution of an infinite system of equations. At the same time, within the framework of our approach, the Hamiltonian can be explicitly written directly through the parameters m and C of the Gaussian random field. In addition, we obtained Hamiltonian as a result of direct calculations, and we do not need to represent it as a sum of interactions since it is indeed a Hamiltonian in the sense of axiomatic definition introduced in [5].

Thus, the approach proposed in the present paper allows one to obtain a Gibbs representation of the conditional densities of a given Gaussian random field quite simply by means of direct calculations. The conditions of Gibbsianess of a Gaussian random field are given in terms of its covariances and follows directly from the suggested Pr–definition of a Gibbs random field.

3.2.1 Gaussian Markov random fields

There are several works in which the problem of Gibbsianness was considered for Gaussian Markov random fields (see, for example, [10, 14–16]). To solve this problem, one has to prove the fact that Gaussian Markov random fields are Gibbsian with a suitable finite-range potential. Within the framework of Pr–definition, the answer to this question directly follows from the definition of a Gibbs random field: all Gaussian Markov random fields are Gibbsian.

Let us remind the definition of Markov random fields. It is based on the notion of neighborhood system in $\mathbb{Z}^d$, which is a collection $\partial = \{\partial t, t \in \mathbb{Z}^d\}$ of finite subsets of the lattice $\mathbb{Z}^d$ such that $t \notin \partial t$ and $s \in \partial t$ if and only if $t \in \partial s$, $s \in \mathbb{Z}^d$.

A random field P is called a *Markov random field* (with respect to the neighborhood system ∂) if for any filtration $\Im$, its one-point conditional densities $G_1(P, \mathscr{F})$ posses the following (Markov) property: for all $t \in \mathbb{Z}^d$ and $\bar{x} \in \mathscr{X}_t(\mathscr{F})$, it holds

$$g_t^{\bar{x}}(x) = g_t^{\bar{x}_{\partial t}}(x), \qquad x \in X^t.$$

The equivalent condition for a random field to be Markovian can be formulated in terms of finite-conditional densities (see, for example, Proposition 12 in [11]) as follows: for any $t \in \mathbb{Z}^d$ and $\Lambda \in W(t^c)$ such that $\partial t \subset \Lambda$, it holds

$$g_t^{z}(x) = g_t^{z_{\partial t}}(x), \qquad x \in X^t, z \in X^\Lambda.$$

Note that for any Markov random field P, all systems of conditional distributions constructed with respect to different filtrations $\mathscr{F}$ coincide, and the sets of admissible boundary conditions are $\mathscr{X}_t(\mathscr{F}) = X^{t^c}$, $t \in \mathbb{Z}^d$. Thus, the system of one-point conditional densities of the Markov random field P is the set $G_1(P) = \{g_t^{\bar{x}}, \bar{x} \in X^{t^c}, t \in \mathbb{Z}^d\}$.

It is clear that conditional densities of any Markov random field satisfy (2), i.e., are quasilocal. Thus, by Criterioan 2, any Markov random field with positive conditional densities is Gibbsian. For Gaussian Markov random fields, this result can also be obtained by application of Theorem 4, which will be shown in the remainder of the subsection.

Let us present the conditions under which a Gaussian random field is Markovian (see also Proposition 13.7 in [10]).

Theorem 5 *Gaussian random field P is Markovian if and only if the elements of corresponding inverse matrices A_Λ, $\Lambda \in W$, satisfy the following conditions: for all $t \in \mathbb{Z}^d$ and $\Lambda \in W(t^c)$ such that $\partial t \subset \Lambda$, it holds*

$$a_{tt}^{t\cup\Lambda} = a_{tt}^{t\cup\partial t}, \qquad a_{ts}^{t\cup\Lambda} = \begin{cases} a_{ts}^{t\cup\partial t}, & s \in \partial t, \\ 0, & s \in \Lambda \backslash \partial t. \end{cases} \tag{11}$$

Proof. Let P be a Gaussian random field such that the elements of corresponding inverse matrices A_Λ, $\Lambda \in W$, satisfy conditions (11). Then for all $t \in \mathbb{Z}^d$ and $\Lambda \in W(t^c)$ such that $\partial t \subset \Lambda$ and any $x \in \mathbb{R}^t$, $z \in \mathbb{R}^\Lambda$, we can write

$$g_t^z(x) = \sqrt{\frac{a_{tt}^{t\cup\Lambda}}{2\pi}} \exp\Big\{ -\frac{a_{tt}^{t\cup\Lambda}}{2}\Big((x-m_t) + \frac{1}{a_{tt}^{t\cup\Lambda}} \sum_{s\in\Lambda} a_{ts}^{t\cup\Lambda}(z_s - m_s)\Big)^2\Big\} =$$
$$= \sqrt{\frac{a_{tt}^{t\cup\partial t}}{2\pi}} \exp\Big\{ -\frac{a_{tt}^{t\cup\partial t}}{2}\Big((x-m_t) + \frac{1}{a_{tt}^{t\cup\partial t}} \sum_{s\in\partial t} a_{ts}^{t\cup\partial t}(z_s - m_s)\Big)^2\Big\} = g_t^{z_{\partial t}}(x).$$

Hence, P is Markovian.

Conversely, let the Gaussian random field P be Markovian. Then for any $t \in \mathbb{Z}^d$ and $\Lambda \in W(t^c)$ such that $\partial t \subset \Lambda$, we have

$$g_t^z(x) = g_t^{z_{\partial t}}(x), \qquad x \in \mathbb{R}^t, z \in \mathbb{R}^\Lambda.$$

Taking into account the positivity of finite-conditional distributions of Gaussian random field, from here for any $x, u \in \mathbb{R}^t$, we obtain

$$\ln \frac{g_t^z(x)}{g_t^z(u)} = \ln \frac{g_t^{z_{\partial t}}(x)}{g_t^{z_{\partial t}}(u)},$$

or, equivalently,

$$\frac{a_{tt}^{t\cup\Lambda}}{2}(u^2 - x^2) + (u-x)\sum_{s\in\Lambda} a_{ts}^{t\cup\Lambda} z_s = \frac{a_{tt}^{t\cup\partial t}}{2}(u^2 - x^2) + (u-x)\sum_{s\in\partial t} a_{ts}^{t\cup\partial t} z_s.$$

Since the last equality must be satisfied for all $z \in \Lambda$, and, particularly, for $z_s = 0$, $s \in \Lambda$, we conclude that $a_{tt}^{t\cup\Lambda} = a_{tt}^{t\cup\partial t}$. Therefore, for any $z \in \mathbb{R}^\Lambda$, it holds

$$\sum_{s\in\Lambda} a_{ts}^{t\cup\Lambda} z_s = \sum_{s\in\partial t} a_{ts}^{t\cup\partial t} z_s.$$

Choosing different values for z, one can obtain the required relations (11). $\square$

Thus, for the Gaussian Markov random field P, conditions of Theorem 4 are satisfied for any filtration $\mathscr{F} = \{\mathscr{B}^{\Lambda_n}\}_{n\geq 1}$ and sets of admissible boundary conditions $\mathscr{Y}_t(\mathscr{F}) = \mathbb{R}^{t^c}$, $t \in \mathbb{Z}^d$. Therefore, P is Gibbsian. The elements of its Hamiltonian can be written in the following form:

$$H_t^{\bar{x}}(x) = \frac{a_{tt}^{t\cup\partial t}}{2}(x - m_t)^2 + (x - m_t)\sum_{s\in\partial t} a_{ts}^{t\cup\partial t}(\bar{x}_s - m_s), \; x \in \mathbb{R}^t, \bar{x} \in \mathbb{R}^{t^c}, t \in \mathbb{Z}^d.$$

Appendix

Proof of Proposition 3 For any $t, s \in \mathbb{Z}^d$ and $\Lambda \in W(\{t, s\}^c)$, denote by $E_{t,s,\Lambda}$ the matrix which is obtained from the identity matrix $I_{t\cup s\cup\Lambda}$ by permuting first two rows indexed by t and s:

$$E_{t,s,\Lambda} = \begin{pmatrix} 0 & 1 & \mathbf{0}_\Lambda \\ 1 & 0 & \mathbf{0}_\Lambda \\ \mathbf{0}_\Lambda^T & \mathbf{0}_\Lambda^T & I_\Lambda \end{pmatrix},$$

where $\mathbf{0}_\Lambda$ is the zero vector of length $|\Lambda|$. Then

$$\begin{aligned} A_{s\cup t\cup\Lambda} = C_{s\cup t\cup\Lambda}^{-1} = (E_{t,s,\Lambda} C_{t\cup s\cup\Lambda} E_{t,s,\Lambda})^{-1} = \\ = E_{t,s,\Lambda}^{-1} C_{t\cup s\cup\Lambda}^{-1} E_{t,s,\Lambda}^{-1} = E_{t,s,\Lambda} A_{t\cup s\cup\Lambda} E_{t,s,\Lambda}, \end{aligned}$$

since $E_{t,s,\Lambda}^{-1} = E_{t,s,\Lambda}$. $\square$

Proof of Proposition 4 Let $A_\Lambda = C_\Lambda^{-1}$, $\Lambda \in W$, and let us compute $A_{t\cup\Lambda} = C_{t\cup\Lambda}^{-1}$, $t \in \Lambda^c$.

Enumerate the points in Λ: $1, 2, ..., n$, $n = |\Lambda|$, and assign number 0 to point t. Then $C_\Lambda = C_n = \{c_{ij}, 1 \le i, j \le n\}$, $A_\Lambda = A_n = \{a_{ij}^{(n)}, 1 \le i, j \le n\}$, and $C_{t\cup\Lambda} = C_{n+1} = \{c_{ij}, 0 \le i, j \le n\}$, $A_{t\cup\Lambda} = A_{n+1} = \{a_{ij}^{(n+1)}, 0 \le i, j \le n\}$. Matrix C_{n+1} can be represented as follows:

$$C_{n+1} = \begin{pmatrix} c_{00} & (c_{0j}) \\ (c_{0j})^T & C_n \end{pmatrix},$$

where $(c_{0j}) = (c_{0j}, 1 \le j \le n)$ is a $1 \times n$ row vector and $(c_{0j})^T = (c_{0j}, 1 \le j \le n)^T$ is an $n \times 1$ column vector.

First note that according to the formula for calculating the determinant of a block matrix (see, for example, [9]), we have

$$\det C_{n+1} = \det C_n \det K,$$

where

$$K = c_{00} - (c_{0j}) \cdot C_n^{-1} \cdot (c_{0j})^T = c_{00} - \sum_{i,j=1}^{n} a_{ij}^{(n)} c_{0i} c_{0j},$$

and hence,

$$\frac{\det C_{n+1}}{\det C_n} = \det K = K = c_{00} - \sum_{i,j=1}^{n} a_{ij}^{(n)} c_{0i} c_{0j}.$$

According to the Frobenius formula for the inversion of a block matrix (see, for example, [9])

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} K^{-1} & -K^{-1}BD^{-1} \\ -D^{-1}CK^{-1} & D^{-1} + D^{-1}CK^{-1}BD^{-1} \end{pmatrix},$$

where $K = A - BD^{-1}C$, we have

$$A_{n+1} = C_{n+1}^{-1} =$$

$$= \begin{pmatrix} K^{-1} & -K^{-1} \cdot (c_{0j}) \cdot C_n^{-1} \\ -C_n^{-1} \cdot (c_{0j})^T \cdot K^{-1} & C_n^{-1} + C_n^{-1} \cdot (c_{0j})^T \cdot K^{-1} \cdot (c_{0j}) \cdot C_n^{-1} \end{pmatrix} =$$

$$= \frac{\det C_n}{\det C_{n+1}} \cdot \begin{pmatrix} 1 & -(c_{0j}) \cdot A_n \\ -A_n \cdot (c_{0j})^T & K \cdot A_n + A_n \cdot (c_{0j})^T \cdot (c_{0j}) \cdot A_n \end{pmatrix}.$$

Further,

$$(c_{0j}) \cdot A_n = \left(b_{0j}^{(n)}, 1 \le j \le n\right), \qquad A_n \cdot (c_{0j})^T = \left(b_{0j}^{(n)}, 1 \le j \le n\right)^T,$$

where

$$b_{0j}^{(n)} = \sum_{i=1}^{n} a_{ij}^{(n)} c_{0i}, \qquad 1 \le j \le n.$$

Thus, matrix A_{n+1} takes the form

$$\begin{pmatrix} \dfrac{\det C_n}{\det C_{n+1}} & -\dfrac{\det C_n}{\det C_{n+1}} b_{01}^{(n)} & \dots & -\dfrac{\det C_n}{\det C_{n+1}} b_{0n}^{(n)} \\ -\dfrac{\det C_n}{\det C_{n+1}} b_{01}^{(n)} & a_{11}^{(n)} + \dfrac{\det C_n}{\det C_{n+1}} \left(b_{01}^{(n)}\right)^2 & \dots & a_{1n}^{(n)} + \dfrac{\det C_n}{\det C_{n+1}} b_{01}^{(n)} b_{0n}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{\det C_n}{\det C_{n+1}} b_{0n}^{(n)} & a_{1n}^{(n)} + \dfrac{\det C_n}{\det C_{n+1}} b_{0n}^{(n)} b_{01}^{(n)} & \dots & a_{nn}^{(n)} + \dfrac{\det C_n}{\det C_{n+1}} \left(b_{0n}^{(n)}\right)^2 \end{pmatrix}.$$

Therefore,

$$a_{00} = \frac{\det C_n}{\det C_{n+1}}, \tag{12}$$

$$a_{0k}^{(n+1)} = -\frac{\det C_n}{\det C_{n+1}} \sum_{j=1}^{n} a_{kj}^{(n)} c_{0j}, \quad 1 \le k \le n, \tag{13}$$

$$a_{km}^{(n+1)} = a_{km}^{(n)} + \frac{\det C_n}{\det C_{n+1}} \sum_{i,j=1}^{n} a_{ki}^{(n)} a_{mj}^{(n)} c_{0i} c_{0j}, \quad 1 \le k, m \le n. \tag{14}$$

It remains to note that according to Proposition 3, the elements of matrix A_{n+1} do not depend on the enumeration of points in Λ. Hence, formulas (12)–(14) can be written in the form (5)–(7). $\square$

Acknowledgments. The author is grateful to Boris S. Nahapetian for the attention to the work and valuable remarks. The author also thanks the anonymous reviewers for their careful reading of the paper and important comments. The work was supported by the Science Committee of the Republic of Armenia in the frames of the research project 21AG-1A045.

References

[1] M.B. Averintsev, One method of describing random fields with a discrete argument. Problems Inform. Transmission, **6** (1970), no. 2, pp. 169–175.

[2] M.B. Averintsev, Description of Markovian random fields by Gibbsian conditional probabilities. Theory Probab. Appl., **17** (1973), no. 1, pp. 20–33 https://doi.org/10.1137/1117002

[3] S. Dachian and B.S. Nahapetian, Description of specifications by means of probability distributions in small volumes under condition of very week positivity. J. Stat. Phys., **117** (2004), pp. 281–300 . https://doi.org/10.1023/b:joss.0000044069.91072.0b

[4] S. Dachian and B.S. Nahapetian, On Gibbsiannes of random fields, Markov Process. Relat. Fields, **15** (2009), no. 1, pp. 81–104.

[5] S. Dachian and B.S. Nahapetian, On the relationship of energy and probability in models of classical statistical physics. Markov Processes Relat. Fields, **25** (2019), no. 4, pp. 649–681.

[6] R.L. Dobrushin, Gibbs random fields for lattice systems with pairwise interaction. Funct. Anal. and Appl., **2** (1968), pp. 292–301. https://doi.org/10.1007/BF01075681

[7] R.L. Dobrushin, Gaussian random fields — Gibbsian point of view. In: *Multicomponent random systems*, eds. R.L. Dobrushin, Ya. G. Sinai, New York: M. Dekker, 1980, pp. 119–151 (Translation of *Multicomponent random systems* (in Russian), Nauka, Moscow, 1978)

[8] R. Fernández and G. Maillard, Construction of a specification from its singleton part. ALEA Lat. Am. J. Probab. Math. Stat., **2** (2006), pp. 297–315.

[9] F.R. Gantmacher, *The Theory of Matrices*. Chelsea Publishing Co., 1960.

[10] H.-O. Georgii, *Gibbs measures and phase transitions*, De Gruyter, Berlin, 1988.

[11] L.A. Khachatryan, Description of lattice random fields by systems of conditional distributions. Armen. J. Math., **14** (2022), no. 8, pp. 1–40. https://doi.org/10.52737/18291163-2022.14.8-1-40

[12] L.A. Khachatryan and B.S. Nahapetian, Gibbs scheme in the theory of random fields. Ann. Henri Poincare (2025), 27 pp. https://doi.org/10.1007/s00023-025-01573-z

[13] O.K. Kozlov, Gibbs description of a system of random variables. Problems Inform. Transmission, **10** (1974), pp. 258–265.

[14] H. Künsch, Gaussian Markov random fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math., **26** (1979), pp. 53–73.

[15] H. Künsch, Reellwertige Zufallsfelder auf einem Gitter: Interpolationsprobleme, Variationsprinzip und statistische Analyse. Dissertation, 1980.

[16] H. Künsch, Thermodynamics and statistical analysis of Gaussian random fields. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **58** (1981), pp. 407–421. https://doi.org/10.1007/BF00542645

[17] Ya.G. Sinai, *Theory of Phase Transitions — Rigorous Results*. Pergamon, Oxford, 1982.

[18] W.G. Sullivan, Potentials for almost Markovian random fields. Commun. Math. Phys., **33** (1973), pp. 61–74. http://dx.doi.org/10.1007/BF01645607

Linda A. Khachatryan
Institute of Mathematics,
National Academy of Sciences of the Republic of Armenia,
24/5 Marshal Baghramian ave., Yerevan, 0019, Armenia
linda@instmath.sci.am

Please, cite to this paper as published in
Armen. J. Math., V. **17**, N. 13(2025), pp. 1–20
https://doi.org/10.52737/18291163-2025.17.13-1-20