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Isotropic Basis in a Real Symmetric Bilinear Space: Existence and Construction

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Abstract. A basis of a real symmetric bilinear space is called an isotropic basis if all its elements are isotropic. In this paper, we provide both necessary and sufficient conditions for the existence of such an isotropic basis. We present one geometric method and two linear algebraic methods for constructing isotropic bases. Additionally, we address a question arising from the properties of symmetric bilinear forms. As a consequence, we explore various properties of the vector space spanned by the preimage set of a point under a real-valued continuous function. We also demonstrate some applications of these properties within the context of real symmetric bilinear spaces.

Key Words: Symmetric Bilinear Form, Symmetric Bilinear Space, Isotropic Basis, Preimage Dimension
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Introduction

The theory of symmetric bilinear forms, closely related to quadratic forms, is one of the most intriguing areas of linear algebra (see, for example, [10, 14, 18]). A real vector space equipped with a symmetric bilinear form is referred to as a real symmetric bilinear space or simply a symmetric bilinear space (see Definition 1). Within this context, a non-zero element in a symmetric bilinear space can be classified as isotropic, positive, or negative (see Definition 2). A basis of a symmetric bilinear space is called isotropic (positive or negative) if all elements in that basis are isotropic (positive or negative, respectively).

Pseudo-Euclidean spaces are a specific type of symmetric bilinear space that plays a central role in geometry and physics (see [10, Chapter IX, §4]). In [9, Lemma 1.2.4], the author provides sufficient conditions for the existence of positive and negative bases in a pseudo-Euclidean space. The proof of Lemma 1.2.4 in [9] suggests a method for constructing these positive and negative bases. These results can be generalized for any symmetric bilinear space (see Theorem 3 in Section 1). This leads us to address the following questions:

1. Which symmetric bilinear spaces possess isotropic bases?
2. How can we construct isotropic bases?

Answering the first question will help us in tackling Problem 13 in [14, Chapter 1], which inquires about the existence of isotropic bases in a regular quadratic space. Beyond its theoretical significance, there is a strong motivation to pursue this investigation due to the applications of isotropic bases in sphere geometries. For instance, in [8], the author demonstrates the existence of isotropic bases in certain pseudo-Euclidean spaces, which is essential for proving various theorems (e.g., see Lemmas 3.6 and 3.10 in [8, Chapter 3]) related to Lie sphere transformations. It is noteworthy that the author has established the existence of isotropic bases in a pseudo-Euclidean space by constructing them using the standard basis of that space.

With these motivations in mind, our first objective is to address the above questions 1 and 2, with a particular focus on the second question. After this, we will consider $\mathbb{R}^n$ as a normed space and analyze some properties of symmetric bilinear forms defined on $\mathbb{R}^n$ in relation to its bases. These properties raise a question (see Problem 1) concerning real-valued continuous functions defined on $\mathbb{R}^n$. Our second objective is to provide a positive answer to this question by proving a theorem. Additionally, we will explore various properties of symmetric bilinear forms using this theorem. Finally, we will demonstrate how our results facilitate an analytical examination of the existence of different types of bases in a symmetric bilinear space.

The organization of this paper is as follows: In Section 1, we collect the necessary definitions and results which are known. We will also establish the necessary and sufficient conditions for the existence of positive and negative bases in a symmetric bilinear space. In Section 2, we will present the criteria for the existence of isotropic bases in a symmetric bilinear space. Section 3 is divided into two subsections. In Subsection 3.1, we will offer a geometrical method to construct isotropic bases for Minkowski space, which is one of the most significant symmetric bilinear spaces. In Subsection 3.2, we will introduce a linear algebraic method for constructing isotropic bases in a symmetric bilinear space. In Section 4, we will establish several linear algebraic properties of the preimage set of a point under real-valued continuous functions defined on $\mathbb{R}^n$. We will also demonstrate some applications of these properties in a symmetric bilinear space. Finally, in Section 5, we will conclude our paper with some closing remarks.

1 Preliminaries

To begin, we will establish some notation for the entire paper. For any positive integers m and n, we denote the set of all $m \times n$ real matrices as $\mathbb{R}^{m\times n}$. For a matrix $A \in \mathbb{R}^{m\times n}$, its transpose is denoted by A^{T}, and its determinant is denoted by $\det(A)$.

We now recall some known definitions and results, which will be used to state and prove our main results.

Lemma 1 ([1, 17]) *For a matrix $A \in \mathbb{R}^{m\times n}$, the following properties are equivalent:*

1. *rank of A is k;*
2. *there is a $k \times k$ submatrix of A with non-zero determinant, and no $(k+1)\times(k+1)$ submatrices of A have non-zero determinant;*
3. *maximum number of linearly independent rows (columns) of A is k.*

Lemma 2 ([1, 17]) *Let A be an $n \times n$ real matrix. Let A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and the j-th column of A. Then the Laplace expansion of $\det(A)$ along the p-th row is given by*

$$\det(A) = \sum_{k=1}^{n} (-1)^{p+k} a_{pk} \det(A_{pk}).$$

Theorem 1 ([17, Theorem 8.7.2]) *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$ (i.e., $SS^{\mathsf{T}} = S^{\mathsf{T}}S = I_n$) such that SAS^{T} is a diagonal matrix.*

Definition 1 *Let $\mathcal{V}$ be a finite dimensional vector space over the field $\mathbb{R}$. A symmetric bilinear form on $\mathcal{V}$ is a function $\langle\, ,\rangle : \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ such that for all $\alpha, \beta, \gamma \in \mathcal{V}$ and all $a, b \in \mathbb{R}$,*

$$\langle a\alpha+b\beta,\gamma\rangle = a\langle\alpha,\gamma\rangle+b\langle\beta,\gamma\rangle, \langle\gamma, a\alpha+b\beta\rangle = a\langle\gamma,\alpha\rangle+b\langle\gamma,\beta\rangle, \langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle.$$

The pair $(\mathcal{V}, \langle\, ,\rangle)$ is called a real symmetric bilinear space (or symmetric bilinear space in short).

Let $(\mathcal{V}, \langle\, ,\rangle)$ be a symmetric bilinear space. We say that $(\mathcal{V}, \langle\, ,\rangle)$ is n-dimensional if $\dim(\mathcal{V}) = n$. We will now define three different types of elements and bases in $\mathcal{V}$.

Definition 2 *A non-zero element α in $\mathcal{V}$ is called isotropic or positive or negative if $\langle\alpha,\alpha\rangle = 0$ or $\langle\alpha,\alpha\rangle > 0$ or $\langle\alpha,\alpha\rangle < 0$, respectively.*

Definition 3 *Let $\mathcal{B}$ be a basis of $\mathcal{V}$. Then $\mathcal{B}$ is called isotropic (respectively, positive or negative) if all the elements in $\mathcal{B}$ are isotropic (respectively, positive or negative).*

We will now define the matrix of a symmetric bilinear space, as well as isometric bilinear spaces, and recall some related results which can be found in [18, Chapter 1, §2].

Definition 4 *Let $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a basis of the vector space $\mathcal{V}$. If $\langle\, , \rangle$ is a symmetric bilinear form on $\mathcal{V}$, then the $n \times n$ symmetric matrix $A = (a_{ij}) = (\langle \beta_i, \beta_j \rangle)$ is called the matrix of $\langle\, , \rangle$ with respect to the basis $\mathcal{B}$.*

Note that if A is the matrix of $\langle\, , \rangle$ with respect to the basis $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ of $\mathcal{V}$, then for $\alpha = \sum_{k=1}^{n} a_k \beta_k$ and $\beta = \sum_{k=1}^{n} b_k \beta_k$ in $\mathcal{V}$, we have $\langle \alpha, \beta \rangle = \alpha A \beta^{\mathsf{T}}$, where we identify α and β with $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, respectively.

Definition 5 *Two symmetric bilinear spaces $(\mathcal{V}_1, \langle\, , \rangle_1)$ and $(\mathcal{V}_2, \langle\, , \rangle_2)$ are said to be isometric if there exists a bijective linear function $\sigma : \mathcal{V}_1 \to \mathcal{V}_2$ satisfying $\langle \sigma(\alpha), \sigma(\beta) \rangle_2 = \langle \alpha, \beta \rangle_1$ for all $\alpha, \beta \in \mathcal{V}_1$. In this case, the function σ is called an isometry.*

Definition 6 *Two matrices A and B in $\mathbb{R}^{n \times n}$ are said to be congruent if there exists a matrix $P \in \mathbb{R}^{n \times n}$ satisfying $\det(P) \neq 0$ such that $A = PBP^{\mathsf{T}}$.*

Theorem 2 *Two symmetric bilinear spaces are isometric if and only if their associated symmetric matrices (with respect to arbitrary bases) are congruent.*

We conclude this section with the following theorem, which provides the necessary and sufficient conditions for a symmetric bilinear space to have positive and negative bases. The proof of this theorem is adapted from the proof of [9, Lemma 1.2.4].

Theorem 3 *A symmetric bilinear space $(\mathcal{V}, \langle\, , \rangle)$ possesses a positive (respectively, negative) basis if and only if $(\mathcal{V}, \langle\, , \rangle)$ contains positive (respectively, negative) elements.*

Proof. The necessity is straightforward. We will now demonstrate the sufficiency. Let α_1 be a positive element in $\mathcal{V}$. Choose a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $\mathcal{V}$ and consider $\alpha_i + x\alpha_1$, where $x \in \mathbb{R}$. Since $\langle \alpha_i + x\alpha_1, \alpha_i + x\alpha_1 \rangle = \langle \alpha_i, \alpha_i \rangle + 2x\langle \alpha_i, \alpha_1 \rangle + x^2 \langle \alpha_1, \alpha_1 \rangle$ and the coefficient of x^2 is positive, there exists $c_i \in \mathbb{R}$ such that $\langle \alpha_i + c_i\alpha_1, \alpha_i + c_i\alpha_1 \rangle > 0$ for each $i = 2, 3, \ldots, n$. Therefore, the set $\{\alpha_1, \alpha_2 + c_2\alpha_1, \alpha_3 + c_3\alpha_1, \ldots, \alpha_n + c_n\alpha_1\}$ forms a positive basis of $\mathcal{V}$. In a similar manner, one can show that a negative basis exists in $\mathcal{V}$ when $\mathcal{V}$ contains negative elements. This completes the proof. □

Note that the proof above outlines a method for constructing both positive and negative bases for a symmetric bilinear space.

2 Existence of isotropic bases in a symmetric bilinear space

In this section, we present the necessary and sufficient conditions for the existence of an isotropic basis in a symmetric bilinear space. Additionally, we discuss a method for constructing isotropic bases. We begin with the following lemma.

Lemma 3 *Let $(\mathcal{V}, \langle\,,\rangle)$ be a symmetric bilinear space and let α and β be its positive and negative element, respectively. Then there exist two non-zero real numbers, c_1 and c_2, such that $\alpha + c_1\beta$ and $\alpha + c_2\beta$ are linearly independent isotropic elements in $(\mathcal{V}, \langle\,,\rangle)$.*

Proof. Consider polynomial $p(x) = \langle \alpha + x\beta, \alpha + x\beta\rangle = \langle \alpha, \alpha\rangle + 2x\langle \alpha, \beta\rangle + x^2\langle \beta, \beta\rangle$. Since $\langle \alpha, \alpha\rangle\langle \beta, \beta\rangle < 0$, polynomial $p(x)$ has two distinct non-zero real roots, denoted as c_1 and c_2. Because α and β are linearly independent in $\mathcal{V}$, the isotropic elements $\alpha+c_1\beta$ and $\alpha+c_2\beta$ are also linearly independent in $\mathcal{V}$. □

In the following theorem, we present the necessary and sufficient conditions for the existence of an isotropic basis in a symmetric bilinear space. The proof outlines a method for constructing an isotropic basis.

Theorem 4 *A symmetric bilinear space $(\mathcal{V}, \langle\,,\rangle)$ possesses an isotropic basis if and only if all the elements in $(\mathcal{V}, \langle\,,\rangle)$ are isotropic or $(\mathcal{V}, \langle\,,\rangle)$ contains both positive and negative elements.*

Proof. If all the elements in $\mathcal{V}$ are isotropic, then any basis of $\mathcal{V}$ is isotropic. We now show that $\mathcal{V}$ possesses an isotropic basis when $\mathcal{V}$ contains both positive and negative elements. Consider a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $\mathcal{V}$, which contains both positive and negative elements. Assume that $\langle \alpha_i, \alpha_i\rangle > 0$ when $i \in \{1, 2, \ldots, p\}$, $\langle \alpha_j, \alpha_j\rangle = 0$ when $j \in \{p+1, p+2, \ldots, q-1\}$, and $\langle \alpha_k, \alpha_k\rangle < 0$ when $k \in \{q, q+1, \ldots, n\}$. We will now construct an isotropic basis of $\mathcal{V}$ using the basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. According to Lemma 3, for each positive element α_i, there exist two non-zero real numbers c_{i1} and c_{i2} such that $\alpha_i + c_{i1}\alpha_n$ and $\alpha_i + c_{i2}\alpha_n$ are linearly independent isotropic elements in $\mathcal{V}$. Similarly, for each negative element α_k, there exist two non-zero real numbers c_{k1} and c_{k2} such that $\alpha_1 + c_{k1}\alpha_k$ and $\alpha_1 + c_{k2}\alpha_k$ are linearly independent isotropic elements in $\mathcal{V}$. Define $\beta_i := \alpha_i + c_{i1}\alpha_n$ when $i \in \{1, 2, \ldots, p\}$, $\beta_j := \alpha_j$ when $j \in \{p+1, p+2, \ldots, q-1\}$, $\beta_k := \alpha_1 + c_{k1}\alpha_k$ when $k \in \{q, q+1, \ldots, n-1\}$, and $\beta_n := \alpha_1 + c_{12}\alpha_n$. Since the set $\{\beta_1, \beta_n\}$ is linearly independent, we can express α_1 and α_n as linear combinations of β_1 and β_n. Therefore, α_i can be expressed as a linear combination of β_i, β_1, and β_n for each $i = 1, 2, \ldots, n$. Consequently, the set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is a basis of $\mathcal{V}$. Furthermore, we have $\langle \beta_i, \beta_i \rangle = 0$ for all $i = 1, 2, \ldots, n$. This concludes that $\mathcal{V}$ possesses an isotropic basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$.

Conversely, if $\mathcal{V}$ possesses an isotropic basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$, then $\mathcal{V}$ must contain isotropic elements. Suppose all the elements in $\mathcal{V}$ are not isotropic. Let $\alpha = \sum_{i=1}^{n} c_i \beta_i$ be a non-isotropic element in $\mathcal{V}$. We claim that there exists at least one β_k such that $\langle \alpha, \beta_k \rangle \neq 0$. If this is not the case, we have

$$\langle \alpha, \alpha \rangle = \left\langle \alpha, \sum_{i=1}^{n} c_i \beta_i \right\rangle = \sum_{i=1}^{n} c_i \langle \alpha, \beta_i \rangle = 0,$$

which contradicts the fact that α is non-isotropic. Let us consider the element $\alpha + x\beta_k$ in $\mathcal{V}$, where $x \in \mathbb{R}$. We see that $\langle \alpha + x\beta_k, \alpha + x\beta_k \rangle = \langle \alpha, \alpha \rangle + 2x\langle \alpha, \beta_k \rangle$. Therefore, $\alpha + x\beta_k$ can be positive or negative depending on the value of $x \in \mathbb{R}$. This shows that $\mathcal{V}$ contains both positive and negative elements. This completes the proof. □

The following corollary directly follows from Theorems 3 and 4, which provides the necessary and sufficient conditions for $(\mathcal{V}, \langle\,,\rangle)$ to possess isotropic, positive and negative bases.

Corollary 1 *A symmetric bilinear space $(\mathcal{V}, \langle\,,\rangle)$ possesses isotropic, positive and negative bases if and only if $(\mathcal{V}, \langle\,,\rangle)$ contains both positive and negative elements.*

We observe that using the method discussed in the proof of Theorem 4, it is possible to construct isotropic bases for symmetric bilinear spaces. However, this method relies on the choice of the basis. In the next section, we will present additional methods for constructing isotropic bases for symmetric bilinear spaces that do not depend on the basis.

3 Construction of isotropic bases for symmetric bilinear space

In this section, we outline methods for constructing isotropic bases of symmetric bilinear spaces. In Subsection 3.1, we present a geometric method for constructing isotropic bases in Minkowski space, and in Subsection 3.2, we introduce a linear algebraic method for constructing isotropic bases for any symmetric bilinear space. We begin by proving a proposition that will simplify the process of constructing these isotropic bases.

Proposition 1 *Let $(\mathcal{V}, \langle\,,\rangle_1)$ be an n-dimensional symmetric bilinear space. Then $(\mathcal{V}, \langle\,,\rangle_1)$ is isometric to $(\mathbb{R}^n, \langle\,,\rangle_2)$, where $\langle \alpha, \beta \rangle_2 = \sum_{i=1}^{n} d_i a_i b_i$, $d_1 \geq d_2 \geq \cdots \geq d_n$ and $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$.*

Proof. Let A be the matrix of the symmetric bilinear form $\langle\, ,\rangle_1$ with respect to the basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of $\mathcal{V}$. Since $A \in \mathbb{R}^{n\times n}$ is symmetric, by Theorem 1, there exists an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that UAU^{T} is a diagonal matrix. By permuting the rows of U, we can find another orthogonal matrix $V \in \mathbb{R}^{n\times n}$ such that the diagonal entries of matrix VAV^{T} are arranged in decreasing order. Specifically, we have $d_1 \geq d_2 \geq \cdots \geq d_n$, where d_i represents the (i,i)-th entry of VAV^{T}. We now define a symmetric bilinear form $\langle\, ,\rangle_2$ on $\mathbb{R}^n$ as follows: $\langle \alpha, \beta\rangle_2 = \sum_{i=1}^n d_i a_i b_i$, where $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_n)$. It is clear that VAV^{T} is the matrix of $\langle\, ,\rangle_2$ with respect to the standard basis of $\mathbb{R}^n$. Therefore, by Theorem 2, $(\mathcal{V}, \langle\, ,\rangle_1)$ and $(\mathbb{R}^n, \langle\, ,\rangle_2)$ are isometric. $\square$

Let us make some observations based on the proof of Proposition 1.

Remark 1 Consider a bijective linear function $f : \mathbb{R}^n \to \mathcal{V}$, defined by $f(\alpha) = \sum_{i=1}^n a_i\alpha_i$ where $\alpha = (a_1, a_2, \dots, a_n)$. It is evident that

$$\langle f(\alpha), f(\beta)\rangle_1 = \alpha A \beta^{\mathsf{T}}.$$

Next, define a function $\sigma : \mathbb{R}^n \to \mathcal{V}$ by $\sigma(\alpha) = f(\alpha V)$. Since f is a bijective linear function and $\det(V) \neq 0$, it follows that σ is also a bijective linear function. Furthermore, we have $\langle \sigma(\alpha), \sigma(\beta)\rangle_1 = \langle f(\alpha V), f(\beta V)\rangle_1 = \alpha VAV^{\mathsf{T}}\beta^{\mathsf{T}} = \langle \alpha, \beta\rangle_2$. Therefore, by Definition 5, we conclude that σ is an isometry. Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be an isotropic basis of $(\mathbb{R}^n, \langle\, ,\rangle_2)$. It follows immediately that $\{\sigma(\beta_1), \sigma(\beta_2), \dots, \sigma(\beta_n)\}$ is an isotropic basis of $(\mathcal{V}, \langle\, ,\rangle_1)$.

In view of Proposition 1 and Remark 1, it is sufficient to find the isotropic bases of symmetric bilinear space $(\mathbb{R}^n, \langle\, ,\rangle)$, where $\langle \alpha, \beta\rangle = \sum_{i=1}^n e_i a_i b_i$, $e_1 \geq e_2 \geq \cdots \geq e_n$ and $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$. In the remainder of this section, we will prove the results for $(\mathbb{R}^n, \langle\, ,\rangle)$. We will specify the values of n and $e_1, e_2, \dots, e_n$ as necessary. For instance, in the following subsection, we will set $n = 4$, $e_1 = e_2 = e_3 = 1$, and $e_4 = -1$.

3.1 Geometrical construction of isotropic bases for Minkowski space

Minkowski space is fundamental in the theory of special relativity and in two-dimensional conformal geometry (see, for example, [8, 11, 16]). Two-dimensional conformal geometry establishes a connection between Minkowski space and the set of circles, lines, and points in $\mathbb{R}^2$ (see [8, Chapter 2, Section 2.2] and [11, Chapter 4, Section 4.2]). In this section, we will construct isotropic bases for Minkowski space by utilizing this relationship. To begin, let us revisit the definition of Minkowski space.

Definition 7 *Minkowski space, denoted by* $\mathbb{R}^{3,1}$*, is the vector space* $\mathbb{R}^4$*, equipped with the symmetric bilinear form* $\langle \alpha, \beta \rangle := a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4$*, where* $\alpha = (a_1, a_2, a_3, a_4)$ *and* $\beta = (b_1, b_2, b_3, b_4)$ *in* $\mathbb{R}^4$*.*

A circle or a point (which can be viewed as a circle of zero radius) in $\mathbb{R}^2$ can be represented by an equation of the form $(x-a)^2 + (y-b)^2 = r^2$, where $r \geq 0$. A line in $\mathbb{R}^2$ can be represented by an equation of the form $cx + dy = h$, where $c^2 + d^2 = 1$. The lemma below provides a relation between $\mathcal{C}$ and $\mathbb{R}^{3,1}$, where $\mathcal{C}$ represents the set of all circles, lines, and points in $\mathbb{R}^2$.

Lemma 4 *Let* $\phi : \mathcal{C} \to \mathbb{R}^{3,1}$ *be a function defined by*

$$\phi(\mathfrak{C}) = \begin{cases} (2a, 2b, 1 - a^2 - b^2 + r^2, 1 + a^2 + b^2 - r^2) & \text{if } \mathbf{C1} \text{ holds,} \\ (c, d, -h, h) & \text{if } \mathbf{C2} \text{ holds,} \end{cases}$$

where $\mathbf{C1} : \mathfrak{C}$ *is* $(x-a)^2 + (y-b)^2 = r^2$*, and* $\mathbf{C2} : \mathfrak{C}$ *is* $cx + dy = h$*. Then* ϕ *is an injective function which maps a circle or a line in* $\mathbb{R}^2$ *to a positive element in* $\mathbb{R}^{3,1}$ *and a point in* $\mathbb{R}^2$ *to an isotropic element in* $\mathbb{R}^{3,1}$*.*

Proof. It is straightforward to see that the function ϕ is injective. We will now prove the other assertion. Let $\mathfrak{C} \in \mathcal{C}$ be a circle or a point defined by the equation $(x-a)^2 + (y-b)^2 = r^2$. It is easy to see using the definition of ϕ that

$$\langle \phi(\mathfrak{C}), \phi(\mathfrak{C}) \rangle = 4a^2 + 4b^2 + (1 - a^2 - b^2 + r^2)^2 - (1 + a^2 + b^2 - r^2)^2 = 4r^2.$$

Here, $\mathfrak{C}$ is a circle when $r > 0$ and a point when $r = 0$. Thus, ϕ maps a circle to a positive element and a point to an isotropic element in $\mathbb{R}^{3,1}$. Now, if $\mathfrak{C} \in \mathcal{C}$ is a line given by the equation $cx + dy = h$, we find that $\langle \phi(\mathfrak{C}), \phi(\mathfrak{C}) \rangle = c^2 + d^2 = 1$. Therefore, ϕ maps a line to a positive element in $\mathbb{R}^{3,1}$. $\square$

The following lemma provides the necessary and sufficient conditions for a circle or a line to pass through a given point.

Lemma 5 *Let* $\mathfrak{C}_1$ *be a point and* $\mathfrak{C}_2$ *be a circle or a line in* $\mathbb{R}^2$*. Then* $\mathfrak{C}_2$ *passes through the point* $\mathfrak{C}_1$ *if and only if* $\langle \phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2) \rangle = 0$*.*

Proof. Let $\mathfrak{C}_1$ represent the point (a_1, b_1), which can be expressed by equation $(x - a_1)^2 + (y - b_1)^2 = 0$. Let $\mathfrak{C}_2$ be the circle defined by $(x - a_2)^2 + (y - b_2)^2 = r_2^2$. According to the definition of ϕ (see Lemma 4), we have $\phi(\mathfrak{C}_1) = (2a_1, 2b_1, 1 - a_1^2 - b_1^2, 1 + a_1^2 + b_1^2)$ and $\phi(\mathfrak{C}_2) = (2a_2, 2b_2, 1 - a_2^2 - b_2^2 + r_2^2, 1 + a_2^2 + b_2^2 - r_2^2)$. A direct calculation shows that $\langle \phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2) \rangle = 2r_2^2 - 2(a_1 - a_2)^2 - 2(b_1 - b_2)^2$. Thus, the circle $\mathfrak{C}_2$ passes through the point $\mathfrak{C}_1$ if and only if $\langle \phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2) \rangle = 0$. Now, suppose that $\mathfrak{C}_2$ represents the line expressed by $c_2x + d_2y = h_2$. In this case, we have $\phi(\mathfrak{C}_2) = (c_2, d_2, -h_2, h_2)$. For this scenario, we find that $\langle \phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2) \rangle = 2a_1c_2 + 2b_1d_2 - 2h_2$, which equals 0 if and only if the line $\mathfrak{C}_2$ passes through the point $\mathfrak{C}_1$. $\square$

Remark 2 For any two points $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathbb{R}^2$, we have $\langle \phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2)\rangle \neq 0$. One can see this by setting $r_2 = 0$ in the proof of Lemma 5.

We now provide a method to construct an isotropic basis for $\mathbb{R}^{3,1}$ using circles, lines, and points of $\mathbb{R}^2$.

Theorem 5 *Let $\mathfrak{C}$ be a circle or a line in $\mathbb{R}^2$. We choose three points $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ which lie on $\mathfrak{C}$ and a point $\mathfrak{C}_4$ which does not lie on $\mathfrak{C}$. Then $\{\phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2), \phi(\mathfrak{C}_3), \phi(\mathfrak{C}_4)\}$ is an isotropic basis of $\mathbb{R}^{3,1}$.*

Proof. It follows immediately from Lemma 4 that $\langle \phi(\mathfrak{C}_i), \phi(\mathfrak{C}_i)\rangle = 0$ for all $i = 1, 2, 3, 4$. Note also that for any real number k, $\phi(\mathfrak{C}_i) \neq k\phi(\mathfrak{C}_j)$ when $i \neq j$; otherwise, it would contradict the fact $\langle \phi(\mathfrak{C}_i), \phi(\mathfrak{C}_j)\rangle \neq 0$ (cf. Remark 2). Now, if $\phi(\mathfrak{C}_1)$, $\phi(\mathfrak{C}_2)$ and $\phi(\mathfrak{C}_3)$ are linearly dependent, then there exist two real numbers k_1 and k_2 such that $k_1\phi(\mathfrak{C}_1) + k_2\phi(\mathfrak{C}_2) = \phi(\mathfrak{C}_3)$. Thus, we have $\langle k_1\phi(\mathfrak{C}_1) + k_2\phi(\mathfrak{C}_2), k_1\phi(\mathfrak{C}_1) + k_2\phi(\mathfrak{C}_2)\rangle = \langle \phi(\mathfrak{C}_3), \phi(\mathfrak{C}_3)\rangle = 0$. Consequently, $\langle \phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2)\rangle = 0$, which contradicts Remark 2. Hence, the set $\{\phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2), \phi(\mathfrak{C}_3)\}$ is linearly independent in $\mathbb{R}^{3,1}$. Since $\mathfrak{C}$ passes through the point $\mathfrak{C}_i$, by Lemma 5, we have $\langle \phi(\mathfrak{C}_i), \phi(\mathfrak{C})\rangle = 0$ for all $i = 1, 2, 3$. Now, let us assume that $k_1\phi(\mathfrak{C}_1) + k_2\phi(\mathfrak{C}_2) + k_3\phi(\mathfrak{C}_3) = \phi(\mathfrak{C}_4)$ for some real numbers k_1, k_2, and k_3. Then $\langle \phi(\mathfrak{C}_4), \phi(\mathfrak{C})\rangle = k_1\langle \phi(\mathfrak{C}_1), \phi(\mathfrak{C})\rangle + k_2\langle \phi(\mathfrak{C}_2), \phi(\mathfrak{C})\rangle + k_3\langle \phi(\mathfrak{C}_3), \phi(\mathfrak{C})\rangle = 0$. This contradicts Lemma 5 since $\mathfrak{C}$ does not pass through the point $\mathfrak{C}_4$. Hence, the set $\{\phi(\mathfrak{C}_1), \phi(\mathfrak{C}_2), \phi(\mathfrak{C}_3), \phi(\mathfrak{C}_4)\}$ is linearly independent, and therefore, forms an isotropic basis of $\mathbb{R}^{3,1}$. $\square$

In the following example, we construct an isotropic basis for $\mathbb{R}^{3,1}$ using the method provided in the Theorem 5.

Example 1 Let us consider the circle $\mathfrak{C}$ defined by $(x - 1)^2 + y^2 = 1$. We choose three points $(0, 0)$, $(2, 0)$, and $(1, 1)$, which lie on the circle $\mathfrak{C}$. Denote these points as $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_3$, respectively. Choose another point $(0, 1)$ that does not lie on $\mathfrak{C}$ and denote it by $\mathfrak{C}_4$. Now, represent the point $\mathfrak{C}_1$ by the equation $x^2 + y^2 = 0$, and using the definition of ϕ given in Lemma 4, we get $\phi(\mathfrak{C}_1) = (0, 0, 1, 1)$. In a similar way, we find $\phi(\mathfrak{C}_2) = (4, 0, -3, 5)$, $\phi(\mathfrak{C}_3) = (2, 2, -1, 3)$, and $\phi(\mathfrak{C}_4) = (0, 2, 0, 2)$. Now, using Theorem 5, we conclude that the set $\{(0, 0, 1, 1), (4, 0, -3, 5), (2, 2, -1, 3), (0, 2, 0, 2)\}$ forms an isotropic basis of $\mathbb{R}^{3,1}$.

Although this method allows for the straightforward construction of isotropic bases in Minkowski space, it becomes challenging to apply to other symmetric bilinear spaces. In the following section, we will present a technique that enables the construction of isotropic bases for any symmetric bilinear space.

3.2 Construction of isotropic basis for $(\mathbb{R}^n, \langle\,,\rangle)$

The proof of the following theorem provides a method to construct an isotropic basis for $(\mathbb{R}^n, \langle\,,\rangle)$. Recall that $\langle \alpha, \beta\rangle = \sum_{i=1}^n e_i a_i b_i$ and $e_1 \geq e_2 \geq \cdots \geq e_n$.

Theorem 6 *Let $(\mathbb{R}^n, \langle\,,\rangle)$ contain both positive and negative elements. Then any set $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathbb{R}^n$ of n elements can be transformed into an isotropic basis of $(\mathbb{R}^n, \langle\,,\rangle)$ by changing the values of at most three coordinates of each α_i, $i = 1, 2, \dots, n$.*

Proof. For ease of notation, we write $\mathbb{R}^n := (\mathbb{R}^n, \langle\,,\rangle)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a subset of $\mathbb{R}^n$. We transform this set into an isotropic basis of $\mathbb{R}^n$ in n steps. First, we provide a brief overview of this transformation process. In the first step, we transform α_1 into an isotropic element $\beta_1 \in \mathbb{R}^n$. In the k-th step, we transform α_k into an isotropic element $\beta_k \in \mathbb{R}^n$, ensuring that the set $\{\beta_1, \beta_2, \dots, \beta_k\}$ remains linearly independent. Finally, in the n-th step, we transform α_n into an isotropic element $\beta_n \in \mathbb{R}^n$. At this point, the set $\{\beta_1, \beta_2, \dots, \beta_n\}$ will constitute an isotropic basis of $\mathbb{R}^n$. We now describe the procedure for transforming α_k into an isotropic element β_k during the k-th step of this process.

Suppose we have constructed the set $\{\beta_1, \beta_2, \dots, \beta_{k-1}\}$ of $(k-1)$ linearly independent isotropic elements in $\mathbb{R}^n$, $k > 1$. Now, we want to transform $\alpha_k = (a_{k1}, a_{k2}, \dots, a_{kn})$ into an isotropic element $\beta_k = (b_{k1}, b_{k2}, \dots, b_{kn})$ in $\mathbb{R}^n$ such that the set $\{\beta_1, \beta_2, \dots, \beta_k\}$ remains linearly independent. To achieve this, we will use two functions: ψ_k, which will transform α_k into β_k, and F_k, which will maintain the linear independence of the set $\{\beta_1, \beta_2, \dots, \beta_k\}$. It is important to note that since $\mathbb{R}^n$ contains positive and negative elements, there exist positive integers p and q (with $q > p$) such that $e_i > 0$ for all $i \in \{1, 2, \dots, p\}$ and $e_j < 0$ for all $j \in \{q, q+1, \dots, n\}$.

Let $\gamma_k \in \mathbb{R}^n$ denote the element whose k-th coordinate is 1 and rest of the coordinates are 0. Define function $\psi_k : \mathbb{R}^3 \to \mathbb{R}^n$ by

$$\psi_k(x, y, z) = \begin{cases} \alpha_k + x\gamma_k + y\gamma_n & \text{if } k \in \{1, 2, \dots, p\}, \\ \alpha_k + x\gamma_k + y\gamma_1 + z\gamma_n & \text{if } k \in \{p+1, p+2, \dots, q-1\}, \\ \alpha_k + x\gamma_k + y\gamma_1 & \text{if } k \in \{q, q+1, \dots, n\}. \end{cases}$$

Now, let $M_k = (m_{ij})$ be the $k \times k$ matrix given by

$$m_{ij} = \begin{cases} b_{ij} & \text{if } i \neq k, \\ a_{ij} & \text{if } i = k, \end{cases}$$

(recall that a_{ij} and b_{ij} are the j-th coordinate of α_i and β_i, respectively) and let $E_{ij}^{(k)}$ denote the $k \times k$ matrix, whose (i, j)-th entry is 1 and rest of the entries are 0. Define function $F_k : \mathbb{R}^3 \to \mathbb{R}^{k\times k}$ by

$$F_k(x,y,z) = \begin{cases} M_k + xE_{kk}^{(k)} & \text{if } k \in \{1,2,\ldots,p\}, \\ M_k + xE_{kk}^{(k)} + yE_{k1}^{(k)} & \text{if } k \in \{p+1,p+2,\ldots,n\}. \end{cases}$$

Put $\mathcal{S}_{k1} := \{(x,y,z) \in \mathbb{R}^3 : \langle \psi_k(x,y,z), \psi_k(x,y,z)\rangle = 0\}$ and $\mathcal{S}_{k2} := \{(x,y,z) \in \mathbb{R}^3 : \det(F_k(x,y,z)) = 0\}$, choose a point (x_k, y_k, z_k) from the set $\mathcal{S}_{k1} \setminus \mathcal{S}_{k2}$, and define $\beta_k := \psi_k(x_k, y_k, z_k)$.

It remains to show that $\mathcal{S}_{k1} \setminus \mathcal{S}_{k2}$ is non-empty and $\{\beta_1, \beta_2, \ldots, \beta_k\}$ is linearly independent. The following observations will assist us in doing this.

1. Note that the i-th coordinate of $\psi_k(x,y,z)$ is the (k,i)-th entry of $F_k(x,y,z)$ for $i = 1,2,\ldots,k$. Thus, (i,j)-th entry of $F_k(x_k,y_k,z_k)$ is b_{ij}, where $i,j = 1,2,\ldots,k$.
2. For $k > 1$, $F_{k-1}(x_{k-1}, y_{k-1}, z_{k-1})$ is a $(k-1)\times(k-1)$ submatrix of $F_k(x,y,z)$, which can be obtained by deleting the k-th row and the k-th column of $F_k(x,y,z)$.

Let us show that $\mathcal{S}_{k1} \setminus \mathcal{S}_{k2}$ is non-empty. To do this, first note that if $k \in \{1,2,\ldots,p\}$, then

$$\begin{aligned}&\langle \psi_k(x,y,z), \psi_k(x,y,z)\rangle \\ &\qquad = \sum_{i=1}^{k-1} e_i a_{ki}^2 + e_k(a_{kk}+x)^2 + \sum_{i=k+1}^{p} e_i a_{ki}^2 + \sum_{i=q}^{n-1} e_i a_{ki}^2 + e_n(a_{kn}+y)^2.\end{aligned}$$

If $k \in \{p+1, p+2, \ldots, q-1\}$, then

$$\langle \psi_k(x,y,z), \psi_k(x,y,z)\rangle = e_1(a_{k1}+y)^2 + \sum_{i=2}^{p} e_i a_{ki}^2 + \sum_{i=q}^{n-1} e_i a_{ki}^2 + e_n(a_{kn}+z)^2.$$

If $k \in \{q, q+1, \ldots, n\}$, then

$$\begin{aligned}&\langle \psi_k(x,y,z), \psi_k(x,y,z)\rangle \\ &\qquad = e_1(a_{k1}+y)^2 + \sum_{i=2}^{p} e_i a_{ki}^2 + \sum_{i=q}^{k-1} e_i a_{ki}^2 + e_k(a_{kk}+x)^2 + \sum_{i=k+1}^{n} e_i a_{ki}^2.\end{aligned}$$

By considering all these cases, it is evident that $\langle \psi_k(x,y,z), \psi_k(x,y,z)\rangle$ can be expressed in one of the following forms: either as $\pm v_1(x+u_1)^2 \mp v_2(y+u_2)^2 + u_3$ or as $v_3(y+u_4)^2 - v_4(z+u_5)^2 + u_6$. Here u_i, $i = 1,2,\ldots,6$, are real numbers, and v_i, $i = 1,2,3,4$, are positive real numbers. We also observe that if $k = 1$, then $\det(F_1(x,y,z)) = a_{11} + x$. If $k > 1$, the Laplace expansion (see Lemma 2) of $\det(F_k(x,y,z))$ along the k-th row leads to the expression $\det(F_k(x, y, z)) = w_1x + w_2y + w_3$. It is important to note that x appears only in the (k, k)-th entry of $F_k(x, y, z)$. Hence, by observation 2, we find that $w_1 = \det(F_{k-1}(x_{k-1}, y_{k-1}, z_{k-1}))$. From the construction, it is clear that $(x_{k-1}, y_{k-1}, z_{k-1}) \in \mathcal{S}_{(k-1)1} \setminus \mathcal{S}_{(k-1)2}$, and therefore, $w_1 = \det(F_{k-1}(x_{k-1}, y_{k-1}, z_{k-1})) \neq 0$. Consequently, $\det(F_k(x, y, z)) = w_1x + w_2y + w_3$ for some real numbers $w_1(\neq 0)$, w_2 and w_3. Therefore, the set $\mathcal{S}_{k1} \setminus \mathcal{S}_{k2}$ is non-empty.

We now show that the set $\{\beta_1, \beta_2, \ldots, \beta_k\}$ of k isotropic elements is linearly independent in $\mathbb{R}^n$. Consider the $k \times n$ matrix A whose i-th row is β_i, $i = 1, 2, \ldots, k$. From observation 1, we can see that the (i, j)-th entry of $F_k(x_k, y_k, z_k)$ is b_{ij}, where $i, j = 1, 2, \ldots, k$. Therefore, $F_k(x_k, y_k, z_k)$ is a $k \times k$ submatrix of A. Now, since $(x_k, y_k, z_k) \in \mathcal{S}_{k1} \setminus \mathcal{S}_{k2}$, $\det(F_k(x_k, y_k, z_k)) \neq 0$. Thus, applying Lemma 1, we conclude that A has rank k. As a result, the set $\{\beta_1, \beta_2, \ldots, \beta_k\}$ is linearly independent.

If we carry out this procedure for $k = 1, 2, \ldots, n$, we obtain the isotropic basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ of $\mathbb{R}^n$. We complete the proof by noting that β_k has been obtained by changing the values of at most three coordinates of α_k for each $k = 1, 2, \ldots, n$. $\square$

Remark 3 The method described above can generate an isotropic basis for $(\mathbb{R}^n, \langle\, , \rangle)$ from any n elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not necessarily distinct) in $\mathbb{R}^n$.

The following example illustrates the method provided in the proof of Theorem 6.

Example 2 Let $\mathcal{P}_3$ be the set of all polynomials in x with real coefficients and of degree less than 4. It is easy to see that $\mathcal{P}_3$ forms a 4-dimensional real vector space. On $\mathcal{P}_3$, we define a symmetric bilinear form as follows: $\langle \alpha(x), \beta(x) \rangle_1 = 2a_3b_4 + 2a_4b_3 - 3a_4b_4$, where $\alpha(x) = a_1 + a_2x + a_3x^2 + a_4x^3$ and $\beta(x) = b_1 + b_2x + b_3x^2 + b_4x^3$. Our goal is to find an isotropic basis for the symmetric bilinear space $(\mathcal{P}_3, \langle\, , \rangle_1)$. To do this, first note, using Definition 4, that matrix A of the symmetric bilinear form $\langle\, , \rangle_1$ with respect to the basis $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3$ is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -3 \end{bmatrix},$$

and then,

$$SAS^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \text{ where } S = \begin{bmatrix} 0 & 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Consider now the symmetric bilinear space $(\mathbb{R}^4, \langle\, , \rangle_2)$, where $\langle \alpha, \beta \rangle_2 = a_1 b_1 - 4a_4 b_4$, $\alpha = (a_1, a_2, a_3, a_4)$ and $\beta = (b_1, b_2, b_3, b_4)$. Using Definition 4, note that the matrix of $\langle\, , \rangle_2$ with respect to the standard basis of $\mathbb{R}^4$ is given by SAS^{T}. By applying Theorem 2, we conclude that the symmetric bilinear spaces $(\mathcal{P}_3, \langle\, , \rangle_1)$ and $(\mathbb{R}^4, \langle\, , \rangle_2)$ are isometric.

Define a linear function $f : \mathbb{R}^4 \to \mathcal{P}_3$ as follows: $f(\alpha) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$, where $\alpha = (a_1, a_2, a_3, a_4)$. Also, consider the function $\sigma : \mathbb{R}^4 \to \mathcal{P}_3$ given by $\sigma(\alpha) = f(\alpha S)$. It is clear from Remark 1 that σ is an isometry. We will now construct an isotropic basis for $(\mathbb{R}^4, \langle\, , \rangle_2)$ using the procedure described in the proof of Theorem 6. We will then convert this basis into an isotropic basis of $(\mathcal{P}_3, \langle\, , \rangle_1)$ using the isometry σ.

We begin by choosing four elements $\alpha_1 = \alpha_2 = (0, 0, 0, 0)$, $\alpha_3 = (0, 0, 1, 0)$, and $\alpha_4 = (0, 0, 0, 1)$ in $\mathbb{R}^4$ (cf. Remark 3). Now, consider the functions $\psi_1(x, y, z) = (x, 0, 0, y)$ and $F_1(x, y, z) = \begin{bmatrix} x \end{bmatrix}$, and put $\mathcal{S}_{11} = \{(x, y, z) \in \mathbb{R}^3 : \langle \psi_1, \psi_1 \rangle_2 = x^2 - 4y^2 = 0\}$ and $\mathcal{S}_{12} = \{(x, y, z) \in \mathbb{R}^3 : \det(F_1) = x = 0\}$. Choose point $(x_1, y_1, z_1) = (2, 1, 0) \in \mathcal{S}_{11} \setminus \mathcal{S}_{12}$, and let $\beta_1 = \psi_1(2, 1, 0) = (2, 0, 0, 1)$. Consider functions

$$\psi_2(x, y, z) = (y, x, 0, z) \text{ and } F_2(x, y, z) = \begin{bmatrix} 2 & 0 \\ y & x \end{bmatrix}.$$

Now, $\mathcal{S}_{21} = \{(x, y, z) \in \mathbb{R}^3 : \langle \psi_2, \psi_2 \rangle_2 = y^2 - 4z^2 = 0\}$ and $\mathcal{S}_{22} = \{(x, y, z) \in \mathbb{R}^3 : \det(F_2) = 2x = 0\}$. Choose point $(x_2, y_2, z_2) = (1, 2, 1) \in \mathcal{S}_{21} \setminus \mathcal{S}_{22}$, and let $\beta_2 = \psi_2(1, 2, 1) = (2, 1, 0, 1)$. Next, consider the following functions:

$$\psi_3(x, y, z) = (y, 0, 1 + x, z) \text{ and } F_3(x, y, z) = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ y & 0 & 1 + x \end{bmatrix},$$

and put $\mathcal{S}_{31} = \{(x, y, z) \in \mathbb{R}^3 : \langle \psi_3, \psi_3 \rangle_2 = y^2 - 4z^2 = 0\}$, $\mathcal{S}_{32} = \{(x, y, z) \in \mathbb{R}^3 : \det(F_3) = 2 + 2x = 0\}$. Choose point $(x_3, y_3, z_3) = (1, 2, 1) \in \mathcal{S}_{31} \setminus \mathcal{S}_{32}$ and let $\beta_3 = \psi_3(1, 2, 1) = (2, 0, 2, 1)$. Finally, consider functions

$$\psi_4(x, y, z) = (y, 0, 0, 1 + x) \text{ and } F_4(x, y, z) = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ y & 0 & 0 & 1 + x \end{bmatrix}.$$

Now, $\mathcal{S}_{41} = \{(x, y, z) \in \mathbb{R}^3 : \langle \psi_4, \psi_4 \rangle_2 = y^2 - 4(1 + x)^2 = 0\}$ and $\mathcal{S}_{42} = \{(x, y, z) \in \mathbb{R}^3 : \det(F_4) = 4 + 4x - 2y = 0\}$. Choose point $(x_4, y_4, z_4) = (0, -2, 0) \in \mathcal{S}_{41} \setminus \mathcal{S}_{42}$ and let $\beta_4 = \psi_4(0, -2, 0) = (-2, 0, 0, 1)$. It is clear from the proof of Theorem 6 that the set $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ forms an isotropic basis of $(\mathbb{R}^4, \langle\, , \rangle_2)$. Therefore, the set $\{\sigma(\beta_1), \sigma(\beta_2), \sigma(\beta_3), \sigma(\beta_4)\}$ is an isotropic basis of $(\mathcal{P}_3, \langle\, , \rangle_1)$. Here $\sigma(\beta_1) = f(\beta_1 S) = (3/\sqrt{5})x^2 + (4/\sqrt{5})x^3$, $\sigma(\beta_2) = f(\beta_2 S) = 1 + (3/\sqrt{5})x^2 + (4/\sqrt{5})x^3$, $\sigma(\beta_3) = f(\beta_3 S) = 2x + (3/\sqrt{5})x^2 + (4/\sqrt{5})x^3$, and $\sigma(\beta_4) = f(\beta_4 S) = -\sqrt{5}x^2$.

Remark 4 In the proof of Theorem 6, if we replace the set $\mathcal{S}_{k1}$ with $\mathcal{S}_{k3}$, defined as $\mathcal{S}_{k3} := \{(x, y, z) \in \mathbb{R}^3 : \langle \psi_k(x, y, z), \psi_k(x, y, z)\rangle > 0\}$, we will obtain a positive basis of $(\mathbb{R}^n, \langle\, ,\rangle)$. Similarly, if we substitute $\mathcal{S}_{k1}$ by $\mathcal{S}_{k4}$, where $\mathcal{S}_{k4} := \{(x, y, z) \in \mathbb{R}^3 : \langle \psi_k(x, y, z), \psi_k(x, y, z)\rangle < 0\}$, we will derive a negative basis of $(\mathbb{R}^n, \langle\, ,\rangle)$.

In the following section, we explore a question in topology that emerges naturally from the characteristics of symmetric bilinear spaces and their bases. We provide a positive answer to this question by proving a key result. Additionally, we demonstrate some applications of this result within the context of symmetric bilinear spaces.

4 Applications of topology in symmetric bilinear space

For any positive integer n, let us consider the topology on $\mathbb{R}^n$ induced by the Euclidean norm $\|\alpha\| = \sqrt{\sum_{i=1}^{n} a_i^2}$, where $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. We now make two observations.

Remark 5 Let $\langle\, ,\rangle$ be a symmetric bilinear form on $\mathbb{R}^n$. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\alpha) = \langle \alpha, \alpha\rangle$. Clearly, function f is continuous (see [7, Chapter VI. 1, Proposition 5]). Moreover, from Theorems 3 and 4, we know that the space $(\mathbb{R}^n, \langle\, ,\rangle)$ possesses at least one basis that is either positive, isotropic, or negative. If $(\mathbb{R}^n, \langle\, ,\rangle)$ possesses an isotropic basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, then $f(\alpha_i) = 0$ for all $i = 1, 2, \ldots, n$. If $(\mathbb{R}^n, \langle\, ,\rangle)$ possesses a positive basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, then for a positive real number c, we define $\beta_i = (\sqrt{c/f(\alpha_i)})\alpha_i$. It is evident that the set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ forms a basis of $\mathbb{R}^n$, and we have $f(\beta_i) = c$ for all $i = 1, 2, \ldots, n$. Similarly, if $(\mathbb{R}^n, \langle\, ,\rangle)$ possesses a negative basis, then for any negative real number c, there exists a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $f(\alpha_i) = c$ for all $i = 1, 2, \ldots, n$. Therefore, there exists a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $\mathbb{R}^n$ such that $f(\alpha_i) = f(\alpha_j)$ for all $i, j = 1, 2, \ldots, n$.

Now, let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of $\mathbb{R}^n$. Consider function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\alpha) = \inf\{\|\alpha - \alpha_i\| : i = 1, 2, \ldots, n\}$. Clearly, f is continuous (cf. [15, §27]) and $f(\alpha_i) = f(\alpha_j)$ for all $i, j = 1, 2, \ldots, n$.

Based on the above two observations, it is natural to ask the following question.

Problem 1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Does there exists a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $\mathbb{R}^n$ such that $f(\alpha_i) = f(\alpha_j)$ for all $i, j = 1, 2, \ldots, n$?*

We will answer this question in the affirmative by proving Theorem 7. To do this, we first need to establish some additional notations and terminology. We will say that an interval $\mathcal{I} \subset \mathbb{R}$ is non-degenerate if it contains more than one element. Under a function $f : \mathcal{X} \to \mathcal{Y}$, the preimage of $y \in \mathcal{Y}$, denoted by $f^{-1}(y)$, is the set $\{x : f(x) = y\}$, and the preimage of $\mathcal{Z} \subset \mathcal{Y}$, denoted by $f^{-1}(\mathcal{Z})$, is the set $\{x : f(x) \in \mathcal{Z}\}$. When we use the term *subspace*, we mean *vector subspace*. For all topological terminology, such as interior point, open set, and connected set, we refer to [15]. Now, we proceed to prove the following theorem.

Theorem 7 *Let f be a continuous function from $\mathbb{R}^n$ onto a non-degenerate interval $\mathcal{I} \subset \mathbb{R}$. Then for each interior point c of $\mathcal{I}$, $\dim(\mathrm{span}(f^{-1}(c)))$ is either $n-1$ or n. Furthermore, $\dim(\mathrm{span}(f^{-1}(c))) = n-1$ holds for at most one interior point c of $\mathcal{I}$.*

Proof. First, let us show that if $\mathcal{S}$ is a proper subset of an $(n-1)$-dimensional subspace $\mathcal{S}_1$ of $\mathbb{R}^n$, then set $\mathbb{R}^n \setminus \mathcal{S}$ is connected. To see this, consider a basis $\{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ of $\mathcal{S}_1$ and extend it to a basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of $\mathbb{R}^n$. Now, consider linear function $g : \mathbb{R}^n \to \mathbb{R}$ which satisfies $g(\alpha_i) = 0$, $i = 1, 2, \dots, n-1$, and $g(\alpha_n) = 1$. Observe that $g^{-1}(0) = \mathcal{S}_1$. We will now show that the set $\mathcal{S}_2 := g^{-1}((0, \infty))$ is connected. If we take any two elements α and β from $\mathcal{S}_2$, we find that $g(x\alpha + (1-x)\beta) = xg(\alpha) + (1-x)g(\beta) > 0$ for $0 \le x \le 1$. This shows that $x\alpha + (1-x)\beta \in \mathcal{S}_2$, which implies that $\mathcal{S}_2$ is a connected set. Similarly, we can show that $\mathcal{S}_3 := g^{-1}((-\infty, 0))$ is also connected. Note that $\mathbb{R}^n \setminus \mathcal{S} = (\mathcal{S}_1 \setminus \mathcal{S}) \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Suppose $\mathbb{R}^n \setminus \mathcal{S}$ is not connected. Then there exist two disjoint non-empty open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathbb{R}^n \setminus \mathcal{S} = \mathcal{U}_1 \cup \mathcal{U}_2$. Choose an element α in $\mathcal{S}_1 \setminus \mathcal{S}$ and assume that α lies in $\mathcal{U}_1$. Consequently, there exists an open ball $\mathsf{B}(\alpha; r)$, centered at α with radius $r > 0$ such that $\mathsf{B}(\alpha; r) \subset \mathcal{U}_1$. Now, $\|(\alpha + x\alpha_n) - \alpha\| = |x|\|\alpha_n\|$. Hence, $\alpha + x\alpha_n$ lies in $\mathcal{U}_1$ if $|x| < (r/\|\alpha_n\|)$. Furthermore, we find that $g(\alpha + x\alpha_n) = g(\alpha) + xg(\alpha_n) = x$. Therefore, $\alpha + x\alpha_n$ is in $\mathcal{S}_2$ when $x > 0$ and in $\mathcal{S}_3$ when $x < 0$. As a result, $\mathcal{U}_1$ has non-empty intersections with the connected sets $\mathcal{S}_2$ and $\mathcal{S}_3$. This implies that both $\mathcal{S}_2$ and $\mathcal{S}_3$ are contained in $\mathcal{U}_1$. Consequently, the set $\mathcal{S}_1 \setminus \mathcal{S}$ is also contained in $\mathcal{U}_1$. Therefore, $\mathcal{U}_2$ is empty, which is a contradiction. Hence, we conclude that $\mathbb{R}^n \setminus \mathcal{S}$ is connected.

Assume now that there exists an interior point c of $\mathcal{I}$ such that $f^{-1}(c)$ is a proper subset of an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Then $\mathcal{S}_4 := \mathbb{R}^n \setminus f^{-1}(c)$ is connected. It is a known result that continuous image of a connected set is connected (see [15, §23, Theorem 23.5]). Thus, $f(\mathcal{S}_4)$ is connected. However, we also have $f(\mathcal{S}_4) = \mathcal{I} \setminus \{c\}$, which is not connected. This leads to a contradiction, and we conclude that $\dim(\mathrm{span}(f^{-1}(c)))$ must be either $n-1$ or n. Now, suppose there are two interior points c and d in $\mathcal{I}$

such that $\dim(\mathrm{span}(f^{-1}(c))) = \dim(\mathrm{span}(f^{-1}(d))) = n-1$. From the previous discussions, it is evident that $\mathrm{span}(f^{-1}(c)) = f^{-1}(c)$ and $\mathrm{span}(f^{-1}(d)) = f^{-1}(d)$. Therefore, $f^{-1}(c) \cap f^{-1}(d)$ is non-empty, which leads to a contradiction. This completes the proof. $\square$

We now demonstrate the above result by an example.

Example 3 Consider continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x$. It is easy to verify that $\dim(\mathrm{span}(f^{-1}(0))) = 1$ and $\dim(\mathrm{span}(f^{-1}(c))) = 2$ when $c \neq 0$.

Remark 6 It is clear from the proof of Theorem 7 that if $\dim(\mathrm{span}(f^{-1}(c))) = n-1$ holds at an interior point $c \in \mathcal{I}$, then $\mathrm{span}(f^{-1}(c)) = f^{-1}(c)$.

We now explore a property of symmetric bilinear forms using Theorem 7 and Remark 6.

Lemma 6 *Let $\langle\, , \rangle$ be a symmetric bilinear form on $\mathbb{R}^n$. Then function f from $\mathbb{R}^n$ onto an interval $\mathcal{I} \subset \mathbb{R}$ given by $f(\alpha) = \langle \alpha, \alpha \rangle$ is continuous. Also, for each interior point c of $\mathcal{I}$,* $\dim(\mathrm{span}(f^{-1}(c))) = n$.

Proof. The continuity of function f follows from the continuity of $\langle\, , \rangle$. Now, assume that $\dim(\mathrm{span}(f^{-1}(c))) \neq n$ for some interior point c of $\mathcal{I}$. It is evident from Remark 6 that $f^{-1}(c)$ is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. If $c \neq 0$, we choose a non-zero element α in $f^{-1}(c)$. Note that for any real number x, $f(x\alpha) = x^2 f(\alpha) = x^2 c$. Therefore, $x\alpha$ is not in $f^{-1}(c)$ when $x^2 \neq 1$, which contradicts the fact that $f^{-1}(c)$ is a subspace. On the other hand, if $c = 0$, we can choose two elements α and β such that $f(\alpha) > 0$ and $f(\beta) < 0$. Consider now polynomial $f(\alpha + x\beta) = f(\alpha) + 2x\langle \alpha, \beta \rangle + x^2 f(\beta)$ in x. Since $f(\alpha)f(\beta) < 0$, polynomial $f(\alpha + x\beta)$ has two distinct real roots, denoted as x_1 and x_2. Therefore, $\alpha + x_1\beta$ and $\alpha + x_2\beta$ are in $f^{-1}(c)$, and consequently, α and β are also in $f^{-1}(c)$. Thus, we derive a contradiction and we conclude that $\dim(\mathrm{span}(f^{-1}(c))) = n$. $\square$

Remark 7 Theorems 3 and 4, along with Corollary 1, are immediate consequences of Lemma 6 and Proposition 1. Additionally, Lemma 6 applies to bilinear forms that are not symmetric as well. The proof for this case is similar to that of Lemma 6.

In Theorem 7 and Lemma 6, if c is not an interior point, we cannot conclude anything about $\dim(\mathrm{span}(f^{-1}(c)))$. We illustrate this with the following example.

Example 4 Let us consider the function $f : \mathbb{R}^n \to [0, \infty)$ given by $f(\alpha) = \sum_{i=1}^{k} a_i^2$, where $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. $f(\alpha) = 0$ if and only if $a_i = 0$ for all $i = 1, 2, \ldots, k$. Thus, $f^{-1}(0)$ is an $(n-k)$-dimensional subspace of $\mathbb{R}^n$.

In conclusion, we propose the following important problem related to Theorem 7.

Problem 2 *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a non-constant continuous function, where $m > 1$ and $n > 1$. For $\alpha \in \mathbb{R}^n$, determine all possible values of* $\dim(\mathrm{span}(f^{-1}(\alpha)))$.

5 Conclusion

In this paper, our primary focus lies on isotropic bases of real symmetric bilinear space. However, it is important to note that bilinear forms can also be defined on vector spaces over other fields. Thus, understanding the existence of different types of bases in bilinear spaces, along with the methods for their construction, is of significant interest from the theoretical perspective. Also, this knowledge could be beneficial in the fields where bilinear spaces play a crucial role; for example, see [2–6, 12–14, 16].

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