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On Growth of Free Burnside Groups

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Abstract. In his seminal monograph, S. I. Adian established a lower bound on the growth of the free Burnside group $B(2,n)$ for odd $n \geq 665$. Building on Adian's methods, we extend this result to free Burnside groups $B(m,n)$ of arbitrary rank $m \geq 2$. As a consequence, we obtain growth estimates for a broad class of finitely generated groups.

Key Words: Growth of Groups, Free Burnside Group, Exponential Growth, n-Torsion Group
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Introduction

The growth of a finitely generated group measures how quickly the number of its elements increases with their word length. To define this formally, let G be a finitely generated group, and S be a finite generating set for G. For every element $g \in G$, denote by $\ell_S(g)$ the word length of g, i.e., the smallest integer $r \geq 0$ for which there exist $s_1, \ldots, s_r \in S$ such that $g = s_1^{\varepsilon_1} \ldots s_r^{\varepsilon_r}$, where $\varepsilon_i \in \{1, -1\},\ i \in \{1, \ldots, r\}$. The *growth function* of a group G with respect to a generating set S, denoted by $\gamma_{G,S}(r)$, is the number of elements $g \in G$ such that $\ell_S(g) \leq r$. Although the growth function $\gamma_{G,S}$ depends on the particular choice of generating set S, its asymptotic behavior is well-defined up to a natural equivalence relation. Two growth functions f and g are said to be equivalent, $f \sim g$, if there exists a constant $C > 0$ such that for all sufficiently large r,

$$f(r) \leq g(Cr) \quad \text{and} \quad g(r) \leq f(Cr).$$

It is straightforward to show that for any two finite generating sets S and S' of the same group G, the corresponding growth functions $\gamma_{G,S}$ and $\gamma_{G,S'}$ are equivalent in this sense. Consequently, one can speak of the growth type of G, regardless of the generating set used, as the equivalence class of its growth function under this relation.

A group G is said to have *polynomial growth* if $\gamma_{G,S}(r) \sim r^d$ for some $d \geq 0$. If $\gamma_{G,S}(r) \sim e^r$, then G is said to have *exponential growth*. Finally, we say that G has *intermediate growth* if its growth is faster than any polynomial and slower than any exponential function. It is known that finitely generated groups have either polynomial, exponential or intermediate growth.

The growth of finitely generated groups has been studied extensively in recent decades. It is well known that finitely generated solvable groups have either polynomial or exponential growth (see, for example, [12, 15]). The same dichotomy applies to linear groups by the Tits alternative (see [14]). One of the most striking results in geometric group theory is Gromov's theorem [11] asserting that finitely generated groups of polynomial growth are exactly the virtually nilpotent groups. In 1984, Grigorchuk [10] constructed the first example of a group of intermediate growth, providing a negative answer to a long-standing question by Milnor on the possible types of growth.

The notion of *growth rate* is important to differentiate between types of growth. For a finitely generated group G with a finite generating set S, its growth rate is defined to be

$$\lambda(G, S) = \lim_{r\to\infty} \gamma_{G,S}^{\frac{1}{r}}(r).$$

It is clear that having exponential growth type is equivalent to $\lambda(G, S) > 1$. If

$$\inf_S \lambda(G, S) > 1,$$

where the infimum is taken over all finite generating sets S, then we say that the group G has *uniform exponential growth*.

Groups with exponential growth include free groups, non-elementary hyperbolic groups, certain Baumslag-Solitar groups, and the free Burnside groups $B(m, n)$. The growth of free Burnside groups is of particular interest and constitutes the main focus of this paper. This focus is motivated, in part, by the use of a lower bound on the growth function $\gamma_{B(3,n)}$ in the computations presented in [7].

We now recall the definition of the free Burnside group. For integers $m \geq 1$ and $n \geq 2$, the free Burnside group $B(m, n)$ is defined as the quotient of the free group of rank m by the normal subgroup generated by all n-th powers. That is,

$$B(m, n) = \langle a_1, \ldots, a_m \mid X^n = 1\rangle,$$

where X ranges over all elements in the free group.

Adian [1] showed that $B(m, n)$ has exponential growth for odd $n \geq 665$ and $m > 1$. For the case $m = 2$, he obtained an explicit lower bound on the growth function. These groups exhibit uniform exponential growth for odd $n \geq 1003$. Moreover, as it was shown in [5], any non-cyclic subgroup of $B(m, n)$ has uniform exponential growth (see also [4]). In [2], it was proven that n-periodic products of groups also have uniform exponential growth for odd $n \geq 1003$. For free Burnside groups, a different approach to this problem can be found in [13]. For an exposition of the growth of periodic quotients of hyperbolic groups, we refer to [9].

In this paper, we generalize Adian's estimate to free Burnside groups of arbitrary rank $m \geq 2$. More precisely, we establish new explicit lower bounds on the growth function $\gamma_{B(m,n),S}$ with respect to a set of free generators S for odd exponents $n \geq 665$.

1 The Main Result

Let $B(m,n)$ be a free Burnside group. Denote by $S = \{a_1, \ldots, a_m\}$ its generating set and let $\gamma = \gamma_{B(m,n),S}$ be the growth function of $B(m,n)$ with respect to S.

Theorem 1 *For all $m \geq 2$, odd $n \geq 665$ and any natural number s, the following inequality holds:*

$$\gamma(s) > \frac{m}{m-1}(2m-1-2\cdot 10^{-3})^s - 1.$$

Proof. Let $\overline{\gamma}(s)$ denote the number of elements of exact length s in $B(m,n)$, and let $\beta(s)$ denote the number of reduced words of length s that do not contain subwords of the form A^8. By Definition 4.34 in [1, Ch.I], all such words are absolutely reduced. Hence, such different words represent distinct elements of $B(m,n)$. Therefore, as in the proof of Theorem 2.15 in [1, Ch.VI], we conclude that

$$\overline{\gamma}(s) \geq \beta(s).$$

For $s \geq 8i$, denote by $\delta_i(s)$ the number of reduced words of the form XB^8, where $i = |B|, s = |XB^8|$, and X does not contain a subword of the form A^8. Obviously,

$$\delta_i(s) \leq \beta(s-8i)(2m)^i.$$

To estimate $\beta(s+1)$, we note that every reduced word of length $s+1$ is obtained by juxtaposing to a reduced word Y of length s one of the $2m-1$ letters from $S \cup S^{-1}$ that is not the inverse of the last letter of that word. Also, if no subword of the form A^8 occurs in a word Y, then such a subword can occur only at the end of Ya, where $a \in S \cup S^{-1}$. Thus,

$$\beta(s+1) \geq (2m-1)\beta(s) - \sum_{i=1}^{[\frac{s+1}{8}]} \delta_i(s+1) \geq$$

$$\geq (2m-1)\beta(s) - \sum_{i=1}^{[\frac{s+1}{8}]} \beta(s+1-8i)\cdot(2m)^i, \quad (1)$$

where $[c]$ denotes the integer part of number c.

Let $\alpha = 2 \cdot 10^{-3}$. Now we prove that

$$\beta(s+1) > (2m-1-\alpha)\beta(s) \tag{2}$$

by induction on s. For $0 < s < 7$, the inequality is obvious, since in this case $\beta(s+1) = (2m-1)\beta(s)$. Assume that the inequality is true for all $s < t$. Then for $1 \le i \le [(t+1)/8]$,

$$\beta(t) > (2m-1-\alpha)\beta(t-1) > \cdots > (2m-1-\alpha)^{8i-1}\beta(t-(8i-1)),$$

therefore,

$$\beta(t-(8i-1)) < \frac{\beta(t)}{(2m-1-\alpha)^{8i-1}}.$$

From the last inequality and (1), we get

$$\beta(t+1) \ge (2m-1)\beta(t) - \sum_{i=1}^{[\frac{t+1}{8}]} \frac{\beta(t)\cdot(2m)^i}{(2m-1-\alpha)^{8i-1}} \ge$$

$$\ge \beta(t)\left(2m-1-(2m-1-\alpha)\sum_{i=1}^{\infty}\left(\frac{2m}{(2m-1-\alpha)^8}\right)^i\right) =$$

$$= \beta(t)\left(2m-1-\frac{2m(2m-1-\alpha)}{(2m-1-\alpha)^8-2m}\right).$$

It is easy to see that the inequality

$$\frac{2m(2m-1-\alpha)}{(2m-1-\alpha)^8-2m} < \alpha$$

is equivalent to

$$2m(2m-1) < \alpha(2m-1-\alpha)^8, \tag{3}$$

which holds for any $m \ge 2$. Since $\beta(1) = 2m$, from (2) we obtain

$$\beta(s) \ge 2m \cdot (2m-1-\alpha)^{s-1},$$

and hence,

$$\overline{\gamma}(s) \ge 2m \cdot (2m-1-\alpha)^{s-1}$$

for arbitrary $s > 0$. Finally, we get

$$\gamma(s) = 1+\sum_{i=1}^{s}\overline{\gamma}(i) \ge 1+2m\sum_{i=1}^{s}(2m-1-\alpha)^{i-1} = 1+2m\frac{(2m-1-\alpha)^s-1}{2m-2-\alpha} =$$

$$= \frac{2m(2m-1-\alpha)^s-2-\alpha}{2m-2-\alpha} > \frac{m}{m-1}(2m-1-\alpha)^s - 1.$$

This completes the proof. □

Remark 1 *The proof of the result shows that the inequality*

$$\gamma(s) > \frac{m}{m-1}(2m-1-\alpha)^s - 1$$

holds for any $\alpha \in (0,1)$ *satisfying* (3)*.*

2 Some Applications

We will now show that Theorem 1 leads to explicit lower bounds on the growth functions of a broad class of finitely generated groups. The argument relies on the observation that the growth of a group G is bounded below by the growth of any of its homomorphic images. Consequently, if an m-generated group G admits a surjective homomorphism onto the free Burnside group $B(m,n)$ for an odd $n \geq 665$, Theorem 1 immediately provides a lower bound for the growth of G. This proves the following general result.

Corollary 1 *Let G be an m-generated group, where $m \geq 2$. If G admits a surjective homomorphism onto the free Burnside group $B(m,n)$ for some odd integer $n \geq 665$, then there exists an m-element generating set S for G such that its growth function satisfies*

$$\gamma_{G,S}(s) > \frac{m}{m-1}(2m-1-2\cdot 10^{-3})^s - 1.$$

An important class of groups satisfying the condition of Corollary 1 is that of n-torsion groups. Let S be an arbitrary group alphabet, $\mathcal{R}$ be a set of words over this alphabet, and let $n > 1$ be a fixed natural number. A group G defined by the presentation

$$G = \langle S \mid R^n = 1,\ R \in \mathcal{R}\rangle. \tag{4}$$

is called an *n-torsion group* (see [3], Definition 1.1) if for any element $y \in G$, either $y^n = 1$ or y has infinite order. By Proposition 1.1 in [3], G admits a surjective homomorphism onto the free Burnside group $B(S,n) \cong B(m,n)$. Thus, our corollary shows that all finitely generated non-cyclic n-torsion groups for odd $n \geq 665$ exhibit exponential growth at a rate bounded below by the growth of the corresponding free Burnside group.

Our main theorem can also be used to strengthen existing results on product set growth within Burnside groups. For any finite subset $S \subset B(m,n)$, define

$$S^s = \{a_1 a_2 \dots a_s \mid a_i \in S\}.$$

In [8], it was proved that for any finite symmetric subset S of the free Burnside group $B(m,n)$ that generates a non-cyclic subgroup, the inequality

$$|S^s| \geq 4 \cdot 2.9^{s/(400d)^3 - 1} \tag{5}$$

takes place, where d is the least odd divisor of the number n satisfying the inequality $d \geq 1003$. In the proof of (5), Adian's growth estimate for free Burnside groups was used. The same reasoning allows us to strengthen the aforementioned inequality.

Corollary 2 *For any finite symmetric subset S of the free Burnside group $B(m,n)$ of odd exponent $n \geq 1003$ that generates a non-cyclic subgroup, the inequality*

$$|S^s| \geq 4 \cdot 2.998^{s/(400d)^3-1}$$

takes place, where d is the least odd divisor of the number n satisfying the inequality $d \geq 1003$.

This result reinforces the fact that non-cyclic subgroups of free Burnside groups of sufficiently large odd exponent have uniform exponential growth.

Analogously, we can strengthen Theorem 1 from the paper [6].

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