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On a convergence of the Fourier-Pade interpolation

Arnak Poghosyan

Institute of Mathematics,
National Academy of Sciences of Armenia
Bagramian ave. 24b, 0019 Yerevan, Armenia
arnak@instmath.sci.am

Abstract

We investigate convergence of the rational-trigonometric-polynomial interpolation that performs convergence acceleration of the classical trigonometric interpolation by sequential application of polynomial and rational correction functions. Unknown parameters of the rational corrections are determined along the ideas of the Fourier-Pade approximations. The resultant interpolation we call as Fourier-Pade interpolation and investigate its convergence in the regions away from the endpoints. Comparison with other rational-trigonometric-polynomial interpolations outlines the convergence properties of the Fourier-Pade interpolation.

Key Words: Convergence Acceleration, Fourier-Pade interpolation, Rational interpolation, Krylov-Lanczos interpolation
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Introduction

In this paper we continue investigations of [9] where we considered convergence acceleration of the classical trigonometric interpolation

$$
\begin{aligned}
I_N(f;x) &= \sum_{n=-N}^{N} \check{f}_n e^{i\pi nx}, \\
\check{f}_n &= \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) e^{-i\pi n x_k}, \ x_k = \frac{2k}{2N+1}
\end{aligned}
$$

via sequential application of rational and polynomial corrections. We recap some details from [9].

Let $r_N(f;x)$ be the error of the classical trigonometric interpolation

$$r_N(f;x) = f(x) - I_N(f;x).$$

We have

$$r_N(f) = \sum_{n=-N}^{N} (f_n - \check{f}_n) e^{i\pi nx} + \sum_{n=N+1}^{\infty} f_n e^{i\pi nx} + \sum_{n=-\infty}^{-N-1} f_n e^{i\pi nx},$$

where f_n is the n-th Fourier coefficient of f

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi nx} dx.$$

Consider a finite sequence of complex numbers $\theta = \{\theta_k\}_{|k|=1}^{p}$ and denote

$$\delta_n^0(\theta, c_n) = c_n,$$

$$\delta_n^k(\theta, c_n) = \delta_n^{k-1}(\theta, c_n) + \theta_{-k}\delta_{n-1}^{k-1}(\theta, c_n) + \theta_k\left(\delta_{n+1}^{k-1}(\theta, c_n) + \theta_{-k}\delta_n^{k-1}(\theta, c_n)\right)$$

for some sequence c_n. By $\delta_n^k(c_n)$ we denote the sequence that corresponds to the choice $\theta \equiv 1$. It is easy to check that

$$\delta_n^k(c_n) = \Delta_{n+k}^{2k}(c_n),$$

where $\Delta_n^k(c_n)$ are the classical backward finite differences defined by the recurrence relation

$$\begin{aligned} \Delta_n^0(c_n) &= c_n, \\ \Delta_n^k(c_n) &= \Delta_n^{k-1}(c_n) + \Delta_{n-1}^{k-1}(c_n). \end{aligned}$$

Reiterations of the Abel transformation up to p times leads to the following expansion of the error

$$\begin{aligned} r_N(f) &= (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_{-k}\delta_N^{k-1}(\theta, \check{f}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_k\delta_{-N}^{k-1}(\theta, \check{f}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ &+ \frac{1}{\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta_n^p(\theta, f_n) e^{i\pi nx} \\ &+ \frac{1}{\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{n=-N}^{N} \delta_n^p(\theta, f_n - \check{f}_n) e^{i\pi nx}, \end{aligned}$$

where the first two terms in the right-hand side can be viewed as rational corrections and the last two terms as the actual error. This viewing leads to the following rational-trigonometric interpolation

$$\begin{aligned} I_N^p(f;x) &= I_N(f,x) + (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_{-k}\delta_N^{k-1}(\theta, \check{f}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \\ &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\theta_k\delta_{-N}^{k-1}(\theta, \check{f}_n)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \end{aligned}$$

with the error

$$r_N^p(f;x) = \frac{1}{\prod_{s=1}^p (1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta_n^p(\theta, f_n) e^{i\pi n x}$$

$$+ \frac{1}{\prod_{s=1}^p (1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{n=-N}^{N} \delta_n^p(\theta, f_n - \check{f}_n) e^{i\pi n x}. \quad (1)$$

Additional acceleration of the rational-trigonometric interpolation can be achieved by application of the polynomial corrections.

Let $f \in C^{q-1}[-1,1]$. By $A_k(f)$ denote the jumps of f at the end points of the interval

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1),\ k = 0, \dots, q-1.$$

The polynomial correction method is based on the following representation of the interpolated function

$$f(x) = \sum_{k=0}^{q-1} A_k(f) B_k(x) + F(x), \quad (2)$$

where B_k are 2-periodic Bernoulli polynomials

$$B_0(x) = \frac{x}{2},\ B_k(x) = \int B_{k-1}(x)dx,\ \int_{-1}^{1} B_k(x)dx = 0,\ x \in [-1,1]$$

with the Fourier coefficients

$$B_n(k) = \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}},\ n \neq 0,\ B_0(k) = 0.$$

Function F is a 2-periodic and relatively smooth function on the real line ($F \in C^{q-1}(R)$) with the discrete Fourier coefficients

$$\check{F}_n = \check{f}_n - \sum_{k=0}^{q-1} A_k(f) \check{B}_n(k).$$

Approximation of F in (2) by the classical trigonometric interpolation leads to the Krylov-Lanczos (KL-) interpolation

$$I_{N,q}(f;x) = \sum_{k=0}^{q-1} A_k(f) B_k(x) + I_N(F;x)$$

and approximation of F by the rational-trigonometric interpolation leads to the rational-trigonometric-polynomial (RTP-) interpolation

$$I_{N,q}^p(f;x) = \sum_{k=0}^{q-1} A_k(f) B_k(x) + I_N^p(F;x)$$

with the errors $r_{N,q}(f;x)$ and $r_{N,q}^p(f;x)$, respectively.

The Krylov-Lanczos polynomial correction approach was suggested in 1906 by Krylov [6] and later in 1964 by Lanczos [7], [8] (see also [1], [3]-[5], [10] with references therein).

The next results we need for further comparisons.

Denote

$$\phi_m = \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^m}.$$

Theorem 1 *[10] Let $q \geq 2$ be even, $f \in C^{q+1}[-1,1]$ and $f^{(q+1)} \in AC[-1,1]$. Then the following estimate holds for $|x| < 1$*

$$r_{N,q}(f;x) = A_q(f)\frac{(-1)^{N+\frac{q}{2}}}{2\pi^{q+1}N^{q+1}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}}\phi_{q+1} + o(N^{-q-1}),\ N \to \infty.$$

Theorem 2 *[10] Let $q \geq 1$ be odd, $f \in C^{q+2}[-1,1]$ and $f^{(q+2)} \in AC[-1,1]$. Then the following estimate holds for $|x| < 1$*

$$\begin{aligned} r_{N,q}(f;x) &= A_q(f)\frac{(-1)^{N+\frac{q+1}{2}+1}(q+1)}{4\pi^{q+1}N^{q+2}}\frac{\sin\frac{\pi x}{2}\sin\frac{\pi x}{2}(2N+1)}{\cos^2\frac{\pi x}{2}}\phi_{q+2} \\ &+ A_{q+1}(f)\frac{(-1)^{N+\frac{q+1}{2}}}{2\pi^{q+2}N^{q+2}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos\frac{\pi x}{2}}\phi_{q+2} + o(N^{-q-2}),\ N \to \infty. \end{aligned}$$

The approach of rational corrections applied for convergence acceleration of the classical trigonometric interpolation was suggested in [13]. The idea of such corrections is coming from the Fourier-Pade approximation (see [2]) where rational corrections are applied for convergence acceleration of the truncated Fourier series.

Rational-trigonometric-polynomial interpolation is undefined nevertheless parameters θ are unknown. One approach was considered in [9], [11] and [12] where

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N},\ k = 1, \cdots, p \tag{3}$$

and τ_k are independent of N.

Let θ_k be defined as in (3). Let $\gamma_k(\tau)$ be the coefficients of polynomial

$$\prod_{s=1}^{p}(1+\tau_s x) = \sum_{s=0}^{p}\gamma_s(\tau)x^s.$$

Also denote

$$\psi_{m,p} = \sum_{s=0}^{p}(-1)^s\gamma_s(\tau)\sum_{k=0}^{p}\gamma_k(\tau)(2p-k-s+m)!\phi_{2p-k-s+m+1}.$$

Theorem 3 *[9] Let $q \geq 2$ be even and $f \in C^{q+2p+1}[-1,1]$ with $f^{(q+2p+1)} \in AC[-1,1]$ for some $p \geq 1$. Let parameters θ_k be chosen as in (3). Then the following estimate holds for $|x| < 1$*

$$r^p_{N,q}(f;x) = A_q(f)\frac{(-1)^{N+p+\frac{q}{2}}}{2^{2p+1}\pi^{q+1}q!N^{2p+q+1}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q,p} + o(N^{-2p-q-1}),\ N \to \infty.$$

Theorem 4 *[9] Let $q \geq 1$ be odd and $f \in C^{q+2p+2}[-1,1]$ with $f^{(q+2p+2)} \in AC[-1,1]$ for some $p \geq 1$. Let parameters θ_k be chosen as in (3). Then the following estimate holds for $|x| < 1$*

$$\begin{aligned} r_{N,q}^{p}(f;x) &= A_q(f)\frac{(-1)^{N+p+\frac{q+1}{2}+1}}{2^{2p+2}\pi^{q+1}q!N^{2p+q+2}}\frac{\sin\frac{\pi x}{2}\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+2}\frac{\pi x}{2}}\psi_{q+1,p} \\ &+ A_{q+1}(f)\frac{(-1)^{N+p+\frac{q+1}{2}}}{2^{2p+1}\pi^{q+2}(q+1)!N^{2p+q+2}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q+1,p} \\ &+ o(N^{-2p-q-2}),\ N\to\infty. \end{aligned}$$

Comparison with Theorems 1 and 2 shows that for smooth functions RTP-interpolation is asymptotically more precise than the KL-interpolation and improvement in precision is by the factor $O(N^{2p})$ as $N \to \infty$.

Theorems 3 and 4 are valid nonetheless parameters τ_k are still undefined and this allows to achieve additional accuracy in different frameworks. One such approach was realized in [11] where the L_2-minimal RTP-interpolation was introduced and investigated for $p = 1$. Another approach was considered in [9] where parameters τ_k are the roots of the associated Laguerre polynomials $L_p^q(x)$.

In this paper we consider essentially different approach for determination of parameters θ which we determine by the following system

$$\delta_n^p(\theta, \check{F}_n) = 0,\ |n| = N-p+1,\cdots,N. \tag{4}$$

Taking into account that the discrete Fourier coefficients are periodic

$$\check{f}_{n+s(2N+1)} = \check{f}_n$$

then system (4) can be rewritten in the equivalent form

$$\delta_n^p(\theta, \check{F}_n) = 0,\ n = N-p+1,\cdots,N+p. \tag{5}$$

This leads to the *Fourier-Pade interpolation* ([13]). We investigate convergence of such interpolations for smooth functions and show its fast convergence compared to the choice as in (3). In particular, we show additional improvement in accuracy by the factor $O(N^{2p})$ compared to Theorems 3 and 4 and hence by the factor $O(N^{4p})$ compared to Theorems 1 and 2.

1 Pointwise Convergence of the Fourier-Pade Interpolation away from the Endpoints

Throughout the paper we consider that interpolated function is smooth on $[-1,1]$ and the exact values of jumps $A_k(f)$ are known. Parameter θ is supposed to be determined from system (4). Let $\gamma_s(\theta^{\pm})$ be the coefficients of the polynomials

$$\prod_{k=1}^{p}(1+\theta_k x) = \sum_{k=0}^{p}\gamma_k(\theta^+)x^k$$

and

$$\prod_{k=1}^{p}(1+\theta_{-k} x) = \sum_{k=0}^{p}\gamma_k(\theta^-)x^k.$$

Note that generalized finite differences $\delta_n^p(\theta,\check{F}_n)$ can be represented by the coefficients $\gamma_s(\theta^{\pm})$

$$\delta_n^p(\theta,\check{F}_n) = \sum_{s=0}^{p}\gamma_s(\theta^+)\sum_{k=0}^{p}\gamma_k(\theta^-)\check{F}_{n+s-k} \tag{6}$$

and hence system (4) can be rewritten in the form

$$\sum_{s=0}^{p}\gamma_s(\theta^+)\sum_{k=0}^{p}\gamma_k(\theta^-)\check{F}_{n+s-k} = 0,\ \ |n| = N-p+1,\cdots,N$$

or according to (5)

$$\sum_{s=0}^{p}\gamma_s(\theta^+)\sum_{k=0}^{p}\gamma_k(\theta^-)\check{F}_{n+s-k} = 0,\ \ n = N-p+1,\ldots,N+p. \tag{7}$$

We can rewrite (5) using also the finite differences Δ_n^k

$$\Delta_{N+p}^{w}(\delta_n^p(\theta,\check{F}_n)) = 0,\ \ w = 0,\ldots,2p-1. \tag{8}$$

Denote

$$\begin{aligned} X_{w,m}(j,z) &= \binom{j+w+m}{m}\binom{j+w}{z}\sum_{y=0}^{j-z}(-1)^{j-z-y}\binom{j+w-z}{y+w}\alpha_{w,y+w} \\ &\times \sum_{r=-\infty}^{\infty}(-1)^r\frac{(p+r)^{j-z-y}}{(2r+1)^{j+w+m+1}}, \end{aligned}$$

where

$$\alpha_{w,y} = \sum_{\ell=0}^{w}(-1)^{\ell}\binom{w}{\ell}\ell^y.$$

In particular

$$X_{w,m}(j,j) = (-1)^w w!\binom{j+w+m}{m}\binom{j+w}{j}\phi_{j+w+m+1}.$$

It is easy to verify from definition of ϕ_m that for even values of m we have

$$X_{2w,m}(2j+1,2j+1) = X_{2w+1,m}(2j,2j) = 0$$

and for odd values

$$X_{2w,m}(2j,2j) = X_{2w+1,m}(2j+1,2j+1) = 0.$$

We are widely using these facts in coming subsections an that is why in further investigations we need to separate even and odd values of q.

1.1 Even values of q

First we prove some lemmas concerning the properties of the generalized finite differences.

Lemma 1 *Let* $f \in C^{4p+q+1}[-1,1]$ *and* $f^{(4p+q+1)} \in AC[-1,1]$*. Let the coefficients* $\gamma_s(\theta^{\pm})$ *satisfy the system (7). Then*

$$\delta_n^w(\delta_n^p(\theta, F_n)) = O(n^{-q-1-2w}) + o(n^{-q-4p-2}),\ |n| > N,\ N \to \infty.$$

Proof. According to the smoothness of f and expansion (2) we have

$$F_n = \sum_{m=q}^{q+4p+1} A_m(f) B_n(m) + o(n^{-q-4p-2}),\ |n| > N,\ N \to \infty. \tag{9}$$

Taking into account that $\check{F}_n = \sum_s F_{n+s(2N+1)}$ we get from (9) and system (7) that $\gamma_s(\theta^{\pm})$ have finite limits as $N \to \infty$

$$\gamma_s(\theta^{\pm}) \to \binom{p}{s},\ N \to \infty. \tag{10}$$

Thus, in view of (9) we have

$$\delta_n^w(\delta_n^p(\theta, F_n)) = \sum_{m=q}^{q+4p+1} A_m(f) \delta_n^w(\delta_n^p(\theta, B_n(m))) + o(n^{-q-4p-2}).$$

This completes the proof as

$$\delta_n^w(\delta_n^p(\theta, B_n(m))) = \Delta_{n+w}^{2w}(\delta_n^p(\theta, B_n(m))) = \sum_{s=0}^{p} \gamma_s(\theta^+) \sum_{k=0}^{p} \gamma_k(\theta^-) \Delta_{n+w}^{2w}(B_{n+s-k}(m))$$

and

$$\Delta_{n+w}^{2w}(B_n(m)) = O(n^{-m-1-2w}).$$

□

Lemma 2 *Let the conditions of Lemma 1 be valid. Then*

$$\delta_n^w(\delta_n^p(\theta, F_n - \check{F}_n)) = O(N^{-q-1-2w}) + o(N^{-q-4p-2}),\ |n| \le N,\ N \to \infty.$$

Proof. We proceed as in the proof of Lemma 1 and write

$$\begin{aligned} \delta_n^w(\delta_n^p(\theta, F_n - \check{F}_n)) &= \sum_{m=q}^{q+4p+1} A_m(f) \delta_n^w(\delta_n^p(\theta, B_n(m) - \check{B}_n(m))) \\ &+ o(N^{-q-4p-2}). \end{aligned}$$

Then

$$\begin{aligned} \delta_n^w(\delta_n^p(\theta, B_n(m) - \check{B}_n(m))) &= \Delta_{n+w}^{2w}(\delta_n^p(\theta, B_n(m) - \check{B}_n(m))) \\ &= \sum_{s=0}^{p} \gamma_s(\theta^+) \sum_{k=0}^{p} \gamma_k(\theta^-) \Delta_{n+w}^{2w}(B_{n+s-k}(m) - \check{B}_{n+s-k}(m)). \end{aligned}$$

This completes the proof as

$$\delta_{n+w}^{2w}(\delta_n^p(\theta, B_n(m) - \check{B}_n(m))) = O(N^{-m-1-2w}).$$

□

We need similar but exact estimates for $\delta_{\pm N}^w(\delta_n^p(\theta, \check{F}_n))$. First we prove important properties of the numbers $\beta_z(\ell - j)$ defined in the next lemma.

Lemma 3 *Let q be even, $f \in C^{4p+q+1}[-1,1]$ with $f^{(4p+q+1)} \in AC[-1,1]$ and $A_q(f) \neq 0$. Let the coefficients $\gamma_s(\theta^\pm)$ satisfy system (7). If determinants*

$$\det((X_{2w,q}(2j,2j))_{j,w=0}^M),\ M = 0,\dots,2p-1, \tag{11}$$

$$\det((X_{2w+1,q}(2j+1,2j+1))_{j,w=0}^M),\ M = 0,\dots,2p-1 \tag{12}$$

are nonzero then

$$\beta_z(\ell - j) = 0,\ \ell \leq 2p-1,\ 0 \leq z \leq j,\ 0 \leq j \leq u \tag{13}$$

where

$$\beta_z(u-j) = \sum_{t=0}^{u-j}\sum_{s=0}^{p}(-1)^s\sum_{k=0}^{p}(-1)^k\gamma_{k,t}^-\gamma_{s,u-j-t}^+(k-s)^z$$

and $\gamma_{s,u}^\pm$ are the coefficients of the asymptotic expansions

$$\gamma_s(\theta^+) = \sum_{u=0}^{4p+1}\frac{\gamma_{s,u}^+}{N^u} + o(N^{-4p-1}),\ \gamma_s(\theta^-) = \sum_{t=0}^{4p+1}\frac{\gamma_{k,t}^-}{N^t} + o(N^{-4p-1}).$$

Proof. We will use system (8) and we start with estimation of $\Delta_{N+p}^w(\delta_n^p(\theta, \check{F}_n))$. We proceed as in the proofs of Lemmas 1 and 2 and write

$$\Delta_{N+p}^w(\delta_n^p(\theta, \check{F}_n)) = \sum_{m=q}^{q+4p+1} A_m(f)\Delta_{N+p}^w(\delta_n^p(\theta, \check{B}_n(m))) + o(N^{-q-4p-2}). \tag{14}$$

Then

$$\Delta_{N+p}^w(\delta_n^p(\theta, \check{B}_n(m))) = \sum_{s=0}^{p}\gamma_s(\theta^+)\sum_{k=0}^{p}\gamma_k(\theta^-)\sum_{\ell=0}^{w}\binom{w}{\ell}\check{B}_{N+p+s-k-\ell}(m). \tag{15}$$

We continue by estimation of $\check{B}_{N+p+s-k-\ell}(m)$

$$\begin{aligned}
\check{B}_{N+p+s-k-\ell}(m) &= \sum_{r=-\infty}^{\infty} B_{N+p+s-k-\ell+r(2N+1)}(m)\\
&= \frac{(-1)^{N+p+s+k+\ell+1}}{2(i\pi)^{m+1}}\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(N+p+s-k-\ell+r(2N+1))^{m+1}}\\
&= \frac{(-1)^{N+p+s+k+\ell+1}}{2(i\pi N)^{m+1}}\sum_{j=0}^{\infty}\frac{1}{N^j}\binom{j+m}{m}\sum_{r=-\infty}^{\infty}(-1)^r\frac{(k+\ell-p-s-r)^j}{(2r+1)^{j+m+1}}.
\end{aligned}$$

Substituting this and the asymptotic expansions of the coefficients $\gamma_s(\theta^{\pm})$ into (15) we derive

$$
\begin{aligned}
\Delta^w_{N+p}(\delta^p_n(\theta,\check{B}_n(m))) &= \frac{(-1)^{N+p+1}}{2(i\pi N)^{m+1}}\sum_{j=0}^{\infty}\frac{1}{N^j}\binom{j+m}{m} \\
\times\ & \sum_{s=0}^{p}(-1)^s\left[\sum_{u=0}^{4p+1}\frac{\gamma^+_{s,u}}{N^u}+o(N^{-4p-1})\right]\sum_{k=0}^{p}(-1)^k\left[\sum_{t=0}^{4p+1}\frac{\gamma^-_{k,t}}{N^t}+o(N^{-4p-1})\right] \\
\times\ & \sum_{\ell=0}^{w}(-1)^\ell\binom{w}{\ell}\sum_{r=-\infty}^{\infty}(-1)^r\frac{(k+\ell-p-s-r)^j}{(2r+1)^{j+m+1}} \\
=\ & \frac{(-1)^{N+p+1}}{2(i\pi N)^{m+1}}\sum_{u=0}^{4p+1}\frac{1}{N^u}\sum_{j=0}^{u}\binom{j+m}{m}\sum_{s=0}^{p}(-1)^s\sum_{k=0}^{p}(-1)^k\sum_{t=0}^{u-j}\gamma^-_{k,t}\gamma^+_{s,u-j-t}\sum_{\ell=0}^{w}(-1)^\ell\binom{w}{\ell} \\
\times\ & \sum_{r=-\infty}^{\infty}(-1)^r\frac{(k+\ell-p-s-r)^j}{(2r+1)^{j+m+1}}+o(N^{-4p-m-2}) \\
=\ & \frac{(-1)^{N+p+1}}{2(i\pi N)^{m+1}}\sum_{u=0}^{4p+1}\frac{1}{N^u}\sum_{j=0}^{u}\sum_{z=0}^{j}\beta_z(u-j)\binom{j+m}{m}\binom{j}{z} \\
\times\ & \sum_{\ell=0}^{w}(-1)^\ell\binom{w}{\ell}\sum_{r=-\infty}^{\infty}(-1)^r\frac{(\ell-p-r)^{j-z}}{(2r+1)^{j+m+1}}+o(N^{-4p-m-2}).
\end{aligned}
$$

After some manipulations we get from here

$$
\begin{aligned}
\Delta^w_{N+p}(\delta^p_n(\theta,\check{B}_n(m))) &= \frac{(-1)^{N+p+1}}{2(i\pi N)^{m+1}}\sum_{u=0}^{4p+1}\frac{1}{N^u}\sum_{j=0}^{u}\sum_{z=0}^{j}\beta_z(u-j)\binom{j+m}{m}\binom{j}{z} \\
&\times \sum_{y=0}^{j-z}(-1)^{j-z-y}\binom{j-z}{y}\alpha_{w,y}\sum_{r=-\infty}^{\infty}(-1)^r\frac{(p+r)^{j-z-y}}{(2r+1)^{j+m+1}} \\
&+ o(N^{-4p-m-2}).
\end{aligned}
$$

Taking into account that

$$\alpha_{w,y}=0,\ 0\le y\le w-1$$

we obtain

$$
\begin{aligned}
\Delta^w_{N+p}(\delta^p_n(\theta,\check{B}_n(m))) &= \frac{(-1)^{N+p+1}}{2(i\pi N)^{m+1}N^w}\sum_{u=0}^{4p-w+1}\frac{1}{N^u}\sum_{j=0}^{u}\sum_{z=0}^{j}\beta_z(u-j)X_{w,m}(j,z) \\
&+ o(N^{-4p-m-2}).
\end{aligned}
$$

Substituting this into (14) we write

$$\begin{aligned}
\Delta^w_{N+p}(\delta^p_n(\theta,\check{F}_n)) &= \frac{(-1)^{N+p+1}}{2N^w}\sum_{m=q}^{q+4p+1}\frac{A_m(f)}{(i\pi N)^{m+1}}\sum_{u=0}^{4p-w+1}\frac{1}{N^u}\sum_{j=0}^{u}\sum_{z=0}^{j}\beta_z(u-j)X_{w,m}(j,z)\\
&+ o(N^{-4p-q-2})\\
&= \frac{(-1)^{N+p+1}}{2(i\pi N)^{q+1}N^w}\sum_{\ell=0}^{4p-w+1}\frac{1}{N^\ell}\sum_{u=0}^{\ell}\frac{A_{\ell-u+q}(f)}{(i\pi)^{\ell-u}}\sum_{j=0}^{u}\sum_{z=0}^{j}\beta_z(u-j)X_{w,\ell-u+q}(j,z)\\
&+ o(N^{-q-4p-2})\\
&= \frac{(-1)^{N+p+1}}{2(i\pi N)^{q+1}N^w}\sum_{\ell=0}^{4p-w+1}\frac{1}{N^\ell}\sum_{j=0}^{\ell}\sum_{z=0}^{j}\beta_z(\ell-j)\sum_{u=z}^{j}\frac{A_{q+j-u}(f)}{(i\pi)^{j-u}}X_{w,j-u+q}(u,z)\\
&+ o(N^{-q-4p-2}) \qquad (16)
\end{aligned}$$

Equation (16) together with system (8) implies

$$\sum_{j=0}^{\ell}\sum_{z=0}^{j}\beta_z(\ell-j)\sum_{u=z}^{j}\frac{A_{q+j-u}(f)}{(i\pi)^{j-u}}X_{w,j-u+q}(u,z)=0, \qquad (17)$$
$$w=0,\dots,2p-1;\ \ \ell=0,\dots,4p-w+1.$$

The proof of this lemma is based on (17). We proceed by the mathematical induction. For $\ell=0$ in (17) we have

$$A_q(f)\beta_0(0)X_{w,q}(0,0)=0.$$

Taking $w=0$ and recalling that $A_q(f)\neq 0$ we obtain

$$\beta_0(0)X_{0,q}(0,0)=0.$$

From here we conclude that $\beta_0(0)=0$ as $X_{0,q}(0,0)=\phi_{q+1}\neq 0$ for even values of q.

For $\ell=1$ in (17) we have

$$\beta_0(1)A_q(f)X_{w,q}(0,0)+\beta_0(0)\sum_{u=0}^{1}\frac{A_{q+1-u}(f)}{(i\pi)^{1-u}}X_{w,1-u+q}(u,0)+\beta_1(0)A_q(f)X_{w,q}(1,1)=0.$$

Then, as $\beta_0(0)=0$ and $A_q(f)\neq 0$

$$\beta_0(1)X_{w,q}(0,0)+\beta_1(0)X_{w,q}(1,1)=0.$$

We take $w=0,1$ and write the following system for determination of $\beta_0(1)$ and $\beta_1(0)$

$$\begin{aligned}
\beta_0(1)X_{0,q}(0,0)+\beta_1(0)X_{0,q}(1,1)&=0,\\
\beta_0(1)X_{1,q}(0,0)+\beta_1(0)X_{1,q}(1,1)&=0.
\end{aligned}$$

It is easy to verify that $X_{0,q}(1,1)=X_{1,q}(0,0)=0$ and $X_{0,q}(0,0)\neq 0$, $X_{1,q}(1,1)\neq 0$ and therefore $\beta_0(1)=\beta_1(0)=0$.

Suppose that (13) is true for $\ell \le \ell_0 - 1$ and let us prove for $\ell = \ell_0 \le 2p-1$. From (17) we have

$$\sum_{j=0}^{\ell_0}\sum_{z=0}^{j}\beta_z(\ell_0-j)\sum_{u=z}^{j}\frac{A_{q+j-u}}{(i\pi)^{j-u}}X_{w,j-u+q}(u,z) = \sum_{j=0}^{\ell_0-1}\sum_{z=0}^{j-1}\beta_z(\ell_0-j)\sum_{u=z}^{j}\frac{A_{q+j-u}}{(i\pi)^{j-u}}X_{w,j-u+q}(u,z)$$
$$+A_q(f)\sum_{j=0}^{\ell_0-1}\beta_j(\ell_0-j)X_{w,q}(j,j)+\sum_{z=0}^{\ell_0-1}\beta_z(0)\sum_{u=z}^{\ell_0}\frac{A_{q+\ell_0-u}}{(i\pi)^{\ell_0-u}}X_{w,\ell_0-u+q}(u,z)$$
$$+\beta_{\ell_0}(0)A_q(f)X_{w,q}(\ell_0,\ell_0) = A_q(f)\sum_{j=0}^{\ell_0}\beta_j(\ell_0-j)X_{w,q}(j,j)=0.$$

As $A_q(f)\neq 0$ we get

$$\sum_{j=0}^{\ell_0}\beta_j(\ell_0-j)X_{w,q}(j,j)=0,\ \ w=0,\dots,\ell_0\le 2p-1.$$

Separating even and odd values of j and w we write for $w=0,\dots,p-1$

$$\sum_{j=0}^{[\frac{\ell_0}{2}]}\beta_{2j}(\ell_0-2j)X_{2w,q}(2j,2j)+\sum_{j=0}^{[\frac{\ell_0}{2}-1]}\beta_{2j+1}(\ell_0-2j-1)X_{2w,q}(2j+1,2j+1)=0,$$
$$\sum_{j=0}^{[\frac{\ell_0}{2}]}\beta_{2j}(\ell_0-2j)X_{2w+1,q}(2j,2j)+\sum_{j=0}^{[\frac{\ell_0}{2}-1]}\beta_{2j+1}(\ell_0-2j-1)X_{2w+1,q}(2j+1,2j+1)=0.$$

Taking into account that $X_{2w,q}(2j+1,2j+1) = X_{2w+1,q}(2j,2j) = 0$ and recalling that determinants (11) and (12) are nonzero, we complete the proof. $\Box$

Next lemma presents exact estimates for $\delta^w_{\pm N}(\delta^p_n(\theta,\check{F}_n))$.

Lemma 4 *Let the conditions of Lemma 3 be valid. Then the following estimate holds*

$$\begin{aligned}\delta^w_{\pm N}(\delta^p_n(\theta,\check{F}_n)) &= \pm A_q(f)\frac{(-1)^{N+w+p+1}(2p)!}{2(i\pi N)^{q+1}N^{2w+2p}}\\ &\times \left[X_{2w,q}(2p,2p)-\sum_{j=0}^{p-1}X_{2w,q}(2j,2j)\sum_{w=0}^{p-1}a^{-1}_{jw}X_{2w,q}(2p,2p)\right]\\ &+ O(N^{-q-2w-2p-2})+o(N^{-q-4p-2}),\ \ N\to\infty,\end{aligned}$$

where a^{-1}_{jw} are the elements of the inverse of the matrix $(X_{2w,q}(2j,2j))^{p-1}_{w,j=0}$.

Proof. Note that

$$\delta^w_{\pm N}(\delta^p_n(\theta,\check{F}_n)) = \Delta^{2w}_{\pm N+w}(\delta^p_n(\theta,\check{F}_n)).$$

Hence, we start as at the beginning of the proof of Lemma 3 and instead of (16) derive the following estimate

$$\begin{aligned}\Delta^{2w}_{\pm N+w}(\delta^p_n(\theta,\check{F}_n)) &= \frac{(-1)^{N+w+1}}{2(i\pi N)^{q+1}N^{2w}}\sum_{\ell=0}^{4p-2w+1}\frac{1}{N^\ell}\sum_{j=0}^{\ell}\sum_{z=0}^{j}\beta_z(\ell-j)\\ &\times \sum_{u=z}^{j}\frac{A_{q+j-u}(f)}{(i\pi)^{j-u}}\binom{2w+j+q}{j-u+q}\binom{u+2w}{z}\\ &\times \sum_{y=0}^{u-z}(-1)^{u+z+y}\binom{u+2w-z}{y+2w}\alpha_{2w,y+2w}\sum_{r=-\infty}^{\infty}(-1)^r\frac{(w+r)^{u-z-y}}{(2r\pm 1)^{2w+j+q+1}}\\ &+ o(N^{-q-4p-2}).\end{aligned}$$

Taking into account (13) we get

$$\begin{aligned}\Delta^{2w}_{\pm N+w}(\delta^p_n(\theta,\check{F}_n)) &= A_q(f)\frac{(-1)^{N+w+1}(2w)!}{2(i\pi N)^{q+1}N^{2w+2p}}\sum_{j=0}^{2p}\beta_j(2p-j)\binom{2w+j+q}{q}\binom{2w+j}{j}\\ &\times \sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(2r\pm 1)^{2w+j+q+1}}+O(N^{-q-2w-2p-2})+o(N^{-q-4p-2}).\end{aligned}$$

It is easy to verify that for odd j (q is even)

$$\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(2r\pm 1)^{2w+j+q+1}}=0$$

and for even j

$$\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(2r\pm 1)^{2w+j+q+1}}=\pm\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(2r+1)^{2w+j+q+1}}.$$

Hence

$$\begin{aligned}\Delta^{2w}_{\pm N+w}(\delta^p_n(\theta,\check{F}_n)) &= \pm A_q(f)\frac{(-1)^{N+w+1}}{2(i\pi N)^{q+1}N^{2w+2p}}\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w,q}(2j,2j)\\ &+ O(N^{-q-2w-2p-2})+o(N^{-q-4p-2}).\end{aligned}$$

Finally, it remains to calculate the values of $\beta_{2j}(2p-2j)$, $j=0,\ldots,p$. We use (17) for even values of w and $\ell=2p$

$$\sum_{j=0}^{2p}\sum_{z=0}^{j}\beta_z(2p-j)\sum_{u=z}^{j}\frac{A_{q+j-u}(f)}{(i\pi)^{j-u}}X_{2w,j-u+q}(u,z)=0,\ \ w=0,\ldots,p-1. \tag{18}$$

We simplify (18) in view of (13)

$$\sum_{j=0}^{2p}\sum_{z=0}^{j}\beta_z(2p-j)\sum_{u=z}^{j}\frac{A_{q+j-u}}{(i\pi)^{j-u}}X_{2w,j-u+q}(u,z)=$$
$$\sum_{j=0}^{2p-1}\sum_{z=0}^{j-1}\beta_z(2p-j)\sum_{u=z}^{j}\frac{A_{q+j-u}}{(i\pi)^{j-u}}X_{2w,j-u+q}(u,z)+A_q(f)\sum_{j=0}^{2p-1}\beta_j(2p-j)X_{2w,q}(j,j)$$
$$+\sum_{z=0}^{2p-1}\beta_z(0)\sum_{u=z}^{2p}\frac{A_{q+2p-u}}{(i\pi)^{2p-u}}X_{2w,2p-u+q}(u,z)+\beta_{2p}(0)A_q(f)X_{2w,q}(2p,2p)$$
$$=A_q(f)\sum_{j=0}^{2p}\beta_j(2p-j)X_{2w,q}(j,j)=0.$$

As $A_q(f)\neq 0$ we get

$$\begin{aligned}\sum_{j=0}^{2p}\beta_j(2p-j)X_{2w,q}(j,j) &= \sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w,q}(2j,2j)\\ &+ \sum_{j=0}^{p-1}\beta_{2j+1}(2p-2j-1)X_{2w,q}(2j+1,2j+1)\\ &= \sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w,q}(2j,2j)=0,\ \ w=0,\dots,p-1.\end{aligned}$$

Hence we derived the following system

$$\sum_{j=0}^{p-1}\beta_{2j}(2p-2j)X_{2w,q}(2j,2j)=-\beta_{2p}(0)X_{2w,q}(2p,2p),\ \ w=0,\dots,p-1$$

which has unique solution

$$\beta_{2j}(2p-2j)=-\beta_{2p}(0)\sum_{w=0}^{p-1}a_{jw}^{-1}X_{2w,q}(2p,2p),\ \ j=0,\dots,p-1, \tag{19}$$

where (see 10)

$$\begin{aligned}\beta_{2p}(0) &= \sum_{s=0}^{p}\sum_{k=0}^{p}(-1)^{k+s}\gamma_{k,0}^{-}\gamma_{s,0}^{+}(k-s)^{2p}\\ &= \sum_{s=0}^{p}\sum_{k=0}^{p}(-1)^{k+s}\binom{p}{k}\binom{p}{s}(k-s)^{2p}\\ &= (-1)^p(2p)!.\end{aligned}$$

This completes the proof.

□

Now we prove the main result of this subsection.

Theorem 5 *Let the conditions of Lemma 3 be valid. Then the following estimate holds for* $|x| < 1$ *as* $N \to \infty$

$$\begin{aligned} r_{N,q}^{p}(f;x) &= A_q(f)\frac{(-1)^{N+\frac{q}{2}}(2p)!}{2^{4p+1}\pi^{q+1}N^{q+4p+1}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{4p+1}\frac{\pi x}{2}} \\ &\times \left[X_{2p,q}(2p,2p) - \sum_{j=0}^{p-1}X_{2p,q}(2j,2j)\sum_{w=0}^{p-1}a_{jw}^{-1}X_{2w,q}(2p,2p)\right] \\ &+ o(N^{-q-4p-1}), \end{aligned}$$

where a_{jw}^{-1} *are the elements of the inverse of the matrix* $(X_{2w,q}(2j,2j))_{w,j=0}^{p-1}$.

Proof. Taking into account that

$$r_{N,q}^{p}(f;x) = r_N^p(F;x)$$

we get from (1) by application of the Abel transformation

$$\begin{aligned} r_{N,q}^{p}(f;x) = &\frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{w=0}^{2p}\frac{\delta_N^w(\delta_n^p(\theta,\check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}} \\ &+\frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{w=0}^{2p}\frac{\delta_{-N}^w(\delta_n^p(\theta,\check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}} \\ &+\frac{1}{(2+2\cos\pi x)^{2p+1}\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{|n|=N+1}^{\infty}\delta_n^{2p+1}(\delta_n^p(\theta,F_n))e^{i\pi nx} \\ &+\frac{1}{(2+2\cos\pi x)^{2p+1}\prod_{s=1}^{p}(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})}\sum_{n=-N}^{N}\delta_n^{2p+1}(\delta_n^p(\theta,F_n-\check{F}_n))e^{i\pi nx}. \end{aligned} \tag{20}$$

Taking into account (see (10)) that $\gamma_s(\theta^{\pm}) \to \binom{p}{s}$ and consequently $\theta_{\pm s} \to 1$ as $N \to \infty$ we need to estimate only the sums in the right hand side of (20).

Lemma 2 shows that

$$\begin{aligned} \delta_n^{2p+1}(\delta_n^p(\theta,F_n-\check{F}_n)) &= O(N^{-q-4p-3}) + o(N^{-q-4p-2}) \\ &= o(N^{-q-4p-2}) \end{aligned}$$

and hence the last term in (20) is $o(N^{-q-4p-1})$. According to Lemma 1

$$\begin{aligned} \delta_n^{2p+1}(\delta_n^p(\theta,F_n)) &= O(n^{-q-4p-3}) + o(n^{-q-4p-2}) \\ &= o(n^{-q-4p-2}) \end{aligned}$$

and the third term is also $o(N^{-q-4p-1})$.

Then, according to system (4)

$$\delta_N^w(\delta_n^p(\theta,\check{F}_n)) = \delta_{-N}^w(\delta_n^p(\theta,\check{F}_n)) = 0,\ w = 0,\dots,p-1. \tag{21}$$

Therefore

$$\begin{aligned} r_{N,q}^p(f;x) &= (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \frac{\delta_N^p(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{2p+1}(1+e^{-i\pi x})^{2p+1}} \\ &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \frac{\delta_{-N}^p(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{2p+1}(1+e^{-i\pi x})^{2p+1}} \\ &+ \frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{(1+e^{i\pi x})^p(1+e^{-i\pi x})^p} \sum_{w=p+1}^{2p} \frac{\delta_N^w(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}} \\ &+ \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})^p(1+e^{-i\pi x})^p} \sum_{w=p+1}^{2p} \frac{\delta_{-N}^w(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}} \\ &+ o(N^{-q-4p-1}). \end{aligned}$$

According to Lemma 4

$$\delta_{\pm N}^w(\delta_n^p(\theta, \check{F}_n)) = O(N^{-q-1-2w-2p}) + o(N^{-q-4p-2})$$

and taking into account that parameter w is ranging from $p+1$ to $2p$ we get

$$\delta_{\pm N}^w(\delta_n^p(\theta, \check{F}_n)) = o(N^{-q-4p-2}), \ w = p+1, \ldots, 2p.$$

Thus

$$\begin{aligned} r_{N,q}^p(f;x) &= (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \frac{\delta_N^p(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{2p+1}(1+e^{-i\pi x})^{2p+1}} \\ &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \frac{\delta_{-N}^p(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{2p+1}(1+e^{-i\pi x})^{2p+1}} \\ &+ o(N^{-q-4p-1}). \end{aligned}$$

This completes the proof in view of the exact estimates of Lemma 4.
□

Comparison with Theorem 1 (the KL-interpolation) shows dramatically improved accuracy in the regions away from the endpoints and improvement is $O(N^{4p})$. Compared with the RTP-interpolation where parameter θ is chosen as in (3) (Theorem 3) we have improvement in accuracy by the factor $O(N^{2p})$.

1.2 Odd values of q

In this subsection we prove analog of Theorem 5 for odd values of q. In general we perform identical observations. First we present analogs of Lemmas 1 and 2 without proofs as they are identical to the old ones.

Lemma 5 *Let $f \in C^{4p+q+2}[-1,1]$ and $f^{(4p+q+2)} \in AC[-1,1]$. Let the coefficients $\gamma_s(\theta^{\pm})$ satisfy the system (7). Then*

$$\delta_n^w(\delta_n^p(\theta, F_n)) = O(n^{-q-1-2w}) + o(n^{-q-4p-3}), \ |n| > N, \ N \to \infty.$$

Lemma 6 *Let the conditions of Lemma 5 be valid. Then*

$$\delta_n^w(\delta_n^p(\theta, F_n - \check{F}_n)) = O(N^{-q-1-2w}) + o(N^{-q-4p-3}),\ |n| \le N,\ N \to \infty.$$

Properties of $\beta_z(u-j)$ are explored in the next lemma.

Lemma 7 *Let q be odd, $f \in C^{4p+q+2}[-1,1]$ with $f^{(4p+q+2)} \in AC[-1,1]$ and $A_q(f) \ne 0$. Let the coefficients $\gamma_s(\theta^{\pm})$ satisfy system (7). If determinants*

$$\det((X_{2w,q}(2j+1, 2j+1))_{j,w=0}^M),\ M = 0, \dots, 2p-1, \tag{22}$$

$$\det((X_{2w+1,q}(2j, 2j))_{j,w=0}^M),\ M = 0, \dots, 2p-1 \tag{23}$$

are nonzero then

$$\beta_z(\ell - j) = 0,\ \ell \le 2p-1,\ 0 \le z \le j,\ 0 \le j \le u. \tag{24}$$

Then

$$\beta_{2j+1}(2p-2j-1) = 0,\ j = 0, \dots, p-1, \tag{25}$$

and

$$\beta_{2j+1}(2p-2j) = 0,\ j = 0, \dots, p, \tag{26}$$

where

$$\beta_z(u-j) = \sum_{t=0}^{u-j} \sum_{s=0}^{p} (-1)^s \sum_{k=0}^{p} (-1)^k \gamma_{k,t}^{-} \gamma_{s,u-t-j}^{+} (k-s)^z$$

and $\gamma_{s,u}^{\pm}$ are the coefficients of the asymptotic expansions

$$\gamma_s(\theta^+) = \sum_{u=0}^{4p+2} \frac{\gamma_{s,u}^{+}}{N^u} + o(N^{-4p-2}),\ \gamma_s(\theta^-) = \sum_{t=0}^{4p+2} \frac{\gamma_{k,t}^{-}}{N^t} + o(N^{-4p-2}).$$

Proof. First we prove (24). Equation (16) is valid also for odd values of q with some corrections according to the higher smoothness of f

$$\begin{aligned}\Delta_{N+p}^w(\delta_n^p(\theta, \check{F}_n)) &= \frac{(-1)^{N+p+1}}{2(i\pi N)^{q+1} N^w} \sum_{\ell=0}^{4p-w+2} \frac{1}{N^\ell} \sum_{j=0}^{\ell} \sum_{z=0}^{j} \beta_z(\ell-j) \sum_{u=z}^{j} \frac{A_{q+j-u}}{(i\pi)^{j-u}} X_{w,j-u+q}(u,z) \\ &+ o(N^{-q-4p-3}).\end{aligned}$$

According to (8) from here we derive

$$\sum_{j=0}^{\ell} \sum_{z=0}^{j} \beta_z(\ell-j) \sum_{u=z}^{j} \frac{A_{q+j-u}}{(i\pi)^{j-u}} X_{w,j-u+q}(u,z) = 0, \tag{27}$$
$$w = 0, \dots, 2p-1;\ \ell = 0, \dots, 4p-w+2.$$

We proceed by the mathematical induction and start with $\ell = 0$ in (27) deriving

$$A_q(f)\beta_0(0)X_{w,q}(0,0) = 0.$$

Taking here $w = 1$ and observing that $X_{1,q}(0, 0,) \neq 0$ for odd values of q and $A_q(f) \neq 0$ we conclude that $\beta_0(0) = 0$.

Then, for $\ell = 1$, we have

$$\beta_0(1)X_{w,q}(0, 0) + \beta_1(0)X_{w,q}(1, 1) = 0, \ w = 0, \dots, 2p - 1.$$

Taking here $w = 0, 1$ we get the following system for determination of $\beta_0(1)$ and $\beta_1(0)$

$$\beta_0(1)X_{0,q}(0, 0) + \beta_1(0)X_{0,q}(1, 1) = 0,$$
$$\beta_0(1)X_{1,q}(0, 0) + \beta_1(0)X_{1,q}(1, 1) = 0.$$

It is easy to verify that for odd values of q we have that $X_{0,q}(0, 0) = X_{1,q}(1, 1) = 0$ and consequently $\beta_0(1) = \beta_1(0) = 0$.

Suppose that (24) is true for $\ell = \ell_0 - 1$ and let us prove it for $\ell = \ell_0 \leq 2p - 1$. We put $\ell = \ell_0$ in (27) and simplify it as we did it for even values of q

$$A_q(f) \sum_{j=0}^{\ell_0} \beta_j(\ell_0 - j)X_{w,q}(j, j) = 0.$$

As $A_q(f) \neq 0$

$$\sum_{j=0}^{\ell_0} \beta_j(\ell_0 - j)X_{w,q}(j, j) = 0, \ w = 0, \dots, \ell_0 \leq 2p - 1. \tag{28}$$

We consider separately odd and even values of w and j and write

$$\sum_{j=0}^{[\ell_0/2]} \beta_{2j}(\ell_0 - 2j)X_{2w,q}(2j, 2j) + \sum_{j=0}^{[\ell_0/2-1]} \beta_{2j+1}(\ell_0 - 2j - 1)X_{2w,q}(2j + 1, 2j + 1) = 0,$$
$$\sum_{j=0}^{[\ell_0/2]} \beta_{2j}(\ell_0 - 2j)X_{2w+1,q}(2j, 2j) + \sum_{j=0}^{[\ell_0/2-1]} \beta_{2j+1}(\ell_0 - 2j - 1)X_{2w+1,q}(2j + 1, 2j + 1) = 0.$$

Taking into account that for odd values of q we have that $X_{2w,q}(2j, 2j) = X_{2w+1,q}(2j+1, 2j+1) = 0$ and determinants (22) and (23) are nonzero we end the proof of (24).

Now let us prove (25). We take $\ell = 2p$ in (27) and write for $w = 0, \dots, 2p - 1$

$$\sum_{j=0}^{2p} \sum_{z=0}^{j} \beta_z(2p - j) \sum_{u=z}^{j} \frac{A_{q+j-u}}{(i\pi)^{j-u}} X_{w,j-u+q}(u, z) = A_q(f) \sum_{j=0}^{2p} \beta_j(2p - j)X_{w,q}(j, j) = 0.$$

Therefore

$$\sum_{j=0}^{p} \beta_{2j}(2p - 2j)X_{2w,q}(2j, 2j) + \sum_{j=0}^{p-1} \beta_{2j+1}(2p - 2j - 1)X_{2w,q}(2j + 1, 2j + 1) = 0.$$

As $X_{2w,q}(2j, 2j) = 0$ we get

$$\sum_{j=0}^{p-1} \beta_{2j+1}(2p - 2j - 1)X_{2w,q}(2j + 1, 2j + 1) = 0, \ w = 0, \dots, p - 1.$$

This system has unique solution and hence $\beta_{2j+1}(2p-2j-1)=0$.

Finally, we prove (26). We again use (27) with $\ell=2p$ and for odd values of w obtain

$$\begin{aligned}\sum_{j=0}^{2p}\beta_j(2p-j)X_{2w+1,q}(j,j) &= \sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w+1,q}(2j,2j)\\ &+ \sum_{j=0}^{p-1}\beta_{2j+1}(2p-2j-1)X_{2w+1,q}(2j+1,2j+1)=0.\end{aligned}$$

Taking into account that $X_{2w+1,q}(2j+1,2j+1)=0$ we get

$$\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w+1,q}(2j,2j)=0. \tag{29}$$

From the other side (17) with $\ell=2p+1$ implies

$$\sum_{j=0}^{2p+1}\sum_{z=0}^{j}\beta_z(2p+1-j)\sum_{u=z}^{j}\frac{A_{q+j-u}}{(i\pi)^{j-u}}X_{w,j-u+q}(u,z)=0$$

which gives

$$\sum_{j=0}^{2p+1}\beta_j(2p+1-j)X_{w,q}(j,j)+\sum_{j=0}^{2p}\beta_j(2p-j)X_{w,q}(j+1,j)=0,\ \ w=0,\dots,2p-1.$$

We rewrite it for even values of w as follows

$$\sum_{j=0}^{p}\beta_{2j+1}(2p-2j)X_{2w,q}(2j+1,2j+1)+\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w,q}(2j+1,2j)=0$$

and hence

$$\begin{aligned}\sum_{j=0}^{p-1}\beta_{2j+1}(2p-2j)X_{2w,q}(2j+1,2j+1) &= -\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w,q}(2j+1,2j)\\ &- \beta_{2p+1}(0)X_{2w,q}(2p+1,2p+1).\end{aligned} \tag{30}$$

It is easy to verify that

$$\beta_{2p+1}(0)=\sum_{s=0}^{p}\sum_{k=0}^{p}(-1)^{k+s}\binom{p}{k}\binom{p}{s}(k-s)^{2p+1}=0.$$

Then simple calculations show that

$$X_{2w,q}(2j+1,2j)=(p-w-1/2)X_{2w+1,q}(2j,2j)$$

and hence from (29)

$$\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w,q}(2j+1,2j)=(p-w-1/2)\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w+1,q}(2j,2j)=0.$$

Substituting all these into (30) we get

$$\sum_{j=0}^{p-1} \beta_{2j+1}(2p-2j)X_{2w,q}(2j+1,2j+1) = 0 \tag{31}$$

which completes the proof of (26) as determinant $\det((X_{2w,q}(2j+1,2j+1))_{w,j=0}^{p-1})$ is nonzero and thus (31) has unique zero solution.

□

In the next lemma we investigate behavior of $\delta_{\pm N}^{w}(\delta_n^p(\theta,\check{F}_n))$ for odd values of q.

Lemma 8 *Let the conditions of Lemma 7 be valid. Then the following estimate holds*

$$\begin{aligned}
\delta_{\pm N}^{w}(\delta_n^p(\theta,\check{F}_n)) &= A_q(f)\frac{(-1)^{N+w+p}(2p)!}{4(i\pi)^{q+1}N^{q+2w+2p+2}} \\
&\times \left(X_{2w+1,q}(2p,2p) - \sum_{j=0}^{p-1} X_{2w+1,q}(2j,2j)\sum_{s=0}^{p-1} b_{js}^{-1}X_{2s+1,q}(2p,2p)\right) \\
&\pm A_{q+1}(f)\frac{(-1)^{N+w+p+1}(2p)!}{2(i\pi)^{q+2}N^{q+2w+2p+2}} \\
&\times \left(X_{2w,q+1}(2p,2p) - \sum_{j=0}^{p-1} X_{2w,q+1}(2j,2j)\sum_{s=0}^{p-1} b_{js}^{-1}X_{2s+1,q}(2p,2p)\right) \\
&+ O(N^{-q-2p-2w-3}) + o(N^{-q-4p-3}),
\end{aligned}$$

where b_{jw}^{-1} are the elements of the inverse of the matrix $(X_{2w+1,q}(2j,2j))_{w,j=0}^{p-1}$.

Proof. We again note that

$$\delta_{\pm N}^{w}(\delta_n^p(\theta,\check{F}_n)) = \Delta_{\pm N+w}^{2w}(\delta_n^p(\theta,\check{F}_n))$$

and proceed as in the proof of Lemma 4. Starting as in the proof of Lemma 7 we derive

$$\begin{aligned}
\Delta_{\pm N+w}^{2w}(\delta_n^p(\theta,\check{F}_n)) &= \frac{(-1)^{N+w+1}}{2(i\pi N)^{q+1}N^{2w}}\sum_{\ell=0}^{4p-2w+2}\frac{1}{N^\ell}\sum_{j=0}^{\ell}\sum_{z=0}^{j}\beta_z(\ell-j) \\
&\times \sum_{u=z}^{j}\frac{A_{q+j-u}}{(i\pi)^{j-u}}\binom{2w+j+q}{j-u+q}\binom{u+2w}{z} \\
&\times \sum_{y=0}^{u-z}(-1)^{u+z+y}\binom{u+2w-z}{y+2w}\alpha_{2w,y+2w}\sum_r(-1)^r\frac{(w+r)^{u-z-y}}{(2r\pm1)^{2w+j+q+1}} \\
&+ o(N^{-q-4p-3}).
\end{aligned}$$

Taking into account (24) we get

$$\begin{aligned}
&\Delta^{2w}_{\pm N+w}(\delta^p_n(\theta,\check{F}_n))\\
&= A_q \frac{(-1)^{N+w+1}(2w)!}{2(i\pi)^{q+1}N^{q+2w+2p+1}} \sum_{j=0}^{2p} \beta_j(2p-j) \binom{2w+j+q}{q} \binom{j+2w}{j} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r\pm 1)^{2w+j+q+1}}\\
&+ A_q \frac{(-1)^{N+w+1}(2w)!}{2(i\pi)^{q+1}N^{q+2w+2p+2}} \sum_{j=0}^{2p+1} \beta_j(2p+1-j) \binom{2w+j+q}{q} \binom{j+2w}{j} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r\pm 1)^{2w+j+q+1}}\\
&+ A_q \frac{(-1)^{N+w+1}}{2(i\pi)^{q+1}N^{q+2w+2p+2}} \sum_{j=0}^{2p} \beta_j(2p-j) \binom{2w+j+1+q}{q} \binom{j+1+2w}{j}\\
&\times\left(-(2w+1)!\sum_{r=-\infty}^{\infty}(-1)^r \frac{w+r}{(2r\pm 1)^{2w+j+q+2}} + w(2w+1)!\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r\pm 1)^{2w+j+q+2}}\right)\\
&+ A_{q+1} \frac{(-1)^{N+w+1}(2w)!}{2(i\pi)^{q+2}N^{q+2w+2p+2}} \sum_{j=0}^{2p} \beta_j(2p-j) \binom{2w+j+q+1}{q+1} \binom{j+2w}{j} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r\pm 1)^{2w+j+q+2}}\\
&+ O(N^{-q-2p-2w-3}) + o(N^{-q-4p-3}). \qquad (32)
\end{aligned}$$

We conclude from the properties (25), (26) and $\sum_r (-1)^r/(2r\pm 1)^{2w+2j+q+1} = 0$ that the first two terms in the right hand side of (32) vanish. Thus

$$\begin{aligned}
&\Delta^{2w}_{\pm N+w}(\delta^p_n(\theta,\check{F}_n))\\
=\ & -A_q(f) \frac{(-1)^{N+w+1}(2w+1)!}{2(i\pi)^{q+1}N^{q+2w+2p+2}} \sum_{j=0}^{p} \beta_{2j}(2p-2j) \binom{2w+2j+1+q}{q} \binom{2j+1+2w}{2j}\\
\times\ & \sum_{r=-\infty}^{\infty} (-1)^r \frac{r}{(2r\pm 1)^{2w+2j+q+2}}\\
+\ & A_{q+1}(f) \frac{(-1)^{N+w+1}(2w)!}{2(i\pi)^{q+2}N^{q+2w+2p+2}} \sum_{j=0}^{p} \beta_{2j}(2p-2j) \binom{2w+2j+q+1}{q+1} \binom{2j+2w}{2j}\\
\times\ & \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r\pm 1)^{2w+2j+q+2}} + O(N^{-q-2p-2w-3}) + o(N^{-q-4p-3}).
\end{aligned}$$

Finally, applying the properties

$$\sum_{r=-\infty}^{\infty} (-1)^r \frac{1}{(2r-1)^{2w+2j+q+2}} = -\sum_{r=-\infty}^{\infty} (-1)^r \frac{1}{(2r+1)^{2w+2j+q+2}}$$

and

$$\sum_{r=-\infty}^{\infty} (-1)^r \frac{r}{(2r-1)^{2w+2j+q+2}} = \sum_{r=-\infty}^{\infty} (-1)^r \frac{r}{(2r+1)^{2w+2j+q+2}}$$

we obtain

$$
\begin{aligned}
\Delta^{2w}_{\pm N+w}(\delta^p_n(\theta,\check{F}_n)) &= A_q(f)\frac{(-1)^{N+w}}{4(i\pi)^{q+1}N^{q+2w+2p+2}}\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w+1,q}(2j,2j)\\
&\pm A_{q+1}(f)\frac{(-1)^{N+w+1}}{2(i\pi)^{q+2}N^{q+2w+2p+2}}\sum_{j=0}^{p}\beta_{2j}(2p-2j)X_{2w,q+1}(2j,2j)\\
&+ O(N^{-q-2p-2w-3})+o(N^{-q-4p-3}). \qquad (33)
\end{aligned}
$$

It remains to calculate the values of $\beta_{2j}(2p-2j)$. We use (29) and write

$$
\sum_{j=0}^{p-1}\beta_{2j}(2p-2j)X_{2w+1,q}(2j,2j) = -\beta_{2p}(0)X_{2w+1,q}(2p,2p). \qquad (34)
$$

Taking into account that determinant $\det((X_{2w+1,q}(2j,2j))_{w,j=0}^{p-1})$ is nonzero we get the unique solution of the system

$$
\beta_{2j}(2p-2j) = -\beta_{2p}(0)\sum_{w=0}^{p-1}b^{-1}_{jw}X_{2w+1,q}(2p,2p)
$$

and

$$
\beta_{2p}(0) = \sum_{s=0}^{p}\sum_{k=0}^{p}(-1)^{k+s}\binom{p}{k}\binom{p}{s}(k-s)^{2p} = (-1)^p(2p)!.
$$

This completes the proof. □

Now we prove the main result of this subsection.

Theorem 6 *Let the conditions of Lemma 7 be valid. Then*

$$
\begin{aligned}
r^p_{N,q}(f;x) &= A_q(f)\frac{(-1)^{N+\frac{q+1}{2}}(2p)!}{2^{4p+2}\pi^{q+1}N^{q+4p+2}}\frac{\sin\frac{\pi x}{2}\sin\frac{\pi x}{2}(2N+1)}{\cos^{4p+2}\frac{\pi x}{2}}\\
&\times \left(X_{2p+1,q}(2p,2p)-\sum_{j=0}^{p-1}X_{2p+1,q}(2j,2j)\sum_{s=0}^{p-1}b^{-1}_{js}X_{2s+1,q}(2p,2p)\right)\\
&+ A_{q+1}(f)\frac{(-1)^{N+\frac{q+1}{2}}(2p)!}{2^{4p+1}\pi^{q+2}N^{q+4p+2}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{4p+1}\frac{\pi x}{2}}\\
&\times \left(X_{2p,q+1}(2p,2p)-\sum_{j=0}^{p}X_{2p,q+1}(2j,2j)\sum_{s=0}^{p-1}b^{-1}_{js}X_{2s+1,q}(2p,2p)\right)\\
&+ o(N^{-q-4p-2}),\ N\to\infty,
\end{aligned}
$$

where b^{-1}_{jw} are the elements of the inverse of the matrix $(X_{2w+1,q}(2j,2j))_{w,j=0}^{p-1}$.

Proof. The proof is similar to the one of Theorem 5 and we omit some details. Application of the Abel transformation leads to the following expansion of the error

$$r_{N,q}^p(f;x) = \frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{\prod_{s=1}^p(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{w=0}^{2p+1} \frac{\delta_N^w(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}}$$
$$+\frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{\prod_{s=1}^p(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{w=0}^{2p+1} \frac{\delta_{-N}^w(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{w+1}(1+e^{-i\pi x})^{w+1}}$$
$$+\frac{1}{(2+2\cos\pi x)^{2p+2}\prod_{s=1}^p(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta_n^{2p+2}(\delta_n^p(\theta, F_n))e^{i\pi nx}$$
$$+\frac{1}{(2+2\cos\pi x)^{2p+2}\prod_{s=1}^p(1+\theta_{-s}e^{i\pi x})(1+\theta_s e^{-i\pi x})} \sum_{n=-N}^{N} \delta_n^{2p+2}(\delta_n^p(\theta, F_n - \check{F}_n))e^{i\pi nx}. \quad (35)$$

According to Lemmas 5 and 6 the last two terms are $o(N^{-q-4p-2})$ as $N \to \infty$. Then according to (21) and Lemma 8 we derive

$$\begin{aligned} r_{N,q}^p(f;x) &= (e^{-i\pi Nx} - e^{i\pi(N+1)x})\frac{\delta_N^p(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{2p+1}(1+e^{-i\pi x})^{2p+1}} \\ &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x})\frac{\delta_{-N}^p(\delta_n^p(\theta, \check{F}_n))}{(1+e^{i\pi x})^{2p+1}(1+e^{-i\pi x})^{2p+1}} \\ &+ o(N^{-q-4p-2}). \end{aligned}$$

This completes the proof as Lemma 8 gives exact estimates for $\delta_{\pm N}^p(\delta_n^p(\theta, \check{F}_n))$.
□

Comparison with the corresponding theorems shows that the Fourier-Pade interpolation has improved accuracy also for odd q and improvement in accuracy is $O(N^{4p})$ compared to the KL-interpolation and consequently $O(N^{2p})$ compared to the RTP-interpolations with θ as in (3).

Appendix

Throughout the paper we used the fact that some determinants are nonzero. We calculated the values of determinants for rather large values of q and M and in all cases they are nonzero. Explicit forms of determinants are unknown. Here we show some values.

Let us start with even q. We denote

$$a_M(q) = \det((X_{2w,q}(2j, 2j))_{w,j=0}^M),$$

and

$$b_M(q) = \det((X_{2w+1,q}(2j+1, 2j+1))_{w,j=0}^M).$$

For $q = 2$ we have

$$a_1(2)=\frac{9\pi^{10}}{2048},\ a_2(2)=\frac{675\pi^{21}}{1048576},\ a_3(2)=\frac{1488375\pi^{36}}{4294967296},\ a_4(2)=\frac{37975888125\pi^{55}}{35184372088832},$$

$$a_5(2)=\frac{65135293914796875\pi^{78}}{2305843009213693952},\ a_6(2)=\frac{514920389675800428632812 5\pi^{105}}{604462909807314587353088},$$

$$b_1(2)=\frac{267\pi^{14}}{32768},\ b_2(2)=-\frac{154305\pi^{27}}{67108864},\ b_3(2)\frac{3267068175\pi^{44}}{1099511627776},$$
$$b_4(2)=-\frac{962101041890625\pi^{65}}{36028797018963968},\ b_5(2)=\frac{2227932849402289476562 5\pi^{90}}{9444732965739290427392},$$
$$b_6(2)=-\frac{27250362102859286767460068359375\pi^{119}}{9903520314283042199192993792}.$$

For $q=4$

$$a_1(4)=\frac{89\pi^{14}}{524288},\ a_2(4)=\frac{85725\pi^{27}}{4294967296},\ a_3(4)=\frac{4235088375\pi^{44}}{281474976710656},$$
$$a_4(4)=\frac{3741504051796875\pi^{65}}{36893488147419103232},\ a_5(4)=\frac{317686721118474610546875\pi^{90}}{38685626227668133590597632},$$
$$a_6(4)=\frac{16838032385779102502609585449218 75\pi^{119}}{162259276829213363391578010288128},$$
$$b_1(4)=\frac{32321\pi^{18}}{25165824},\ b_2(4)=-\frac{114620475\pi^{33}}{274877906944},\ b_3(4)=\frac{71686477940175\pi^{52}}{72057594037927936},$$
$$b_4(4)=-\frac{9144882554906087718 75\pi^{75}}{377789318629571617095 68},\ b_5(4)=\frac{126536668039277577063030726 5625\pi^{102}}{1584563250285286751870879006 72},$$
$$b_6(4)=-\frac{1220243485759053085126244046083348144531 25\pi^{133}}{265845599156983174580761412056068 9152}.$$

For $q=6$

$$a_1(6)=\frac{32321\pi^{18}}{7549747200},\ a_2(6)=\frac{1528273\pi^{33}}{6597069766656},$$
$$a_3(6)=\frac{743415326787\pi^{52}}{5764607523034234880},\ a_4(6)=\frac{1138029829054979805\pi^{75}}{1208925819614629174706176},$$
$$a_5(6)=\frac{115476425943992574030854707 5\pi^{102}}{1014120480182583521197362564300 8},$$
$$a_6(6)=\frac{96510714409466638080898393294177303125\pi^{133}}{340282366920938463463374607431768211456},$$
$$b_1(6)=\frac{33082169\pi^{22}}{362387865600},\ b_2(6)=-\frac{589086370561\pi^{39}}{31665934879948800},\ b_3(6)=\frac{309640792926795687\pi^{60}}{737869762948382064640 0},$$
$$b_4(6)=-\frac{1680353443856200312501785\pi^{85}}{1237940039285380274899124224},$$
$$b_5(6)=\frac{330993716290340414389805939184160 65\pi^{114}}{415383748682786210282439706337607 68},$$
$$b_6(6)=-\frac{586244466287120773883755442559865514735708493 75\pi^{147}}{557518629963265578538392956816209037649510 4}.$$

When q is odd we denote

$$c_M(q)=\det((X_{2w,q}(2j+1,2j+1))_{w,j=0}^{M}),$$

and

$$d_M(q) = \det(X_{2w+1,q}(2j,2j))_{w,j=0}^{M}).$$

For $q = 1$ we have

$$c_1(1) = \frac{3\pi^{10}}{512},\ c_2(1) = \frac{45\pi^{21}}{131072},\ c_3(1) = \frac{14175\pi^{36}}{268435456},\ c_4(1) = \frac{40186125\pi^{55}}{1099511627776},$$

$$c_5(1) = \frac{6266021540625\pi^{78}}{36028797018963968},\ c_6(1) = \frac{38104146940156171875\pi^{105}}{4722366482869645213696},$$

$$d_1(1) = \frac{9\pi^{10}}{512},\ d_2(1) = -\frac{675\pi^{21}}{131072},\ d_3(1) = \frac{1488375\pi^{36}}{268435456},\ d_4(1) = -\frac{37975888125\pi^{55}}{1099511627776},$$

$$d_5(1) = \frac{65135293914796875\pi^{78}}{36028797018963968},\ d_6(1) = -\frac{514920389675800428632812 5\pi^{105}}{4722366482869645213696}.$$

For $q = 3$

$$c_1(3) = \frac{89\pi^{14}}{98304},\ c_2(3) = \frac{5715\pi^{27}}{67108864},\ c_3(3) = \frac{40334175\pi^{44}}{1099511627776}, c_4(3) = \frac{3959263546875\pi^{65}}{36028797018963968},$$

$$c_5(3) = \frac{30561493133090390625\pi^{90}}{9444732965739290427392},\ c_6(3) = \frac{1246015642563296148489257812 5\pi^{119}}{9903520314283042199192993792},$$

$$d_1(3) = \frac{89\pi^{14}}{32768},\ d_2(3) = -\frac{85725\pi^{27}}{67108864},\ d_3(3) = \frac{4235088375\pi^{44}}{1099511627776},$$

$$d_4(3) = -\frac{3741504051796875\pi^{65}}{36028797018963968},\ d_5(3) = \frac{3176867211184746105468 75\pi^{90}}{9444732965739290427392},$$

$$d_6(3) = -\frac{16838032385779102502609585449218 75\pi^{119}}{9903520314283042199192993792}.$$

For $q = 5$

$$c_1(5) = \frac{32321\pi^{18}}{629145600},\ c_2(5) = \frac{4584819\pi^{33}}{1374389534720},\ c_3(5) = \frac{2867459117607\pi^{52}}{18014398509481984 00},$$

$$c_4(5) = \frac{292636241756994807\pi^{75}}{377789318629571617095 68},\ c_5(5) = \frac{80983467545137649320339665\pi^{102}}{1584563250285286751870879006 72},$$

$$c_6(5) = \frac{1561911661771587948961592378986685625\pi^{133}}{26584559915698317458076141205606891 52},$$

$$d_1(5) = \frac{32321\pi^{18}}{209715200},\ d_2(5) = -\frac{13754457\pi^{33}}{274877906944},\ d_3(5) = \frac{60216641469747\pi^{52}}{360287970189639680},$$

$$d_4(5) = -\frac{2765412484603600926 15\pi^{75}}{37778931862957161709568},\ d_5(5) = \frac{8418231451317058646849308176 75\pi^{102}}{158456325028528675187087900672},$$

$$d_6(5) = -\frac{2110689324135035374829247861343657619343 75\pi^{133}}{2658455991569831745807614120560689152}.$$

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