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Numerical Blow-Up Analysis of Explicit L1 Scheme for Time-Fractional Partial Differential Equations

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Abstract. In this paper, we consider the blow-up of explicit L1 scheme for time-fractional partial differential equations. Firstly, we discretize the mentioned equation by the explicit L1 scheme and obtain the corresponding matrix form. Secondly, we introduce the concept of discrete energy. Based on Nakagawa's criteria, a suitable adaptive time-stepping strategy is given by the discrete energy. Thirdly, with the help of lower discrete energy, the finite blow-up behaviors of numerical solution are studied. Finally, some numerical examples for verifying the theoretical results are provided.

Key Words: Time-Fractional Partial Differential Equations, Explicit L1 Scheme, Blow-Up
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Introduction

The blow-up behavior of the solutions for fractional differential equations have aroused widespread concern of researchers. In recent years, a lot of related papers have been published, such as [15, 22, 27, 28]. As an important kind of fractional differential equations, time-fractional partial differential equations (TFPDEs) have become a significant object of investigations due to their demonstrated applications in different areas of science and technology. This type of equations is widely used as models to describe many important physical phenomena such as classical, fluid and quantum mechanics, plasma and nuclear physics, solid state physics, chemical physics, chemical kinematics, and so on. Nowadays, the study of blowing-up solutions to TFPDEs received great attention. For example, some authors obtained results on the blow-up of the solutions of time-fractional Burgers equation [1, 25] and space-time fractional evolution equations [2]. For more information, the interest reader can refer the articles [9, 16, 22].

As everyone knows, the analytic solutions to the TFPDEs cannot be easily obtained except for using similarity transformation and the Hirota bilinear technique [3], Laplace residual power series method [10]. Therefore, the development of effective numerical methods for TFPDEs is necessary and important. In the last few years, many numerical approaches, such as finite difference method [17, 21], finite element method [14, 18], spectral method [4, 30], meshless method [6, 13], and wavelet method [23], have been proposed for TFPDEs. Among these numerical methods, the finite difference method has played a very important role. During the approximation of PDEs, for explicit schemes, we know that CFL conditions of time-stepping restrictions have to be used for the stiffness and positivity [26]. Since fully implicit discretizations require significantly higher computational costs when processing a nonlinear algebraic system, this paper considers a numerical scheme with acceptable computational costs, which is specified by an explicit L1 scheme.

Recently, much attention has been paid to the consideration of numerical blow-up for PDEs, for example, the convergence of numerical blow-up time [12, 20] and the numerical blow-up rate [5]. During the reproduction of finite blow-up time, many approaches have been developed to implement the time step, such as the adaptive time-stepping approach [19] and the uniform time-stepping [7, 8]. Since the uniform time-stepping is not suitable for simulating blow-up solutions [24], the main problem is a suitable adaptive time-stepping strategy.

There are no results concerning the numerical blow-up for TFPDEs. In this paper, we consider the blow-up of numerical solutions to TFPDEs on bounded domains with the nonlinear term. To begin this research, we choose a one-dimensional bounded TFPDEs with Dirichlet boundary conditions and finite difference methods; in our future study, we will consider more complex problems. For simulating blow-up solutions, Nakagawa's criteria [19] is usually implemented by a norm of numerical solutions. Different to [19], we adapt the time-stepping by a discrete energy to bound the numerical solutions. The replicative finite blow-up results are proved for any positive numerical solutions by a lower discrete energy. These results the main contributions of this paper.

The rest of the paper is organized as follows. In Section 2, we present the explicit L1 scheme, and the positivity of numerical solutions is studied. In Section 3, we introduce the concept of discrete energy, upper energy and lower energy. In Section 4, the numerical blow-up is proved for any positive solutions to TFPDEs and an adaptive time-stepping strategy based on the discrete energy is provided. To illustrate the main results, some results are given in Section 5.

1 Explicit L1 scheme

Consider the following one-dimensional nonlinear TFPDEs:

$$\begin{aligned} &{}^C\mathcal{D}_t^\alpha u(x,t) = u_{xx}(x,t) + u^p(x,t), && 0 \le x \le 1, t > 0,\\ &u(0,t) = u(1,t) = 0, && t > 0,\\ &u(x,0) = u_0(x) \gneqq 0, && 0 \le x \le 1, \end{aligned} \tag{1}$$

where $0 < \alpha < 1$, $p > 1$ and ${}^C\mathcal{D}_t^\alpha$ is the Caputo fractional derivative of fractional order α which is defined by

$${}^C\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}.$$

Let $h = 1/N$ be the uniform space step with an integer $N \ge 1$ and $\Delta t_n = t_{n+1} - t_n$ be the time-stepping for $n \ge 0$ and $i = 1, 2, \dots, N-1$. The explicit L1 scheme for (1) is given by

$$\begin{aligned} &\sum_{j=0}^{n} \frac{u_i^{j+1} - u_i^j}{\Delta t_j} \omega_{n,j} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + (u_i^n)^p,\\ &u_0^n = u_N^n = 0,\\ &u_i^0 = u_0(ih), \end{aligned} \tag{2}$$

where

$$\omega_{n,j} = \frac{1}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{-\alpha} ds, \quad j = 0, 1, \dots, n. \tag{3}$$

A compact form of (2) is

$$\sum_{j=0}^{n} \frac{U_{j+1} - U_j}{\Delta t_j} \omega_{n,j} = -\mathbf{A}_N U_n + F(U_n), \tag{4}$$

where

$$U_n = (u_1^n, u_2^n, \cdots, u_{N-1}^n)^T, F(U_n) = ((u_1^n)^p, (u_2^n)^p, \dots, (u_{N-1}^n)^p)^T,$$

and the triple-diagonal matrix $\mathbf{A}_N$ is represented by

$$\mathbf{A}_N := \frac{1}{h^2} \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}.$$

From (4), we have

$$U_{n+1} = \frac{(\gamma_{n,n} - \gamma_{n,n-1})\mathbf{I} - \mathbf{A}_N}{\gamma_{n,n}} U_n + \ldots + \frac{\gamma_{n,1} - \gamma_{n,0}}{\gamma_{n,n}} U_1 + \frac{\gamma_{n,0}}{\gamma_{n,n}} U_0 + \frac{1}{\gamma_{n,n}} F(U_n), \tag{5}$$

where

$$\gamma_{n,j} = \frac{\omega_{n,j}}{\Delta t_j} = \frac{1}{\Gamma(1-\alpha)} \int_0^1 (t_{n+1} - t_j - \theta \Delta t_j)^{-\alpha} d\theta, \quad j = 0, 1, \ldots, n. \tag{6}$$

Before the detailed discussion, we present the preservation of the positivity of numerical solutions which can be proved by the mean value theorem for the definite integral. From (6), we get the following lemma.

Lemma 1 *For $j = 1, 2, \ldots, n$, one has $\gamma_{n,j} - \gamma_{n,j-1} > 0$.*

Define the mesh ratio

$$r_\alpha = \frac{\Delta t_n^\alpha}{h^2}.$$

Lemma 2 *Assume that the time-stepping is not increasing and*

$$r_\alpha < \frac{1 - 2^{2-\alpha}}{\Gamma(2-\alpha)}, \tag{7}$$

then the numerical solutions $U_n > 0$ for all $n \geq 0$.

Proof. By Lemma 1, the coefficients of U_k $(k = 0, 1, \ldots, n+1)$ in (5) are positive, therefore, we only have to consider the coefficient of U_n, that is, to show that

$$\left(1 - \frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right) \mathbf{I} - \frac{1}{\gamma_{n,n}} \mathbf{A}_N > 0.$$

From the expression for $\mathbf{A}_N$ and

$$\gamma_{n,n} = \frac{1}{\Delta t_n^\alpha \Gamma(2-\alpha)},$$

we have

$$1 - \frac{\gamma_{n,n-1}}{\gamma_{n,n}} - 2r_\alpha \Gamma(2-\alpha) > 0,$$

that is,

$$r_\alpha < \frac{1}{2\Gamma(2-\alpha)} \left(1 - \frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right). \tag{8}$$

From (3), we get

$$\begin{aligned}\frac{\gamma_{n,n-1}}{\gamma_{n,n}} &= \frac{\int_0^1 (t_{n+1} - t_{n-1} - \theta\Delta t_{n-1})^{-\alpha} d\theta}{\int_0^1 (t_{n+1} - t_n - \theta\Delta t_n)^{-\alpha} d\theta} \\ &= \left(\frac{\Delta t_n}{\Delta t_{n-1}}\right)^{\alpha} \left(\left(1 + \frac{\Delta t_n}{\Delta t_{n-1}}\right)^{1-\alpha} - \left(\frac{\Delta t_n}{\Delta t_{n-1}}\right)^{1-\alpha}\right) \\ &\leq 2^{1-\alpha} - 1.\end{aligned}$$

This relation together with (8) complete the proof. □

In the process of simulating blow-up solutions, an adaptive time-stepping strategy is needed. The main reason is that the numerical accuracy can be controlled by posteriori error estimate. Generally speaking, the increments of numerical solutions must be linearly controllable, that is,

$$\sup_{n\geqslant 0} \frac{\|U_{n+1}\|_\infty}{\|U_n\|_\infty} < \infty. \tag{9}$$

In this paper, we consider the blow-up behavior of numerical solutions such that the criteria (9) holds. Based on (9), the following definition for blow-up of numerical solutions is naturally introduced.

Definition 1 *A numerical solution of* (4) *blows up in finite time if there is an adaptive time-stepping such that the criteria* (9) *holds,* $\lim_{n\to\infty} \|U_n\|_\infty = \infty$ *and* $\lim_{n\to\infty} t_n < \infty$*.*

2 Discrete energy

In this section, we introduce the energy function to study the blow-up of the numerical solutions.

By the positive definiteness of $\mathbf{A}_N$, there is an eigenvalue $\lambda_N > 0$ and its corresponding eigenvector $\phi_N \in \mathbb{R}^{N-1}$ ($\phi_N^i > 0, i = 1, 2, \ldots, N-1$) such that

$$\mathbf{A}_N \phi_N = \lambda_N \phi_N,$$

where $\sum\limits_{i=1}^{N-1} \phi_N^i = 1$.

Next, we introduce a discrete energy

$$E_n := \sum_{i=1}^{N-1} \phi_N^i u_n^i = \langle \phi_N, U_n \rangle, \tag{10}$$

where $\langle \phi_N, U_n \rangle$ is the inner product of the eigenvector ϕ_N and numerical solution U_n.

The following lemma gives the equivalence of E_n and $\| U_n \|_\infty$. Moreover, we can use this lemma to give the definitions of lower and upper energy.

Lemma 3 *[29] Assume that the time-stepping is not increasing and* (7) *holds, and let* $c_N = \max_{i=1,2,\ldots,N-1} 1/\phi_N^i$. *Then*

$$E_n \leq \| U_n \|_\infty \leq c_N E_n \quad \textit{for all} \quad n \geq 0, \tag{11}$$

and

$$E_n^p \leq \langle \phi_N, F(U_n) \rangle \leq c_N^p E_n^p \quad \textit{for all} \quad n \geq 0. \tag{12}$$

Remark 1 *It follows from* (11) *that the numerical solution* U_n *blows up in finite time if and only if*

$$\sup_{n \geqslant 0} \frac{E_{n+1}}{E_n} < \infty, \lim_{n \to \infty} E_n = \infty, \lim_{n \to \infty} t_n < \infty,$$

which means that the numerical solution and the discrete energy blow up at the same time.

The discrete energy E_n can convert (4) from vector form to scalar form. Thus, by (10), formula (4) becomes

$$\sum_{j=0}^{n} (E_{j+1} - E_j) \gamma_{n,j} = -\langle \phi_N, \mathbf{A}_N U_n \rangle + \langle \phi_N, F(U_n) \rangle.$$

Hence, by Lemma 3 and the property of the inner product, the discrete energy satisfies

$$\sum_{j=0}^{n} (E_{j+1} - E_j) \gamma_{n,j} \geq -\lambda_N E_n + E_n^p \tag{13}$$

and

$$\sum_{j=0}^{n} (E_{j+1} - E_j) \gamma_{n,j} \leq -\lambda_N E_n + c_N^p E_n^p. \tag{14}$$

According to (5), (13) and (14), we define the lower and upper energy, respectively,

$$\gamma_{n,n} \underline{E}_{n+1} = (\gamma_{n,n} - \gamma_{n,n-1}) \underline{E}_n + \ldots + (\gamma_{n,1} - \gamma_{n,0}) \underline{E}_1 + \gamma_{n,0} \underline{E}_0 - \lambda_N \underline{E}_n + \underline{E}_n^p \tag{15}$$

and

$$\gamma_{n,n} \overline{E}_{n+1} = (\gamma_{n,n} - \gamma_{n,n-1}) \overline{E}_n + \ldots + (\gamma_{n,1} - \gamma_{n,0}) \overline{E}_1 + \gamma_{n,0} \overline{E}_0 - \lambda_N \overline{E}_n + c_N^p \overline{E}_n^p, \tag{16}$$

where $\underline{E}_0 = E_0 = \overline{E}_0$.

The following theorem is very important to simplify our blow-up analysis.

Theorem 1 *Assume that the time-stepping is not increasing and*

$$\Delta t_n^\alpha \leq \frac{\alpha}{\lambda_N \Gamma(2-\alpha)}. \tag{17}$$

Then $0 < \underline{E}_n \leq E_n \leq \overline{E}_n$ *for all* $n \geq 0$.

Proof. Firstly, we prove that $a_n = \gamma_{n,n} - \gamma_{n,n-1} - \lambda_N > 0$ under the condition (17). Since

$$\begin{aligned} &a_n + \lambda_N = \gamma_{n,n} - \gamma_{n,n-1} \\ &= \frac{1}{\Delta t_n^\alpha \Gamma(2-\alpha)} - \frac{1}{\Delta t_{n-1}^\alpha \Gamma(2-\alpha)} \left(\left(1 + \frac{\Delta t_n}{\Delta t_{n-1}}\right)^{1-\alpha} - \left(\frac{\Delta t_n}{\Delta t_{n-1}}\right)^{1-\alpha} \right), \end{aligned}$$

we get

$$\Delta t_n^\alpha \Gamma(2-\alpha)(a_n + \lambda_N) = 1 - \left(\frac{\Delta t_n}{\Delta t_{n-1}}\right)^\alpha \left(1 + \frac{\Delta t_n}{\Delta t_{n-1}}\right)^{1-\alpha} + \frac{\Delta t_n}{\Delta t_{n-1}}. \tag{18}$$

Let

$$x = \frac{\Delta t_n}{\Delta t_{n-1}},$$

then

$$f(x) = 1 - x^\alpha (1+x)^{1-\alpha} + x,$$

that is,

$$f(x) = 1 + x\left(1 - \left(1 + \frac{1}{x}\right)^{1-\alpha}\right).$$

Hence, $f(x)$ is decreasing and

$$\lim_{x\to\infty} x\left(1 - \left(1 + \frac{1}{x}\right)^{1-\alpha}\right) = \alpha - 1.$$

Then by (18), we have

$$\Delta t_n^\alpha \Gamma(2-\alpha)(a_n + \lambda_N) > \alpha,$$

thus

$$a_n > \frac{\alpha}{\Delta t_n^\alpha \Gamma(2-\alpha)} - \lambda_N > 0,$$

and (17) is obtained.

Secondly, the method of mathematical induction is used to prove the inequality $0 < \underline{E}_n \leq E_n \leq \overline{E}_n$. For $n = 0$, $\underline{E}_0 = E_0 = \overline{E}_0$ is obvious.

Assume that $0 < \underline{E}_k \le E_k \le \overline{E}_k$ for $k = 1, 2, \ldots, n$. From (14), (16) and $\gamma_{n,n} - \gamma_{n,n-1} - \lambda_N > 0$, we have

$$\begin{aligned}
&(\overline{E}_{n+1} - E_{n+1})\gamma_{n,n} \\
\ge&(\gamma_{n,n} - \gamma_{n,n-1})(\overline{E}_n - E_n) + \ldots + \gamma_{n,0}(\overline{E}_0 - E_0) - \lambda_N(\overline{E}_n - E_n) \\
&+ c_N^p(\overline{E}_n^p - E_n^p) \\
=&(\gamma_{n,n} - \gamma_{n,n-1} - \lambda_N)(\overline{E}_n - E_n) + \ldots + \gamma_{n,0}(\overline{E}_0 - E_0) + c_N^p(\overline{E}_n^p - E_n^p) \\
\ge&0,
\end{aligned}$$

and thus, $\overline{E}_{n+1} \ge E_{n+1}$. Similarity, we can obtain $\underline{E}_{n+1} \le E_{n+1}$ by (13) and (15). The proof is complete. □

3 Blow-up analysis

In contrast to [19], in this section, we present an adaptive time-stepping strategy according to the discrete energy. Moreover, we prove that numerical solutions exhibit the exact finite blow-up behavior for any positive initial function.

The lower energy in (15) can be rewritten as

$$\underline{E}_{n+1} = \underline{\omega}_n + \Delta t_n^{\alpha}\Gamma(2-\alpha)\underline{f}(\underline{E}_n), \tag{19}$$

where $\underline{f}(x) = -\lambda_N x + x^p$ and

$$\underline{\omega}_n = \sum_{l=0}^{n} \frac{\gamma_{n,l} - \gamma_{n,l-1}}{\gamma_{n,n}} \underline{E}_l.$$

We first show the monotonicity of the lower energy.

Lemma 4 *Under the condition* (17) *and*

$$E_0 > \lambda_N^{\frac{1}{p-1}}, \tag{20}$$

the lower energy $\underline{E}_n$ is an increasing sequence.

Proof. The lemma is proved by the method of mathematical induction.

For $n = 0$, by (19) and (20), we have

$$\underline{E}_1 = \underline{\omega}_0 + \Delta t_0^{\alpha}\Gamma(2-\alpha)\underline{f}(\underline{E}_0) > \underline{E}_0.$$

We suppose that $E_0 < \underline{E}_1 < \cdots < \underline{E}_n$. Then $\underline{f}(\underline{E}_n) > 0$ and $\underline{f}(\underline{E}_n) > \underline{f}(\underline{E}_{n-1})$. Hence, from (15) it follows that $\underline{f}(\underline{E}_n) = \sum\limits_{l=0}^{n} \gamma_{n,l}(\underline{E}_{l+1} - \underline{E}_l)$.

Therefore,

$$\begin{aligned}
\underline{f}(\underline{E}_{n-1}) &= \sum_{l=0}^{n-1} \gamma_{n-1,l}(\underline{E}_{l+1} - \underline{E}_l) \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{l=0}^{n-1} (\underline{E}_{l+1} - \underline{E}_l) \int_0^1 (t_n - t_l - \theta\Delta t_l)^{-\alpha} d\theta \\
&> \frac{1}{\Gamma(1-\alpha)} \sum_{l=0}^{n-1} (\underline{E}_{l+1} - \underline{E}_l) \int_0^1 (t_{n+1} - t_l - \theta\Delta t_l)^{-\alpha} d\theta \\
&= \sum_{l=0}^{n-1} \gamma_{n,l}(\underline{E}_{l+1} - \underline{E}_l)
\end{aligned}$$

and

$$\begin{aligned}
\underline{E}_{n+1} &= \sum_{l=0}^{n} \frac{\gamma_{n,l} - \gamma_{n,l-1}}{\gamma_{n,n}} \underline{E}_l + \frac{1}{\gamma_{n,n}} \underline{f}(\underline{E}_n) \\
&> \underline{E}_n - \frac{1}{\gamma_{n,n}} \sum_{l=0}^{n-1} \gamma_{n,l}(\underline{E}_{l+1} - \underline{E}_l) + \frac{1}{\gamma_{n,n}} \underline{f}(\underline{E}_{n-1}) \\
&> \underline{E}_n - \frac{1}{\gamma_{n,n}} \sum_{l=0}^{n-1} \gamma_{n,l}(\underline{E}_{l+1} - \underline{E}_l) + \frac{1}{\gamma_{n,n}} \sum_{l=0}^{n-1} \gamma_{n,l}(\underline{E}_{l+1} - \underline{E}_l) \\
&= \underline{E}_n,
\end{aligned}$$

This completes the proof. □

Assuming that (17) and (20) hold, we define the adaptive time-stepping from t_n to t_{n+1} as follows:

$$\Delta t_n = \min \left\{ \tau, \Delta t_{n-1}, \left(\frac{\tau}{\|U_n\|_\infty^{p-1}} \right)^{\frac{1}{\alpha}} \right\}. \tag{21}$$

The following theorem states that the numerical solutions exhibit the exact finite blow-up behavior.

Theorem 2 *Under the conditions* (17) *and* (20), *let the adaptive time-stepping be defined by* (21). *Then the numerical solution blows up in finite time if*

(i) $\lim_{n\to\infty} \| U_n \|_\infty = \infty$;

(ii) $\lim_{n\to\infty} \Delta t_n = 0$;

(iii) there exist constants n_τ *and* $\rho > 0$ *such that* $\underline{E}_{n+n_\tau} \geq (1 + \rho\tau)\underline{E}_n$;

(iv) $\lim_{n\to\infty} t_n = \sum_{n=0}^{\infty} \Delta t_n < \infty$.

Proof. (i) Suppose that there exists a subsequence E_{n_l} such that $\sup_l E_{n_l} < \infty$, then $\inf_l \Delta t_{n_l} > 0$. By Theorem 1 and Lemma 4, we have $\sup_n \underline{E}_n < \infty$. Therefore, $\lim_{n\to\infty} \underline{E}_n = \underline{E}_\infty < \infty$ and there is a sufficiently small $\epsilon > 0$ such that

$$\Gamma(2-\alpha)(\inf_l \Delta t_{n_l}^\alpha)\underline{f}(\underline{E}_\infty - \epsilon) > 2\epsilon. \tag{22}$$

Let $N_0 > 0$ be such that $\underline{E}_n > \underline{E}_\infty - \epsilon > E_0$ for $n > N_0$. For sufficiently large n, we have

$$\begin{aligned}
\underline{\omega}_n &= \sum_{l=0}^{n} C_{n,l}\underline{E}_l \geq \sum_{l=N_0}^{n} C_{n,l}\underline{E}_l > (\underline{E}_\infty - \epsilon)\sum_{l=N_0}^{n} \frac{\gamma_{n,l}-\gamma_{n,l-1}}{\gamma_{n,n}} \\
&= (\underline{E}_\infty - \epsilon)\left(1 - \Delta t_n^\alpha(1-\alpha)\int_0^1 (t_{n+1} - t_{N_0-1} - \theta\Delta t_{N_0-1})^{-\alpha}d\theta\right) \\
&\geq (\underline{E}_\infty - \epsilon)\left(1 - \tau^\alpha(1-\alpha)\int_0^1 (t_{n+1} - t_{N_0-1} - \theta\Delta t_{N_0-1})^{-\alpha}d\theta\right).
\end{aligned}$$

Taking into account that $0 < \alpha < 1$ and $\lim_{n\to\infty} t_n = \infty$, we conclude that for sufficiently large n, $\underline{\omega}_n > \underline{E}_\infty - 2\epsilon$.

On the other hand, for sufficiently large l, by (22) we get

$$\begin{aligned}
\underline{E}_\infty > \underline{E}_{n_l+1} &= \omega_{n_l} + \Delta t_{n_l}^\alpha \Gamma(2-\alpha)\underline{f}(\underline{E}_{n_l}) \\
&> \underline{E}_\infty - 2\epsilon + \Gamma(2-\alpha)(\inf_l \Delta t_{n_l}^\alpha)\underline{f}(\underline{E}_\infty - \epsilon) \\
&> \underline{E}_\infty - 2\epsilon + 2\epsilon \\
&= \underline{E}_\infty,
\end{aligned}$$

which is a contradiction. Thus, we have $\lim_{n\to\infty} E_n = \infty$, or equivalently, $\lim_{n\to\infty} \|U_n\|_\infty = \infty$.

(ii) Combining (21) and the proof of (i), we can show that $\lim_{n\to\infty} \Delta t_n = 0$.

(iii) Let $\tau > 0$ and n_τ be defined by

$$(n_\tau + 1)^{1-\alpha} - n_\tau^{1-\alpha} < \frac{1}{4}\tau\Gamma(2-\alpha).$$

From Lemma 4, we have

$$\begin{aligned}
\underline{\omega}_{n+n_\tau-1} &= \sum_{l=0}^{n+n_\tau-1} \frac{\gamma_{n+n_\tau-1,l} - \gamma_{n+n_\tau-1,l-1}}{\gamma_{n+n_\tau-1,n+n_\tau-1}}\underline{E}_l \\
&\geq \sum_{l=n}^{n+n_\tau-1} \frac{\gamma_{n+n_\tau-1,l} - \gamma_{n+n_\tau-1,l-1}}{\gamma_{n+n_\tau-1,n+n_\tau-1}}\underline{E}_l \\
&\geq \frac{\gamma_{n+n_\tau-1,n+n_\tau-1} - \gamma_{n+n_\tau-1,n-1}}{\gamma_{n+n_\tau-1,n+n_\tau-1}}\underline{E}_n \\
&\geq \left(1 - \frac{1}{4}\tau\Gamma(2-\alpha)\right)\underline{E}_n.
\end{aligned} \tag{23}$$

Further, assume that

$$\lambda_N \underline{E}_{n+n_\tau} < \frac{1}{2}\underline{E}^p_{n+n_\tau-1}$$

and

$$\Delta t^\alpha_{n+n_\tau-1}\underline{E}^p_{n+n_\tau-1} \geq \tau \underline{E}_n,$$

for sufficiently large n. From (19) and (23), we get

$$\begin{aligned}
\underline{E}_{n+n_\tau} &= \underline{\omega}_{n+n_\tau-1} + \Delta t^\alpha_{n+n_\tau-1}\Gamma(2-\alpha)(-\lambda_N \underline{E}_{n+n_\tau-1} + \underline{E}^p_{n+n_\tau-1}) \\
&\geq \left(1 - \frac{1}{4}\tau\Gamma(2-\alpha)\right)\underline{E}_n + \frac{1}{2}\Delta t^\alpha_{n+n_\tau-1}\Gamma(2-\alpha)\underline{E}^p_{n+n_\tau-1} \\
&\geq \left(1 - \frac{1}{4}\tau\Gamma(2-\alpha)\right)\underline{E}_n + \frac{1}{2}\tau\Gamma(2-\alpha)\underline{E}_n \\
&= (1+\rho\tau)\underline{E}_n,
\end{aligned}$$

where $\rho = \frac{1}{4}\Gamma(2-\alpha) > 0$.

(iv) For sufficiently large n, we assume that $\Delta t^\alpha_n \underline{E}^p_n \geq \tau^\alpha \underline{E}_n$. From Lemma 3 and Theorem 1, we have

$$\begin{aligned}
\Delta t_{n+n_\tau} &\leq \tau\left(\frac{1}{\|U_{n+n_\tau}\|^{p-1}_\infty}\right)^{\frac{1}{\alpha}} \leq \tau\left(\frac{1}{E^{p-1}_{n+n_\tau}}\right)^{\frac{1}{\alpha}} \\
&\leq \tau\left(\frac{1}{\underline{E}^{p-1}_{n+n_\tau}}\right)^{\frac{1}{\alpha}} \leq \tau\left(\frac{1}{(1+\rho\tau)^{p-1}\underline{E}^{p-1}_n}\right)^{\frac{1}{\alpha}} \\
&\leq \tau\left(\frac{1}{(1+\rho\tau)^{p-1}\frac{\tau^\alpha}{\Delta t^\alpha_n}}\right)^{\frac{1}{\alpha}} = \Delta t_n\left(\frac{1}{1+\rho\tau}\right)^{\frac{1}{\alpha(p-1)}},
\end{aligned}$$

thus

$$\sup_n \frac{\Delta t_{n+n_\tau}}{\Delta t_n} \leq \left(\frac{1}{1+\rho\tau}\right)^{\frac{1}{\alpha(p-1)}} < 1.$$

The proof is complete. □

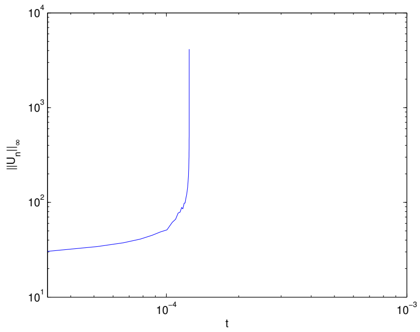
4 Numerical examples

In this section, we present three examples to illustrate the main results. We adapt the time-stepping by (21) and set $N = 10$, $\tau = 0.1$. For the sake of intuitiveness and convenience, we draw the curves of $\|U_n\|_\infty$ under the double logarithmic coordinates.

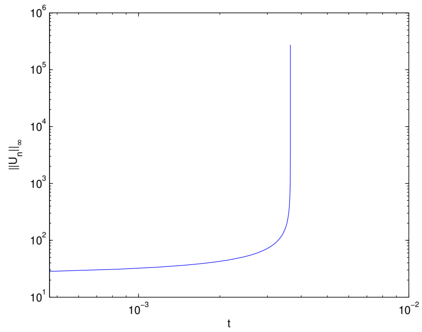
Consider the following TFPDEs:

$$\begin{array}{ll} {}^{C}\mathcal{D}_t^{\alpha} u(x,t) = u_{xx}(x,t) + u^2(x,t), & 0 \le x \le 1, t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 25\sin(\pi x), & 0 \le x \le 1. \end{array} \tag{24}$$

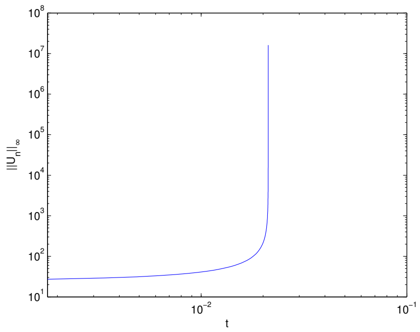
In Figure 1, we draw the curves of $\|U_n\|_\infty$ with different $\alpha \in (0,1)$. It can be seen that the numerical solutions blow up for all selected α. Furthermore, the blow-up times are increasing as the parameter α gets bigger and bigger.



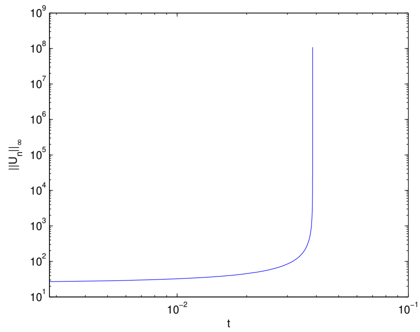
(a) $\alpha = 0.4$



(b) $\alpha = 0.6$



(c) $\alpha = 0.8$



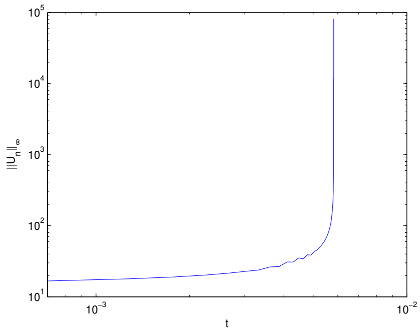
(d) $\alpha = 0.9$

Figure 1: The curves of $\|U_n\|_\infty$ for (24) with different α

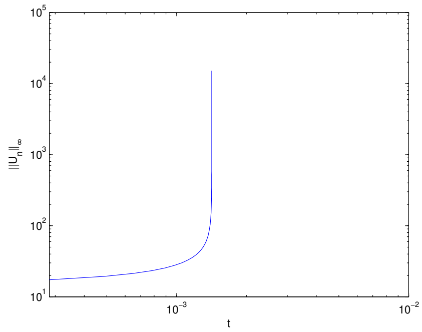
For the TFPDEs

$$\begin{array}{ll} {}^{C}\mathcal{D}_t^{0.6} u(x,t) = u_{xx}(x,t) + u^p(x,t), & 0 \le x \le 1, t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 15\sin(\pi x), & 0 \le x \le 1, \end{array} \tag{25}$$

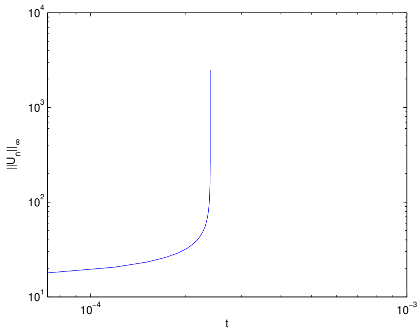
we draw the curves of $\|U_n\|_\infty$ with different p in Figure 2. It can be seen that numerical solutions blow up in finite time. Moreover, the numerical solutions



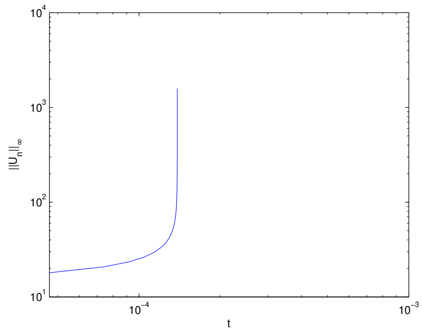
(a) $p = 2.1$



(b) $p = 2.3$



(c) $p = 2.6$



(d) $p = 2.7$

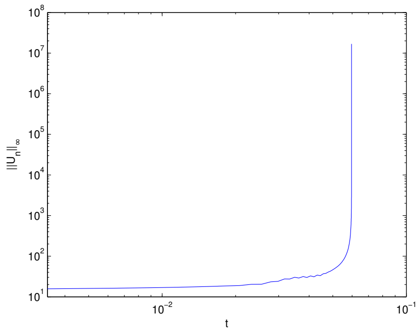
Figure 2: The curves of $\|U_n\|_\infty$ for (25) with different p

blow up faster as p is increasing. As it has been proved, the blow-up results are consistent with Theorem 2.

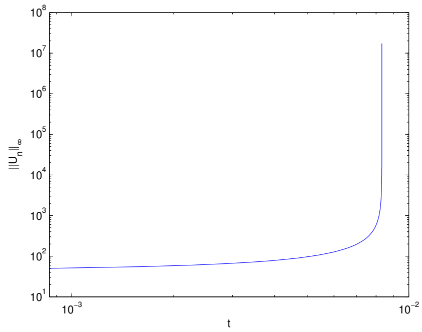
To illustrate the influence of initial condition to the blow-up behaviors, we consider

$$\begin{aligned}
&{}^{C}\mathcal{D}_t^{0.8}u(x,t) = u_{xx}(x,t) + u^2(x,t), && 0 \leq x \leq 1, t > 0,\\
&u(0,t) = u(1,t) = 0, && t > 0,\\
&u(x,0) = u_0(x), && 0 \leq x \leq 1.
\end{aligned} \tag{26}$$

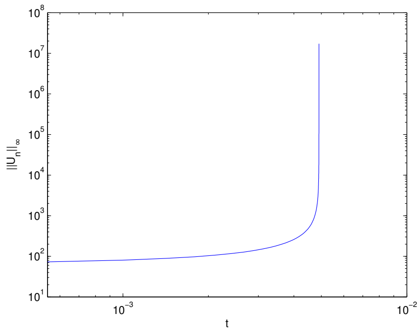
In Figure 3, we draw the curves of $\|U_n\|_\infty$ with different $u_0(x)$. It can be seen that the initial condition has important influence on blowing up of the numerical solutions. Moreover, the blow-up with large initial function is quite fast, which is the same as the analytical case for the nonlinear fractional ordinary differential equations [11], and this phenomena is also in agreement with Theorem 2.



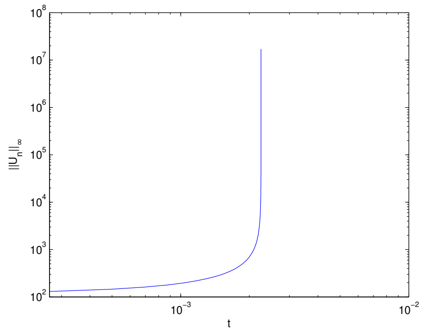
(a) $u_0(x) = 15\sin(\pi x)$



(b) $u_0(x) = 45\sin(\pi x)$



(c) $u_0(x) = 65\sin(\pi x)$



(d) $u_0(x) = 115\sin(\pi x)$

Figure 3: The curves of $\|U_n\|_\infty$ for (26) with different $u_0(x)$

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