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Tauberian Theorems by Weighted Summability Method

Valdete Loku and Naim L. Braha

Abstract. In this paper, we will show a new Tauberian theorems defined by weighted Nörlund-Cesáro summability method.

Key Words: Weighted Nörlund-Cesáro summability; One-sided and two-sided Tauberian conditions.
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Introduction

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums (s_n). The $(C,1)$ (see [3],page 7) transform is defined as the n-th partial sum of $(C,1)$ summability and is given by

$$\frac{s_0 + s_1 + \cdots + s_n}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \quad \text{as} \quad n \to \infty, \tag{1}$$

then the infinite series $\sum_{n=0}^{\infty} a_n$ is summable to the definite number s by $(C,1)$ method.

Let $\{p_n\}$ be a non-negative, non increasing sequence such that

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty, \quad \text{as} \quad n \to \infty, P_{-1} = p_{-1} = 0.$$

Then the series $\sum_{n=0}^{\infty} a_n$ is said to be almost Nörlund summable to S (or (N,p_n)-summable) if

$$\frac{1}{P_n}\sum_{v=0}^{n} p_{n-v} s_v \to S,$$

as $n \to \infty$.

The product of (N,p_n) summability and $(C,1)$ summability defines (N,p_n) $(C,1)$ summability and we denote it by $N_n^p C_n^1$. Thus if

$$N_n^p C_n^1 = \frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{1}{k+1}\sum_{v=0}^{k} s_v \to s \quad \text{as} \quad n \to \infty, \tag{2}$$

where N_n^p denotes the (N, p_n) transform of s_n and C_n^1 denotes the $(C, 1)$ transform of s_n, then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by $(N, p_n)(C, 1)$ means or summable $(N, p_n)(C, 1)$ to a definite number s. The (N, p_n) is a regular method of summability.

$$s_n \to s \Rightarrow C_n^1(s_n) = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s, \quad \text{as} \quad n \to \infty,$$

C_n^1 method is regular,

$$N_n^p(C_n^1(s_n)) = N_n^p C_n^1 \to s, \quad \text{as} \quad n \to \infty,$$

$N_n^p C_n^1$ method is regular.

We say that the sequence (x_n) is Nörlund-Cesáro summable to L by the weighted mean method determined by the sequences (p_n), or briefly $(N, p_n)(C, 1)-$ summable if

$$\lim_n N_n^p C_n^1(x) = L. \tag{3}$$

In this case we will write $L = N_n^p C_n^1 - \lim_n x_n$. We denote by $N_n^p C_n^1$ the set of all sequences which are summable $N_n^p C_n^1$. If

$$\lim_n x_n = a \tag{4}$$

exists, then (3) also exists. However, the converse is not always true. We can show by the following example

Example 1 *Let us consider that $p_n = 1$ for all $n \in \mathbb{N}$. Also we define the following sequence $x = (x_k) = (-1)^k$, then we have*

$$\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k}(-1)^v \to 0 \quad as \quad n \to \infty.$$

And as we know $x = (x_k)$, is not convergent.

Notice that (3) may imply (4) under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of sequences follows from its $(N, p_n)(C, 1)-$ summability and some Tauberian condition is said to be a Tauberian theorem for the $(N, p_n)(C, 1)-$ summability method.

The theory of Tauberian is extensively studied by many authors([1, 2], [4, 5], [7]). In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions convergence of sequences (x_n), follows from $N_n^p C_n^1-$ convergence. This method generalized method given in [5] and [7], it is shown on the following example.

Example 2 *Let us consider that $x_n = n$, then $N_n^p C_n^1$ reduces to the Nörlund method defined in [5] and [7].*

1 Main results

In this paper we will generalized Hardy's Tauberian theorem (see [3]) and obtain new Tauberian theorems for the weighted $(N, p_n)(C, 1)-$ summability method. Let $u = (u_n)$ be a sequence of real numbers. The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n = n(u_n - u_{n-1})$. The general control modulo of the oscillatory behavior of order 1 of (u_n) is defined by

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n^{(1)}(\omega_n^{(0)}(u)),$$

where $\sigma_n^{(1)}(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_k$. And identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}\Delta u,$$

where $V_n^{(0)}\Delta u = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k$, is known as Kronecker identity.
In our case the above definitions are as follows:

$$\omega_{n,p}^{(1)}(u) = \omega_{n,p}^{(0)}(u) - \sigma_{n,p}^{(1)}(\omega_{n,p}^{(0)}(u)),$$

where $\sigma_{n,p}^{(1)}(u) = \frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{1}{k+1}\sum_{v=0}^{k} u_v$.

We will start from theorem of Hardy,

Theorem 1 *([3]) If (x_n) is $(\overline{N}, p)$ summable to x and*

$$\omega_{n,p}^{(0)}(x) = 0(1)$$

then (x_n) converges to x.

Theorem 2 *[7] If $(\sigma_{n,p}^{(1)}(u))$ is $(\overline{N}, p)$ summable to s and the condition*

$$\omega_{n,p}^{(m)}(u) = 0(1)$$

holds, then (u_n) converges to s.

Now we are ready to formulate our results which are generalization of the result given above.

Theorem 3 *If*

$$\liminf_n \frac{P_{t_n}}{P_n} > 1, \quad t > 1 \tag{5}$$

where t_n, denotes the integer parts of the $[t \cdot n]$ for every $n \in \mathbb{N}$, and let (x_k) be a sequence of real numbers which converges to L, via $(N, p_n)(C, 1)-$

summability method. Then (x_k) *is convergent to the same number L if and only if the following two conditions holds:*

$$\sup_{t>1} \liminf_n \frac{1}{P_{t_n} - P_n} \sum_{j=n+1}^{t_n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \geq 0 \tag{6}$$

and

$$\sup_{0<t<1} \liminf_n \frac{1}{P_n - P_{t_n}} \sum_{j=t_n+1}^{n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_n - x_v) \geq 0. \tag{7}$$

In what follows we will show some auxiliary lemmas which are needful in the sequel.

Lemma 1 *Condition given by relation (5) is equivalent to this one:*

$$\liminf_n \frac{P_n}{P_{t_n}} > 1, \quad 0 < t < 1. \tag{8}$$

Proof. We omit it, because it is similar to the lemma 1, given in [6]. □

Proposition 1 *Let us suppose that relation (5) is satisfied and let* $x = (x_k)$ *be a sequence of complex numbers which is Nörlund-Cesáro summable to L. Then*

$$\lim_n \frac{1}{P_{t_n} - P_n} \sum_{j=n+1}^{t_n} p_j \frac{1}{j+1} \sum_{v=0}^{j} x_j = L, \quad \textit{for} \quad t > 1 \tag{9}$$

and

$$\lim_n \frac{1}{P_n - P_{t_n}} \sum_{j=t_n+1}^{n} p_j \frac{1}{j+1} \sum_{v=0}^{j} x_j = L, \quad \textit{for} \quad 0 < t < 1. \tag{10}$$

Proof. (I) Let us consider the case where $t > 1$. Then we obtain

$$\begin{aligned} &\frac{1}{P_{t_n} - P_n} \sum_{k=n+1}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v = \\ &\quad \frac{P_{t_n}}{P_{t_n} - P_n} \frac{1}{P_{t_n}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{P_n}{P_{t_n} - P_n} \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \\ &\qquad = \frac{1}{P_{t_n}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v + \frac{P_n}{P_{t_n} - P_n} \times \\ &\qquad \times \left[\frac{1}{P_{t_n}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right]. \end{aligned} \tag{11}$$

By relation (5) we get

$$\limsup_{n\to\infty} \frac{P_n}{P_{t_n} - P_n} = \frac{1}{\liminf_{n\to\infty} \frac{P_{t_n}}{P_n} - 1} < \infty.$$

Now relation (9), follows from relation (11) and assumed convergence of $N_n^p C_n^1$.

(II) In this case we have that $0 < t < 1$. Then

$$\frac{1}{P_n - P_{t_n}} \sum_{k=t_n+1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v =$$

$$\frac{P_n}{P_n - P_{t_n}} \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{P_{t_n}}{P_n - P_{t_n}} \frac{1}{P_{t_n}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v +$$

$$+ \frac{P_n}{P_n - P_{t_n}} \left[\frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{1}{P_{t_n}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right]. \quad (12)$$

Now relation (10), follows from relations (12), (8) and assumed convergence of $N_n^p C_n^1$. □

Proof of Theorem 3

Proof. Necessity. Let us suppose that $\lim_k x_k = L$, and $\lim_n N_n^p C_n^1(x) = L$. For every $t > 1$ following Proposition 1 we have

$$\lim_n \frac{1}{P_{t_n} - P_n} \sum_{k=n+1}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) =$$

$$\lim_n \left\{ \left(\frac{1}{P_{t_n} - P_n} \sum_{k=n+1}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right) - x_n \right\} = 0.$$

In case where $0 < t < 1$, we find that

$$\lim_n \frac{1}{P_n - P_{t_n}} \sum_{k=t_n+1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} (x_n - x_v) =$$

$$\lim_n \left\{ x_n - \left(\frac{1}{P_n - P_{t_n}} \sum_{k=t_n+1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right) \right\} = 0.$$

Sufficient: Assume that conditions (6) and (7) are satisfied. In what follows we will prove that $\lim_n x_n = L$. Given any $\epsilon > 0$, by relation (6) we can choose $t_1 > 0$ such that

$$\liminf_n \frac{1}{P_{t_{n_1}} - P_n} \sum_{j=n+1}^{t_{n_1}} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \geq -\epsilon, \tag{13}$$

where $t_{n_1} = [t_1 \cdot n]$. By the assumed summability $N_n^p C_n^1$ of (x_n), Proposition 1, for $t > 1$ and taking into account relation (13), we obtain

$$\limsup_n x_n \leq L + \epsilon. \tag{14}$$

On the other hand, if $0 < t < 1$, for every $\epsilon > 0$, we can choose $0 < t_2 < 1$ such that

$$\liminf_n \frac{1}{P_n - P_{t_{n_2}}} \sum_{j=t_{n_2+1}}^{n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_n - x_j) \geq -\epsilon, \tag{15}$$

where $t_{n_2} = [t_2 \cdot n]$. By the assumed summability $N_n^p C_n^1$ of (x_n), Proposition 1, for $0 < t < 1$ and relation (15), we get

$$\liminf_n x_n \geq L - \epsilon. \tag{16}$$

Since $\epsilon > 0$ is arbitrary, combining relations (14) and(16) yields the convergence

$$\lim_n x_n = L.$$

□

In the next result we will consider the case where $x = (x_n)$ is a sequence of complex numbers.

Theorem 4 *Let us suppose that relation (5) is satisfied. And (x_n) be a sequence of complex numbers, which is $N_n^p C_n^1-$ summable to L. Then (x_n) is convergent to the same number L if and only if the following two conditions holds:*

$$\inf_{t>1} \limsup_n \left| \frac{1}{P_{t_n} - P_n} \sum_{j=n+1}^{t_n} p_k \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \right| = 0 \tag{17}$$

and

$$\inf_{0<t<1} \limsup_n \left| \frac{1}{P_n - P_{t_n}} \sum_{j=t_n+1}^{n} p_k \frac{1}{j+1} \sum_{v=0}^{j} (x_n - x_v) \right| = 0. \tag{18}$$

Proof. Necessity: Let us suppose that relations (3) and (4) are satisfied. Than by Proposition 1, we get relation (17), for $t > 1$ and relation (18), fro $0 < t < 1$.

Sufficient: Let us suppose that relation (5), (3) and (17). are satisfied. Then for any given $\epsilon > 0$, there exists a $t_3 > 1$ such that

$$\limsup_n \left| \frac{1}{P_{t_{n_3}} - P_n} \sum_{j=n+1}^{t_{n_3}} p_k \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \right| \leq \epsilon,$$

where $t_{n_3} = [t_3 \cdot n]$. Taking into account fact that (x_n) is $N_n^p C_n^1$ summable we get the following estimation

$$\limsup_n |L - x_n| \leq \limsup_n \left| L - \frac{1}{P_{t_{n_3}} - P_n} \sum_{j=n+1}^{t_{n_3}} p_k \frac{1}{j+1} \sum_{v=0}^{j} x_v \right| +$$

$$\limsup_n \left| \frac{1}{P_{t_{n_3}} - P_n} \sum_{j=n+1}^{t_{n_3}} p_k \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \right| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\lim_n x_n = L$. Second case is similar to the first one and we omit it. □

Remark 1 *Theorem 2.3 is generalization of the theorem 2.2, because in the theorem 2.2 are given conditions for Tauberian theorem for Nörlund-Cesaro summability method* $(N, p_n)(C, 0)$ *and in theorem 2.3 are given conditions for Tauberian theorem for Nörlund-Cesaro summability method* $(N, p_n)(C, 1)$.

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Valdete Loku
Department of Management for Industry,
Street Rexhep Bislimi, University of Applied Sciences Ferizaj, 70000, Kosova
valdeteloku@gmail.com

N. L. Braha
Department of Mathematics and Computer Sciences,
Avenue Mother Teresa, No=5, University of Prishtina, 10000, Kosova
nbraha@yahoo.com

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