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Sums of Positive Integer Powers with Unlike Exponents

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Abstract. Consider the following problem: given a positive integer, what is the minimum number of positive integer powers having unlike exponents greater than one such that their sum is equal to the given number? We deal with this open question by presenting some experimental results, indicating some inequalities and relations, presenting some new integer sequences, obtaining a bivariate generating function, and eventually proposing a conjecture.

Key Words: Number Theory, Sums of Powers, Integer Sequences
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Introduction

What do the numbers 15, 23, 55, 62, 71 have in common? They seem to be the only ones that can be written as sums of not less than five positive integer powers with different exponents greater than one (unlike powers, or UP for short, in the following). Let $\mathbb{Z}$ denote the set of integers, $\mathbb{N}$ the set of non-negative integers and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. In this paper, we look at what happens when facing the problem of understanding the behavior of the function $f : \mathbb{N}^+ \to \mathbb{N}^+$ defined as follows:

$$f(n) = \min_{k\in\mathbb{N}^+} \left\{ k \,\middle|\, n = \sum_{i=1}^{k} x_i^{e_i} \,;\, x_1, \ldots, x_k, e_1, \ldots e_k \in \mathbb{N}^+ \wedge 1 < e_1 < \cdots < e_k \right\}.$$

The following authors found some important results in this sense (note that some of the x_i might possibly be zero in the below cited papers):

– Ford [1] showed that all sufficiently large $n \in \mathbb{N}$ can be represented as $n = x_1^2 + x_2^3 + \cdots + x_k^{k+1}$ with $k = 14$. The circle theorem of Hardy and Littlewood is the main used tool to tackle these kind of problems, and the author points out that in this case the theoretical limit of its applicability is $k = 10$.

– Liu and Zhao [7] improved Ford's result, lowering k to 13.

– Laporta and Wooley [6] proved that almost all (from the density point of view) large integers n can be represented as $n = x_1^3 + \cdots + x_8^{10}$.

– Roth [10] showed that almost all (from the density point of view) $n \in \mathbb{N}$ can be represented as $n = x_1^2 + x_2^3 + x_3^4$. Halberstam [3] proves a stronger result, with x_3 prime.

– Jabara [4] obtained that every $n \in \mathbb{Z}$ can be written as $x_1^2 + x_2^3 + x_3^4 + x_4^5$ with $x_i \in \mathbb{Z}$.

Obviously, if $n = \sum_{i=1}^{k} x_i^{e_i}$ is a valid UP representation (UPr in the following), then $f(n) \leqslant k$, and, by considering base 2 expansion, it is easy to verify that $n = 2^a b$, with b odd, implies

$$1 \leqslant f(n) \leqslant 2 + \lfloor \log_2 b \rfloor$$

(the bound can be refined by considering the Hamming weight of b). The results of Ford and Liu and Zhao show actually that there is a constant $U \in \mathbb{N}$ such that $1 \leqslant f(n) \leqslant U$. Note that the number of 1s in a minimal (with the minimum number of addends) UPr is surely smaller than U (the only $n \in \mathbb{N}$ for which $f(n) = n$ are 1, 2 and 3; for $n > 3$, it is always possible to use at least one addend greater than 1).

While questions concerning sums of powers having equal exponents have been extensively studied and treated — just think to the four-square theorem, or Waring's problem — the problem considered in this paper and some variations of it are still intriguing. In this paper, we present some results obtained by computer search and some "trivial" considerations, together with a bivariate generating function (g.f.), a conjecture and some new sequences recently added in OEIS [9].

An abridged version of this paper [2] also focused on educational aspects, as the problem, although still open, can be easily explained to students of any age.

1 Preliminary results and considerations

Table 1 contains some values of $f(n)$ for $n \leqslant 20$. As $1 = 1^e$ for whatever $e \in \mathbb{N}$, in order to avoid having an infinite number of different equivalent representations, we impose that bases equal to 1, when needed in a UPr, *must* have the smallest possible exponent(s), compatibly with (i.e., different from) the e_i appearing in the other addends, if any. We refer to this agreement with the wording *one-power rule*. In the following, we assume that for a minimal UPr the one-power rule is always satisfied.

n	$f(n)$
$1 = 1^2$	1
$2 = 1^2 + 1^3$	2
$3 = 1^2 + 1^3 + 1^4$	3
$4 = 2^2$	1
$5 = 2^2 + 1^3$	2
$6 = 2^2 + 1^3 + 1^4$	3
$7 = 2^2 + 1^3 + 1^4 + 1^5$	4
$8 = 2^3$	1
$9 = 3^2$	1
$10 = 3^2 + 1^3$	2
$11 = 3^2 + 1^3 + 1^4$	3
$12 = 2^2 + 2^3$	2
$13 = 2^2 + 2^3 + 1^4$	3
$14 = 2^2 + 2^3 + 1^4 + 1^5$	4
$15 = 2^2 + 2^3 + 1^4 + 1^5 + 1^6$	5
$16 = 2^4 = 4^2$	1
$17 = 1^2 + 2^4 = 3^2 + 2^3 = 4^2 + 1^3$	2
$18 = 1^2 + 1^3 + 2^4 = 4^2 + 1^3 + 1^4 = 3^2 + 2^3 + 1^4$	3
$19 = 1^2 + 1^3 + 2^4 + 1^5 = 3^2 + 2^3 + 1^4 + 1^5 = 4^2 + 1^3 + 1^4 + 1^5$	4
$20 = 2^2 + 2^4$	2

Table 1: Values of $f(n)$ for small values of n.

Example 1 *In* $29 = 1^2 + 3^3 + 1^4$*, the exponent* 3 *is "taken" in the power* 3^3 *and two powers of* 1 *are needed to reach* 29*. Their exponents, having to be minimal, can only be* 2 *and* 4*.*

Minimal representations are in general not unique: a "trivial" double one is $16 = 2^4 = 4^2$, easily generalizable for every single power n^e with $3 < e = e_1 e_2$ not a prime number, as $n^e = (n^{e_1})^{e_2}$. For 17 (see Table 1) there are three of them (according to the one-power rule, 1 is written as 1^2 in the first representation and as 1^3 in the last one): the first and last one could possibly, upon agreement, be considered as equivalent, while the second one is "essentially" different. From the table we deduce that $U \geqslant 5$.

One could also consider the number $\mu(n)$ of different minimal UPr of n. It is easy to prove that $\mu(n)$ is not bounded (just consider the numbers n^{p^k} with p prime, having at least k different minimal UPr). We report some random curiosities:

- $246 \leqslant n \leqslant 253$ are representable as sum of two (and not less) UP – all of them but 252 in a unique way.

- $711 \leqslant n \leqslant 721$ are all representable as sum of three (and not less) UP.
- Donald Duck's license plate number – 313 – is the smallest number with ten different representations with three (and not less) UP: $4^2 + 6^3 + 3^4, 5^2 + 2^5 + 2^8, 5^2 + 4^4 + 2^5, 7^2 + 2^3 + 2^8, 7^2 + 2^3 + 4^4, 9^2 + 6^3 + 2^4, 11^2 + 2^6 + 2^7, 11^2 + 4^3 + 2^7, 13^2 + 2^4 + 2^7, 17^2 + 2^3 + 2^4$.
- The above cited "Power rangers": $15 = 2^2 + 2^3 + 1^4 + 1^5 + 1^6$, $23 = 2^2+1^3+2^4+1^5+1^6$, $55 = 5^2+3^3+1^4+1^5+1^6$, $62 = 1^2+3^3+1^4+2^5+1^6$, $71 = 2^2 + 4^3 + 1^4 + 1^5 + 1^6$.

Proposition 1 *For all $n \in \mathbb{N}^+$, $k \in \mathbb{N}$ one has $f(n+k) \leqslant f(n) + k$.*

Proof. We can deduce a UPr of $n+k$ from a representation of n by adding appropriate powers of 1 (with the ad hoc exponents, according to one-power rule). □

Lemma 1 *Let $1 < h < k \in \mathbb{N}^+$ be such that the following conditions are satisfied:*

(1) *$h + k$ is not a prime number,*
(2) *if h is prime, then $h + k \neq h^2$.*

Then there are $1 < d_1 < d_2 \in \mathbb{N}$ with $h + k = d_1 d_2$, and at least one of them is different from both h and k.

Proof. As $k > 2$, we have $k^2 > 2k$ and $h + k < 2k \leqslant hk < k^2$; therefore, $h + k \neq hk \neq k^2$. Let $\gcd(h, k) = d < k$. If $d < h$ or d is composite, there is nothing to prove, otherwise, d is prime and $d = h$ (i.e., h is prime). Then $h + k \neq h^2$ as well and, being it composite and having excluded the three unwanted possibilities $(h, h), (h, k), (k, k)$ for (d_1, d_2), the thesis follows. □

Proposition 2 *If $m = a^h + b^k$, $n = a^k + b^h$ and the hypotheses of Lemma 1 are satisfied, then $f(mn) \leqslant 4$.*

Proof. By applying Lemma 1, $h + k = d_1 d_2$, with d_1 or d_2 different from both h and k. Then $mn = (a^h + b^k)(a^k + b^h)$ and $a^{h+k} + (ab)^h + (ab)^k + (b^{d_1})^{d_2}$ (or $(b^{d_2})^{d_1}$ as last addend) is a valid UPr for mn. □

Proposition 3 *Let $1 < h < k \in \mathbb{N}$ and $\delta = (h, k)$, $n = a^h + b^k$, $m = c^h + d^k$. If δ has two different proper divisors then $f(mn) \leqslant 4$.*

Proof. As there exist $d_1, d_2 \notin \{1, h, k\}$ such that $d_1 \neq d_2$ with $h = d_1 h_1, k = d_1 k_1$ and $h = d_2 h_2, k = d_2 k_2$, we can write

$$\begin{aligned} mn = (a^h + b^k)(c^h + d^k) &= (ac)^h + a^h d^k + b^k c^h + (bd)^k \\ &= (ac)^h + a^{d_1 h_1} d^{d_1 k_1} + b^{d_2 k_2} c^{d_2 h_2} + (bd)^k \\ &= (ac)^h + (a^{h_1} d^{k_1})^{d_1} + (b^{k_2} c^{h_2})^{d_2} + (bd)^k \end{aligned}$$

The exponents h, d_1, d_2, k are all different, hence a UPr is obtained, and the thesis follows. □

Proposition 4 *Let* $n = \sum_{i=1}^{f(n)} x_i^{e_i}$ *be a minimal UPr for* $n \in \mathbb{N}^+$ *and let* $e_0 = \mathrm{lcm}(e_1, \dots e_{f(n)})$. *Then* $f(a^{e_0} n) \leqslant f(n)$ *for all* $a \in \mathbb{N}^+$.

Proof. Let $e'_i = e_0/e_i$, i.e., $e_0 = e'_i e_i$. Then

$$a^{e_0} n = a^{e_0} \sum_{i=1}^{f(n)} x_i^{e_i} = \sum_{i=1}^{f(n)} (x_i a^{e'_i})^{e_i},$$

thus $f(a^{e_0} n) \leqslant f(n)$. □

The inequality may be strict: if $n = 2 = 1^2 + 1^3$, then $e_0 = \mathrm{lcm}(2, 3) = 6$. Let $a = 2$, then $f(a^{e_0} n) = f(2^6 \cdot 2) = f(2^7) = 1 < 2 = f(2)$.

Proposition 5 *Let* $n = \sum_{i=1}^{f(n)} x^{e_i}$. *Then* $f(x^e n) \leqslant f(n)$ *for* $e \in \mathbb{N}$.

Proof. As

$$x^e n = x^e \sum_{i=1}^{f(n)} x^{e_i} = \sum_{i=1}^{f(n)} x^{e_i+e}$$

and $e_i \neq e_j$ imply that $e_i + e \neq e_j + e$, a valid UPr for $x^e n$ is obtained and the thesis follows. □

Note that, in general, the inequality cannot be substituted by an equality. The smallest counterexamples with different decreases are summarized in Table 2.

n	$f(n)$	e	$m = 2^e n$	$f(m)$
$56 = 2^3 + 2^4 + 2^5$	3	5	$1792 = 8^2 + 12^3$	2
$72 = 2^3 + 2^6$	2	1	$144 = 12^2$	1
$120 = 2^3 + 2^4 + 2^5 + 2^6$	4	1	$240 = 12^2 + 4^3 + 2^5$	3
$200 = 2^3 + 2^6 + 2^7$	3	1	$400 = 20^2$	1
$2104 = 2^3 + 2^4 + 2^5 + 2^{11}$	4	3	$16832 = 2^5 + 7^5$	2

Table 2: Smallest strict inequality cases (Proposition 5).

Proposition 6 is an easy corollary of Proposition 1 using the numbers in Table 1.

Proposition 6 *If* $f(n) = 1$, *i.e.,* $n = x^e$, *then* $f(n+1) \leqslant 2$, $f(n+2) \leqslant 3$, $f(n+3) \leqslant 4$. *In addition,*

– *If* $e \geqslant 3$, *then* $f(n+4) \leqslant 2$, $f(n+5) \leqslant 3$, $f(n+6) \leqslant 4$.

– *If* $e > 3$, *the following holds as well:* $f(n+8) \leqslant 2$, $f(n+9) \leqslant 2$, $f(n+10) \leqslant 3$, $f(n+11) \leqslant 4$, $f(n+12) \leqslant 3$, $f(n+13) \leqslant 4$.

2 Generating function

For a generic power series $S = S(x) = \sum_{i\geqslant 0} s_i x^i$, we define $[x^i]S(x) = s_i$. In order to represent the generating function $F(x) = 1+\sum_{i>0} f(i)x^i$ of f, set, first of all, $P_k(x) = \sum_{i\geqslant 0} x^{i^k}$. Let I_r define a sequence $(i_1,\ldots,i_r) \in \mathbb{N}^r$ of integers $1 < i_1 < \cdots < i_r$; let I_r^* be the set of all such sequences, and let $I = \bigcup_{r\geqslant 1} I_r^*$. Setting $P_{I_r}(x) = \prod_{j=1}^{r} P_{i_j}(x)$, we obtain $f(n) = \min_{I_r\in I} \{r \mid [x^n]P_{I_r}(x) > 0\}$ and

$$F(x) = 1 + x + 2x^2 + 3x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + x^8 + x^9 + \cdots .$$

Given a polynomial $p(y) = \sum_{i=0}^{m} a_i y^i \in \mathbb{N}[y]$, let $p^{(j)}(y)$ be its j^{th} derivative, with $p^{(0)}(y) = p(y)$. We define the *order* of $p(y)$ to be the smallest exponent such that the coefficient a_i in the corresponding monomial is not zero and denote it by $\operatorname{ord}(p) = \min_{i\in[0,m]} \{i \mid a_i \neq 0\}$ (having in mind the convention that, if $p(y)$ is the zero polynomial, then $\operatorname{ord}(p) = -\infty$).

One could be tempted to consider the infinite product $P(x) = \prod_{i\geqslant 2} P_i(x)$ as the g.f. counting the number of different possible ways, with no restriction at all, for which a number can be obtained as whatever sum of unlike powers, but, as $x^{1^k} = x$ for every $k \in \mathbb{N}$, infinite "different" instances of x would contribute to the product, making it not well defined. We therefore apply some workaround that will permit to obtain both $f(n)$ and $\mu(n)$ (considering the one-power rule).

First, we introduce a new variable y, associated with the number of addends. Second, we consider the 1^k addends by collecting them in a separate factor, containing all monomials $x^i y^i$ with $i = 0,\ldots,u = U$ representing the sum of i (powers of) 1s. The resulting bivariate g.f. is therefore

$$\begin{aligned} F(x,y) &= \left(\sum_{i=0}^{u} (xy)^i\right) \prod_{k\geqslant 2} \left(1 + y\sum_{i\geqslant 2} x^{i^k}\right) \\ &= \frac{1-(xy)^{u+1}}{1-xy} \prod_{k\geqslant 2} \left(1 + y\sum_{i\geqslant 2} x^{i^k}\right) = 1 + \sum_{n\geqslant 1} c_n(y)x^n \end{aligned}$$

with $c_n(y) = \sum_{i=0}^{n} c_{n,i} y^i$. Then $f(n) = \operatorname{ord}(c_n)$ and $\mu(n) = c_{n,\operatorname{ord}(c_n)}$. The polynomials $c_n(y)$ are, so to say, complete if $u = \infty$. From the obtained results from our experiments we conjecture that it should be $u = 3$: at most three powers of 1 are really needed. In that case $F(x, y)$ would be

$$\begin{aligned}F(x,y) &= 1 + yx + y^2x^2 + y^3x^3 + (y^4+y)x^4 + (y^5+y^2)x^5 + (y^6+y^3)x^6 \\ &+ (y^7+y^4)x^7 + (y^8+y^5+y)x^8 + (y^9+y^6+y^2+y)x^9 \\ &+ (y^{10}+y^7+y^3+y^2)x^{10} + (y^{11}+y^8+y^4+y^3)x^{11} + \cdots\end{aligned}$$

For example, the monomials of the polynomial $c_9(y) = y^9 + y^6 + y^2 + y$ all have unit coefficient and tell therefore that there is only one way to represent 9 as a sum of, respectively, one (y) power, 3^2, two (y^2) powers, $1^2 + 2^3$, six (y^6) powers, $2^2 + 1^3 + 1^4 + 1^5 + 1^6 + 1^7$, or nine ($y^9$) powers, $1^2 + \cdots + 1^{10}$. With $u = 3$, the result would be

$$\begin{aligned}F(x,y) &= 1 + yx + y^2x^2 + y^3x^3 + yx^4 + y^2x^5 + y^3x^6 + y^4x^7 + yx^8 \\ &+ yx^9 + (y^3+y^2)x^{10} + (y^4+y^3)x^{11} + (y^4+y^2)x^{12} + \cdots\end{aligned}$$

Consider $\overline{\zeta}(n) : \mathbb{N} \to \{0, 1\}$ defined as

$$\overline{\zeta}(n) = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{otherwise,} \end{cases}$$

with $\overline{\zeta}(x) = \overline{\zeta}(kx)$ for any $k \in \mathbb{N}^+$.

Proposition 7 *If* $0 \neq p(y) \in \mathbb{N}[y]$, *then*

$$\operatorname{ord}(p) = \sum_{i=1}^{\deg(p)} i\,\overline{\zeta}(p^{(i)}(0)) \prod_{j=0}^{i-1} \left(1 - \overline{\zeta}\left(p^{(i)}(0)\right)\right).$$

Proof. The idea is to look at the coefficients one at a time, starting from the constant term, and find the first one which is not equal to zero. The result is the corresponding exponent of y. If $a_0 \neq 0$, then the result is zero, otherwise, if $a_1 \neq 0$, then the result is one, and so on up to $\deg(p)$, at worst. For example, if $n = 3$, we would have

$$\begin{aligned}\operatorname{ord}(p) &= (1 - \overline{\zeta}(a_0))\Big(\overline{\zeta}(a_1) + (1 - \overline{\zeta}(a_1))\big(2\overline{\zeta}(a_2) + (1 - \overline{\zeta}(a_2))(3\overline{\zeta}(a_3))\big)\Big) \\ &= \overline{\zeta}(a_1)(1 - \overline{\zeta}(a_0)) + 2\overline{\zeta}(a_2)\prod_{i=0}^{1}\big(1 - \overline{\zeta}(a_i)\big) + 3\overline{\zeta}(a_3)\prod_{i=0}^{2}\big(1 - \overline{\zeta}(a_i)\big).\end{aligned}$$

As $a_i = p^{(i)}(0)/i!$, the thesis follows considering that $\overline{\zeta}(a_i) = \overline{\zeta}(a_i i!) = \overline{\zeta}(p^{(i)}(0))$ and by extending the above computation by classical induction. □

By applying Proposition 7 to the polynomials $c_n(y)$ and keeping in mind that $c_{n,0} = 0$, we finally obtain

$$f(n) = \overline{\zeta}(c_n'(0)) + \sum_{i=2}^{n} i\,\overline{\zeta}\big(c_n^{(i)}(0)\big) \prod_{j=1}^{i-1} \big(1 - \overline{\zeta}\big(c_n^{(j)}(0)\big)\big).$$

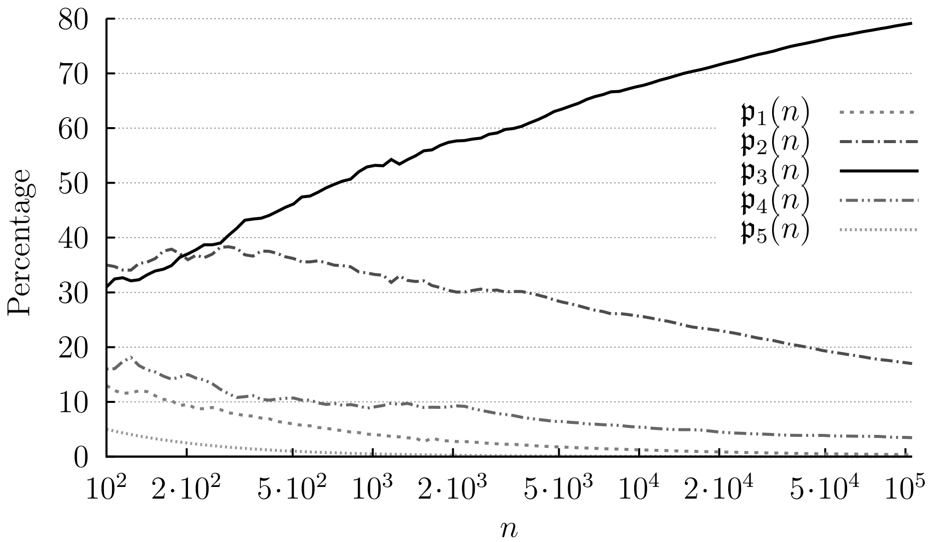

Figure 1: Percentage values $f(n)/n$ for numbers not greater than n for $n \in [10^2, 10^5]$.

3 Experimental results and sequences of sums of unlike powers

Let $v = 1, \ldots, 5$ and $\mathfrak{p}_v(n)$ be the percentage ratio of the number of integers $1 \leqslant x \leqslant n$ such that $f(x) = v$ and n. We report in Figure 1 the graphics of $\mathfrak{p}_v(n)$ for $n \leqslant 100000$ (logarithmic scale is used in abscissas). As one can see, the numbers that can be written with exactly three UP – minimal representation – tend more and more, from the density point of view, to dominate. It is well known that the set of perfect powers (OEIS sequence A001597) has density zero in $\mathbb{N}$ (see Nyblom's paper [8] for more details), that is $\lim\limits_{n\to\infty} \overline{P}(n)/n = 0$, where $\overline{P}(n) = \{m \in \mathbb{N}^+ \mid m \leqslant n \wedge f(m) = 1\}$. From the results of our experiments we suppose that $\mathfrak{p}_5(n) = 500/n$ and the graph, even if very partial, is globally a witness of Roth's result [10].

OEIS sequence A070049 concerns the sum of two perfect powers, but with not necessarily different exponents, while A327609 contains single perfect powers and sums of two of them. The sequence consisting of the integer values $n = x_1^{e_1} + x_2^{e_2}$ (with $x_1, x_2 > 0$ and $1 < e_1 < e_2$) that are not perfect powers,

$$2, 5, 10, 12, 17, 20, 24, 26, 28, 31, 33, 37, 40, 41, 43, 44, 48, 50, 52, \ldots$$

is the sequence A351062. Something similar happens for sequence A074499 (three powers). The sequence, considered in this paper, formed by integers n such that $f(n) = 3$ is A351066, beginning with

$$3, 6, 11, 13, 18, 21, 29, 34, 38, 42, 45, 47, 51, 53, 56, 58, 60, 66, 69, \ldots$$

The sequence given by sums of four — and not less — UP (with unlike exponents greater than 1) is A351063, beginning with

$$7, 14, 19, 22, 30, 35, 39, 46, 54, 61, 67, 70, 78, 87, 94, 99, 103, 110, \ldots$$

not to speak, obviously, about the sequences A351064 and A351065, generated, respectively, by $f(n)$ and $\mu(n)$.

In the website `https://sum-of-unlike-powers.jimdosite.com` minimal UPrs for $n \leqslant 40000$ are freely available. By computer search, we tested for minimal UPrs with at most four addends all $n \leqslant 1.4 \cdot 10^7$: the only missing numbers were the five "Power Rangers", and for all n at most three 1s were needed. This lead us to the following conjecture, similar to Kløve's [5].

Conjecture: For any $n \in \mathbb{N}^+$, $f(n) \leqslant 5$, and $f(n) = 5$ iff $n = 15, 23, 55, 62, 71$.

In a minimal UPr, many different exponents appear. An interesting point is the following one: what is the smallest number $m(e)$ for which exponent $e = 2, 3, \ldots$ is mandatory for the/a minimal UPr of $m(e)$?

If e is a prime number greater than 5, it is clear that $m(e) = 2^e$. If the conjecture is true, as no more than four addends suffice for $n > 71$, exponents with more than three nontrivial divisors are never needed. For example, as x^{12} can be considered also as a square, a third, fourth, and sixth power, and there are (worst case, according to the conjecture) at most other three addends, it is always possible to rewrite a twelfth power as a power with the remaining "free" exponent(s). Two examples follows. We can rewrite the addend 2^{12} in other ways, so that exponent 12 is not really needed:

$$4108 = 2^2 + 2^3 + \boxed{2^{12}} = 2^2 + 2^3 + \boxed{4^6} = 2^2 + 2^3 + \boxed{8^4}$$
$$4331 = 12^2 + 3^3 + 2^6 + \boxed{2^{12}} = 12^2 + 3^3 + \boxed{8^4} + 2^6$$

If the conjecture is true, exponents e with more than three proper divisors will not have to be used necessarily, so that the only ones to be considered are $e = p, p^2, p^3, p^4, pq$ with p, q primes. It is not difficult to answer this question for small values of e by looking at Table 1. These and other interesting particular values are shown in Table 3, with $m(16)$ still to be determined.

e	$m(e)$	e	$m(e)$	e	$m(e)$	e	$m(e)$
2	**1**	8	**1506**	14	**30235**	20	$\times$
3	**2**	9	**520**	15	**32827**	21	**2099366**
4	**3**	10	**1303**	16	?	22	**4383551**
5	**7**	11	**2048**	17	**131072**	23	**8388608**
6	**15**	12	$\times$	18	$\times$	24	$\times$
7	**128**	13	**8192**	19	**524288**	25	**33554464**

Table 3: Values of $m(n)$ for small values of $n \in \mathbb{N}$.

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