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Adomian's Decomposition Method Applied to an Exponential Equation

R. N. Tevzadze

Abstract. The purpose of this study is to obtain a decomposition of the solution to a backward stochastic differential equation used in the dual problem of mathematical finance. Some explicitly solvable equations considered. We convert the equation into a system of recurrent relations. By solving this system and proving convergence of the series the solution to the equation can be determined. In this study, Adomian's method was applied to solve the backward stochastic differential equation. An explicit solution was obtained for some examples.

Key Words: Stochastic Exponential, Martingale, Adomian Series, Brownian Motion
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Introduction

In a number of works [1, 2], Adomian develops a numerical technique using special kinds of polynomials for solving non-linear functional equations. However, Adomian and his collaborators did not develop widely the problem of convergence.

In this article, we will study by Adomian technique some kind of quadratic backward martingale equation and prove the convergence of the series. For example, we consider an equation of the form

$$\mathcal{E}_T(m)\mathcal{E}_T^{\alpha}(m^{\perp}) = c\exp\{\eta\} \tag{1}$$

w.r.t. stochastic integrals $m = \int f_s dW_s$, $m^{\perp} = \int g_s dW_s^{\perp}$ and real number c, where $(W, W^{\perp})$ is two-dimensional Brownian motion and η is a random variable.

Equations of such type are arising in mathematical finance, and they are used to characterize optimal martingale measures (see, for example, Biagini et al. [3], Mania and Tevzadze [11, 12, 13]). Note that equation (1) can be applied also to financial market models with infinitely many assets (see M. De Donno et al. [5]). Biagini et al. [3] considered an exponential equation of the form

$$\frac{\mathcal{E}_T(m)}{\mathcal{E}_T(m^\perp)} = ce^{\int_0^T \lambda_s^2 ds} \tag{2}$$

(which corresponds to the case $\alpha = -1$).

Our goal is to show the solvability of the equation (1) using the Adomian method proving the convergence of series. On the one hand, a simpler proof of solvability than in [11, 9] is obtained. On the other hand, it allows obtaining the approximation of the solution by the partial sums of series. The proof of the convergence is greatly simplified if we represent equation (1) as a backward stochastic differential equation (BSDE)

$$\int_0^T \varphi_s dW_s + \int_0^T \psi_s dW_s = c + \int_0^T \varphi_s^2 ds + \alpha \int_0^T \psi_s^2 ds + \int_0^T \lambda_s^2 ds$$

in the space of BMO-martingales (see Kazamaki [8]) and use the estimations of the BMO-norm. The result is resumed in Theorem 1. This result allows us to find a solution in the form of series with known terms.

Finally, we provide some examples, exactly solvable by Adomian method.

1 The main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\mathbf{F} = (F_t, t \in [0, T])$. We assume that all local martingales with respect to $\mathbf{F}$ are continuous. Here T is a fixed time horizon and $\mathcal{F} = F_T$.

Let $\mathcal{M}$ be a stable subspace of the space of square integrable martingales H^2. Then its ordinary orthogonal $\mathcal{M}^\perp$ is a stable subspace and any element of $\mathcal{M}$ is strongly orthogonal to any element of $\mathcal{M}^\perp$. Thus, we have the decomposition $H^2 = \mathcal{M} \bigoplus \mathcal{M}^\perp$ (see, e.g., [6, 7]).

We consider the following exponential equation

$$\mathcal{E}_T(m)\mathcal{E}_T^\alpha(m^\perp) = c\exp\{\eta\}, \tag{3}$$

where η is a given F_T-measurable random variable and α is a given real number. A solution to equation (3) is a triple $(c, m, m^\perp)$, where c is strictly positive constant, $m \in \mathcal{M}$ and $m^\perp \in \mathcal{M}^\perp$. Here $\mathcal{E}(X) := \exp(X - \langle X\rangle/2)$ denotes the Doleans-Dade exponential of X.

It is evident that if $\alpha = 1$, then equation (3) admits an "explicit" solution. E.g., if $\alpha = 1$ and η is bounded, then using the unique decomposition of the martingale $\mathbb{E}(\exp\{\eta\}/F_t)$,

$$\mathbb{E}(\exp\{\eta\}/F_t) = \mathbb{E}\exp\{\eta\} + m_t(\eta) + m_t^\perp(\eta), \quad m(\eta) \in \mathcal{M}, \quad m^\perp(\eta) \in \mathcal{M}^\perp,$$

it is easy to verify that the triple $c = 1/\mathbb{E}\exp\{\eta\}$,

$$m_t = \int_0^t \frac{1}{\mathbb{E}(\exp\{\eta\}/F_s)} dm_s(\eta), \quad m_t^{\perp} = \int_0^t \frac{1}{\mathbb{E}(\exp\{\eta\}/F_s)} dm_s^{\perp}(\eta)$$

satisfies equation (3).

Our aim is to prove the existence of a series convergent to the unique solution to equation (3) for η satisfying the following boundedness condition:

B) η is an F_T-measurable random variable of the form

$$\eta = \bar{\eta} + \gamma A_T,$$

where $\bar{\eta} \in L^{\infty}$, γ is a constant and $A = (A_t, t \in [0,T])$ is a continuous F-adapted process of finite variation such that

$$\mathbb{E}(\mathrm{var}_{[\tau,T]}(A)/F_{\tau}) \le C$$

for all stopping times τ for a constant $C > 0$.

One can show that equation (3) is equivalent to the following semimartingale backward equation with the square generator:

$$Y_t = Y_0 - \frac{\gamma}{2}A_t - \langle L\rangle_t - \frac{1}{\alpha}\langle L^{\perp}\rangle_t + L_t + L_t^{\perp}, \qquad Y_T = \frac{1}{2}\bar{\eta}. \tag{4}$$

We use also the equivalent equation of the form

$$L_T + L_T^{\perp} = \bar{c} + \langle L\rangle_T + \frac{1}{\alpha}\langle L^{\perp}\rangle_T + \frac{1}{2}\eta$$

w.r.t. $(\bar{c}, L, L^{\perp})$.

We use notations

$$|M|_{\mathrm{BMO}} = \inf\{C : \mathbb{E}^{\frac{1}{2}}(\langle M\rangle_T - \langle M\rangle_{\tau}|F_{\tau}) \le C\} \equiv \operatorname*{ess\,sup}_{\tau} \mathbb{E}^{\frac{1}{2}}(\langle M\rangle_T - \langle M\rangle_{\tau}|F_{\tau})$$

for BMO-norms of martingales, $|A|_{\omega} = \inf\{C : \mathbb{E}(var_t^T(A)|F_t) \le C\}$ for norms of finite variation processes, and $A \cdot M$ for stochastic integrals.

Let us consider the system of semimartingale backward equations

$$\begin{aligned}
Y_t^{(0)} &= Y_0^{(0)} - \frac{\gamma}{2}A_t + L_t^{(0)} + L_t^{(0)\perp}, \qquad Y_T^{(0)} = \frac{1}{2}\bar{\eta}, \\
Y_t^{(n+1)} &= Y_0^{(n+1)} - \sum_{k=0}^{n}\langle L^{(k)}, L^{(n-k)}\rangle_t \\
&\quad - \frac{1}{\alpha}\sum_{k=0}^{n}\langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_t + L_t^{(n+1)} + L_t^{(n+1)\perp}, \\
Y_T^{(n+1)} &= 0.
\end{aligned}$$

The sequence $Y_0^{(n)} = c^{(n)}, L^{(n)} + L^{\perp(n)}$, $n = 0, 1, 2, \ldots$, can be defined consequently by the equations

$$\mathbb{E}(\eta|\mathcal{F}_t) + \frac{\gamma}{2}E(A_T|F_t) = c^{(0)} + L_t^{(0)} + L_t^{\perp(0)},$$
$$\sum_{k=0}^{n}\mathbb{E}(\langle L^{(k)}, L^{(n-k)}\rangle_T|F_t) + \frac{1}{\alpha}\sum_{k=0}^{n}\mathbb{E}(\langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_T|F_t)$$
$$= c^{(n+1)} + L_t^{(n+1)} + L_t^{\perp(n+1)}.$$

Remark 1 If (W, B) is two-dimensional brownian motion and $A_t = \int_0^t a(s, W_s, B_s)ds$, $\bar{\eta} = 0$, then the solution to (4) will be of the form $Y_t = v(t, W_t, B_t)$, where $v(t, x, y)$ is decomposed as series $\sum_n v^n(t, x, y)$ satisfying the system of PDEs

$$(\partial_t + \frac{1}{2}\Delta)v^0(t, x, y) + a(t, x, y) = 0, \qquad v^0(T, x, y) = 0,$$
$$(\partial_t + \frac{1}{2}\Delta)v^n(t, x, y) + \frac{1}{2}\sum_{k=0}^{n-1}(v_x^k(t, x, y)v_x^{n-k-1}(t, x, y)$$
$$+\alpha v_y^k(t, x, y)v_y^{n-k-1}(t, x, y)) = 0,$$
$$v^n(T, x, y) = 0, \quad n \geq 1.$$

Equations for v^n were obtained by equating Ito's formulas for $v^n(t, W_t, B_t)$ and equations for Y_t^n.

Lemma 1 *Let*

$$Y_t = Y_0 + A_t + m_t, \qquad Y_T = \eta,$$

where m is a martingale, $\eta \in L_\infty$ and $|A|_\omega < \infty$. Then $m \in BMO$ and

$$|m|_{\text{BMO}} \leq |\eta|_\infty + |A|_\omega. \tag{5}$$

In particular, if $|A|_\omega < \infty$, then the martingale $\mathbb{E}(A_T|F_t)$ belongs to the BMO space and

$$|\mathbb{E}(A_T|F_\cdot)|_{\text{BMO}} \leq |A|_\omega.$$

Proof. By the Ito formula,

$$Y_t^2 = Y_0^2 + 2\int_0^t Y_s dm_s + 2\int_0^t Y_s dA_s + \langle m\rangle_t.$$

Taking the difference $Y_\tau^2 - Y_T^2$ and conditional expectations, we have that

$$\begin{aligned} Y_\tau^2 + \mathbb{E}(\langle m\rangle_T - \langle m\rangle_\tau|F_\tau) &= \mathbb{E}(\eta^2|F_\tau) - 2\mathbb{E}(\int_\tau^T Y_s dA_s|F_\tau) \\ &\leq |\eta|_\infty^2 + 2|Y|_\infty|A|_\omega. \end{aligned} \tag{6}$$

Here, $\mathbb{E}(\int_\tau^T Y_s dm_s|F_\tau) = 0$ as $Y_t \le \mathbb{E}(\eta + |A_T - A_t||\mathcal{F}_t)$ is bounded and m is a martingale. Since the right-hand side of (6) does not depend on τ, from (6) we obtain

$$|Y|_\infty^2 + |m|_{BMO}^2 \le |\eta|_\infty^2 + |Y|_\infty^2 + |A|_\omega^2.$$

Therefore,

$$||m||_{BMO}^2 \le |\eta|_\infty^2 + |A|_\omega^2,$$

which implies inequality (5). □

Lemma 2 *For the BMO-norms of martingales $L^{(n)} + L^{\perp(n)}$, defined above, the following estimates are true:*

$$|L^{(n)} + L^{\perp(n)}|_{\rm BMO} \le a_n(1+|\beta|)^n |L^{(0)} + L^{\perp(0)}|_{\rm BMO}^{n+1}, \tag{7}$$

where $\beta = 1/\alpha$ and coefficients a_n are calculating recurrently from

$$a_0 = 1, \qquad a_{n+1} = \sum_{k=0}^n a_k a_{n-k}.$$

Proof. Using Lemma 1, it is easy to show that

$$|L^{(1)} + L^{\perp(1)}|_{\rm BMO} \le a_1(1+|\beta|)|L^{(0)} + L^{\perp(0)}|_{\rm BMO}^2,$$

$$|L^{(2)} + L^{\perp(2)}|_{\rm BMO} \le a_2(1+|\beta|)^2|L^{(0)} + L^{\perp(0)}|_{\rm BMO}^3.$$

Assume that inequality (7) is valid for any $k \le n$ and let us show that

$$|L^{(n+1)} + L^{\perp(n+1)}|_{\rm BMO} \le a_{n+1}(1+|\beta|)^{n+1}|L^{(0)} + L^{\perp(0)}|_{\rm BMO}^{n+2}.$$

Applying Lemma 1 for $Y_t^{(n+1)}$ and the Kunita-Watanabe inequality, we have

$$|L^{(n+1)} + L^{\perp(n+1)}|_{\rm BMO} \le$$

$$\le \operatorname{ess\,sup}_\tau \textstyle\sum_{k=0}^n \mathbb{E}(var_\tau^T(\sum_k^n \langle L^{(k)}, L^{(n-k)}\rangle + \beta\langle L^{\perp(k)}, L^{\perp(n-k)}\rangle)|\mathcal{F}_\tau)$$

$$\le \textstyle\sum_{k=0}^n \operatorname{ess\,sup}_\tau \mathbb{E}^{\frac12}(var_\tau^T\langle L^{(k)}\rangle|\mathcal{F}_\tau)\mathbb{E}^{\frac12}(var_\tau^T\langle L^{\perp(n-k)}\rangle|\mathcal{F}_\tau)$$

$$+|\beta| \textstyle\sum_{k=0}^n \operatorname{ess\,sup}_\tau \mathbb{E}^{\frac12}(var_\tau^T\langle L^{\perp(k)}\rangle|\mathcal{F}_\tau)\mathbb{E}^{\frac12}(var_\tau^T\langle L^{\perp(n-k)}\rangle|\mathcal{F}_\tau)$$

$$\le \textstyle\sum_k^n |L^{(k)}|_{\rm BMO}|L^{(n-k)}|_{\rm BMO} + |\beta||L^{\perp(k)}|_{\rm BMO}|L^{\perp(n-k)}|_{\rm BMO}$$

$$\le (1+|\beta|)\textstyle\sum_{k=0}^n |L^{(k)} + L^{\perp(k)}|_{\rm BMO}|L^{(n-k)} + L^{\perp(n-k)}|_{\rm BMO}.$$

Therefore, using inequalities (7) for any $k \le n$, we obtain

$|L^{(n+1)} + L^{\perp(n+1)}|_{\text{BMO}} \le (1+|\beta|)\cdot$

$\cdot \sum_{k=0}^{n} a_k (1+|\beta|)^k |L^{(0)} + L^{\perp(0)}|_{\text{BMO}}^{k+1} a_{n-k}(1+|\beta|)^{n-k} |L^{(n-k)} + L^{\perp(n-k)}|_{\text{BMO}}^{n-k+1}$

$\le (1+|\beta|)^{n+1} |L^{(0)} + L^{\perp(0)}|_{\text{BMO}}^{n+2} \sum_{k=0}^{n} a_k a_{n-k}$

$= a_{n+1}(1+|\beta|)^{n+1}|L^{(0)} + L^{\perp(0)}|_{\text{BMO}}^{n+2},$

and the validity of inequality (7) follows by induction. □

Theorem 1 *The series $\sum_{n\ge0} L^{(n)}$ and $\sum_{n\ge0} L^{\perp(n)}$ are convergent in BMO-space if γ and $|\bar\eta|_\infty$ are small enough and the triple of series*

$$\Big(\sum_{n\ge0} c^{(n)},\ \sum_{n\ge0} L^{(n)},\ \sum_{n\ge0} L^{\perp(n)}\Big)$$

determine a solution to the equation (4).

Proof. From Lemma 1 applied to $Y^{(0)}$, we obtain

$$|L^{(0)} + L^{\perp(0)}|_{\text{BMO}} \le c_0,$$

where $c_0 := \dfrac{1}{2}||\bar\eta||_\infty + \dfrac{\gamma}{2}|A|_\omega$. Using Lemma 2, we get

$$|L^{(n)} + L^{\perp(n)}|_{\text{BMO}} \le a_n(1+|\beta|)^n |L^{(0)} + L^{\perp(0)}|_{\text{BMO}}^{n+1} \le a_n(1+|\beta|)^n c_0^{n+1}.$$

By Lemma 3 of appendix, since

$$\overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{1}{2n+1} C_{n+1}^{2n+2}}$$

$$= \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{(2n)!}{n!n!}} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{(2n)^{2n}}{n^{2n}}} = 4,$$

the series is convergent, when $c_0 < \dfrac{1}{4(1+|\beta|)}$.

Since

$$\max(|L|_{\text{BMO}}, |L^\perp|_{\text{BMO}}) \le |L + L^\perp|_{\text{BMO}} \le |L|_{\text{BMO}} + |L^\perp|_{\text{BMO}},$$

the convergence $\sum_{n\ge0}(L^{(n)} + L^{\perp(n)})$ implies convergence of $\sum_{n\ge0} L^{(n)}$ and $\sum_{n\ge0} L^{\perp(n)}$ and vice versa. □

The existence of the solution for arbitrary bounded η is proven in [11]. We can prove a slightly more general result.

Proposition 1 *There exists a solution to* (3) *for* $\eta = \gamma A_T + \bar{\eta}$, *with sufficiently small* γ *and arbitrary bounded* $\bar{\eta}$.

Proof. Let $(\bar{c}, \bar{m}, \bar{m}^{\perp})$ be a solution to (3), $\bar{m} \in \mathrm{BMO(P)} \cap \mathcal{M}(P), \bar{m}^{\perp} \in \mathrm{BMO(P)} \cap \mathcal{M}^{\perp}(P)$ for γA_T and sufficiently small γ. It is well-known (see [8]) that each BMO-martingale N determines a new probability measure $\tilde{P} = \mathcal{E}_T(N) \cdot P$. Let $\mathcal{M}(\tilde{P})$ and $\mathcal{M}^{\perp}(\tilde{P})$ be images of the Girsanov transformation $n \to \langle n, N\rangle - n$ for $\mathcal{M}(P)$ and $\mathcal{M}^{\perp}(P)$, respectively. From the result of [11], there exists a solution to

$$\mathcal{E}_T(\tilde{m}) \mathcal{E}^{\alpha}_T(\tilde{m}^{\perp}) = \tilde{c} \exp\{\bar{\eta}\},$$

w.r.t $\bar{P} = \mathcal{E}_T(\bar{m} + \bar{m}^{\perp}) \cdot P$, $\tilde{m} \in \mathrm{BMO}(\bar{\mathrm{P}}) \cap \mathcal{M}(\bar{P}), \tilde{m}^{\perp} \in \mathrm{BMO}(\bar{\mathrm{P}}) \cap \mathcal{M}^{\perp}(\bar{P})$. It is easy to verify that $\tilde{m} + \langle \bar{m}, \tilde{m}\rangle \in \mathcal{M}(P)$, $\tilde{m}^{\perp} + \langle \bar{m}^{\perp}, \tilde{m}^{\perp}\rangle \in \mathcal{M}^{\perp}(P)$, and the triple

$$(c, m, m^{\perp}) = (\bar{c}\tilde{c}, \bar{m} + \tilde{m} + \langle \bar{m}, \tilde{m}\rangle, \bar{m}^{\perp} + \tilde{m}^{\perp} + \langle \bar{m}^{\perp}, \tilde{m}^{\perp}\rangle)$$

is a solution to (3) for $\eta = \bar{\eta} + \gamma A_T$. $\square$

The uniqueness of the solution was proved in [11].

Proposition 2 *Let* η *be an* F_T *-measurable random variable. If there exists a triple* $(c, m, m^{\perp})$ *with* $c \in R_+, m \in BMO \cap \mathcal{M}, m^{\perp} \in BMO \cap \mathcal{M}^{\perp}$ *satisfying equation* (3), *then such solution is unique.*

Now we show that without finiteness of $|A|_\omega$ either the solution does not exists or the convergence of series is valid in a week sense.

Example 1 Let $\alpha = -1$, $\gamma = 2$, $\bar{\eta} = 0$, $A_t = \dfrac{1}{2}\int_0^t (W_s^2 + W_s^{\perp 2}) ds$, $\mathbf{F} = (\mathcal{F}_t^{W, W^{\perp}})$, where $W, W^{\perp}$ is 2-dimensional Brownian motion. Then (4) becomes

$$L_T + L_T^{\perp} = c + \langle L\rangle_T - \langle L^{\perp}\rangle_T + \frac{1}{2}\int_0^T (W_s^2 + W_s^{\perp 2}) ds.$$

We have

$$\begin{aligned} L_T^{(0)} + L_T^{(0)\perp} &= c_0 + \int_0^T (T-s) W_s dW_s + \int_0^T (T-s) W_s^{\perp} dW_s^{\perp}, \\ L_T^{n+1)} + L_T^{(n+1)\perp} &= c_n + \sum_{k=0}^{n} \langle L^{(k)}, L^{(n-k)}\rangle_T - \sum_{k=0}^{n} \langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_T, \ n \geq 0. \end{aligned}$$

Let us assume

$$\begin{aligned} L_T^{(n)} &= \int_0^T (T-s)^{2n+1}\alpha_n W_t dW_s, \\ L_T^{(n)\perp} &= \int_0^T (T-s)^{2n+1}\beta_n W_t^\perp dW_s^\perp. \end{aligned}$$

Then $a_0 = 1$, $\beta_0 = 1$ and

$$\begin{aligned} L_T^{(n+1)} &= c_n' + \sum_{k=0}^n \int_0^T (T-s)^{2n+2}\alpha_k\alpha_{n-k}W_s^2 ds \\ L_T^{(n+1)\perp} &= c_n'' - \sum_{k=0}^n \int_0^T (T-s)^{2n+2}\beta_k\beta_{n-k}W_s^2 ds, \ n \geq 0. \end{aligned}$$

Now we will use the formula

$$h = \mathbb{E}h + \int_0^T \mathbb{E}(D_t h|F_t)dW_t + \int_0^T \mathbb{E}(D_t^\perp h|F_t)dW_t^\perp, \quad h \in H^2,$$

to get integrands of the stochastic integral representation. Taking stochastic derivatives $D_t, D_t^\perp$ and conditional expectations on both sides, we obtain

$$\begin{aligned} (T-s)^{2n+3}\alpha_n W_t &= 2\sum_{k=0}^n \alpha_k\alpha_{n-k}W_t\int_t^T (T-s)^{2n+2}ds \\ &= \frac{2}{2n+3}W_t(T-t)^{2n+3}\sum_{k=0}^n \alpha_k\alpha_{n-k}, \\ (T-s)^{2n+3}\beta_n W_t^\perp &= -\frac{2}{2n+3}W_t^\perp(T-t)^{2n+3}\sum_{k=0}^n \beta_k\beta_{n-k}, \end{aligned}$$

which means that

$$\alpha_{n+1} = \frac{2}{2n+3}\sum_{k=0}^n \alpha_k\alpha_{n-k}, \ \beta_{n+1} = -\frac{2}{2n+3}\sum_{k=0}^n \beta_k\beta_{n-k}, n \geq 0.$$

Introducing $\alpha(s) = \sum_{n=0}^\infty \alpha_n s^{2n+1}$ and $\beta(s) = \sum_{n=0}^\infty \beta_n s^{2n+1}$, one obtains

$$\begin{aligned} \alpha'(s) &= \alpha_0 + \sum_{n=0}^\infty (2n+3)\alpha_{n+1}s^{2n+2} \\ &= 1 + 2\sum_{n=0}^\infty\sum_{k=0}^n (\alpha_k\alpha_{n-k})s^{2n+2} = 1 + 2a^2(s), \\ \beta'(s) &= \beta_0 + \sum_{n=0}^\infty (2n+3)\beta_{n+1}s^{2n+2} \\ &= 1 - 2\sum_{n=0}^\infty\sum_{k=0}^n \beta_k\beta_{n-k}s^{2n+2} = 1 - 2\beta^2(s), \end{aligned}$$

i.e.,

$$\alpha'(s) = 1 + 2a^2(s), \qquad \alpha(0) = 0,$$
$$\beta'(s) = 1 - 2\beta^2(s), \qquad \beta(0) = 0.$$

Thus,

$$\alpha(s) = \frac{1}{\sqrt{2}} \tan(\sqrt{2}s), \qquad \beta(s) = -\frac{1}{\sqrt{2}} \tanh(\sqrt{2}s).$$

If $T < \frac{\pi}{2\sqrt{2}}$, series are convergent (not in BMO-space) and $(c, L, L^{\perp})$ is defined as $c = \frac{1}{2} \ln \cos(\sqrt{2}T)\cosh(\sqrt{2}T)$ (by calculations in the appendix),

$$L_t = \frac{1}{\sqrt{2}} \int_0^t \tan(\sqrt{2}s) W_s dW_s, \ L_t^{\perp} = -\frac{1}{\sqrt{2}} \int_0^t \tanh(\sqrt{2}s) W_s^{\perp} W_s^{\perp}.$$

When $T > \frac{\pi}{2\sqrt{2}}$, a local martingale L satisfying $L_T - \langle L \rangle_T = \frac{1}{2} \int_0^T W_t^2 dt$ does not exist (despite the fact that $\int_0^T W_t^2 dt$ is p-integrable for each $p \geq 1$), since from $\mathcal{E}_T(2L) = e^{\int_0^T W_t^2 dt}$, it follows that $\mathbb{E}e^{\int_0^T W_t^2 dt} = \mathbb{E}\mathcal{E}_T(2L) \leq 1$, which contradicts to $\mathbb{E}e^{\int_0^T W_t^2 dt} = \infty$ (see Appendix).

In the next example, exact solution to (4) also exists, however, it does not belong to the extreme cases considered in [12, 13].

Example 2 Let $\alpha = -1$, $\gamma = 2$, $\bar{\eta} = 0$, $A_t = \int_0^t W_s W_s^{\perp} ds$, $\mathbf{F} = (\mathcal{F}_t^{W, W^{\perp}})$, where $W, W^{\perp}$ is a two-dimensional Brownian motion. Then (4) becomes

$$L_T + L_T^{\perp} = c + \langle L \rangle_T - \langle L^{\perp} \rangle_T + \int_0^T W_s W_s^{\perp} ds. \tag{8}$$

We have

$$\begin{aligned} L_T^{(0)} &= \int_0^T (T-s) W_s^{\perp} dW_s, \ L_T^{(0),\perp} = \int_0^T (T-s) W_s dW_s^{\perp}, \\ L_T^{(n+1)} + L_T^{(n+1)\perp} &= c_n + \sum_{k=0}^{n} \langle L^{(k)}, L^{(n-k)} \rangle_T - \sum_{k=0}^{n} \langle L^{(k)\perp}, L^{(n-k)\perp} \rangle_T, \ n \geq 0. \end{aligned}$$

We assert that

$$L_T^{(n)} = \int_0^T (T-s)^{2n+1} (\alpha_n W_s + \beta_n W_s^{\perp}) dW_s,$$
$$L_T^{(n)\perp} = \int_0^T (T-s)^{2n+1} (\beta_n W_s - \alpha_n W_s^{\perp}) dW_s^{\perp},$$

where $\alpha_0 = 0$, $\beta_0 = 1$ and

$$\alpha_{n+1} = \frac{2}{2n+3}\sum_{k=0}^{n}(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k}),\ \beta_{n+1} = \frac{4}{2n+3}\sum_{k=0}^{n}\alpha_k\beta_{n-k}, n \geq 0.$$

Indeed,

$$\begin{aligned}
&L_T^{(n+1)} + L_T^{(n+1)\perp} = c_n \\
&+\sum_{k=0}^{n}\int_0^T (T-s)^{2n+2}(\alpha_k W_s + \beta_k W_s^{\perp})(\alpha_{n-k}W_s + \beta_{n-k}W_s^{\perp})ds \\
&-\sum_{k=0}^{n}\int_0^T (T-s)^{2n+2}(\beta_k W_s - \alpha_k W_s^{\perp})(\beta_{n-k}W_s - \alpha_{n-k}W_s^{\perp})ds \\
&=\sum_{k=0}^{n}\int_0^T (T-s)^{2n+2}[(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_s^2 - (\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_s^{\perp 2} \\
&+2(\alpha_k\beta_{n-k} + \beta_k\alpha_{n-k})W_s W_s^{\perp}]ds + c_n,\ n \geq 0.
\end{aligned}$$

Using representation of integrands by stochastic derivatives, we get

$$(T-t)^{2n+3}(\alpha_{n+1}W_t + \beta_{n+1}W_t^{\perp})$$

$$= \mathbb{E}[D_t(\sum_{k=0}^{n}\langle L^{(k)}, L^{(n-k)}\rangle_T - \sum_{k=0}^{n}\langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_T)|F_t]$$

$$= 2\sum_{k=0}^{n}[(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_t + (\alpha_k\beta_{n-k} + \beta_k\alpha_{n-k})W_t^{\perp}]\int_0^T (T-s)^{2n+2}ds$$

$$= \frac{2(T-t)^{2n+3}}{2n+3}\sum_{k=0}^{n}[(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_t + (\alpha_k\beta_{n-k} + \beta_k\alpha_{n-k})W_t^{\perp}],$$

$$(T-t)^{2n+3}(\beta_{n+1}W_t - \alpha_{n+1}W_t^{\perp})$$

$$= \mathbb{E}[D_t^{\perp}(\sum_{k=0}^{n}\langle L^{(k)}, L^{(n-k)}\rangle_T - \sum_{k=0}^{n}\langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_T)|F_t]$$

$$= 2\sum_{k=0}^{n}[-(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_t^{\perp} + (\alpha_k\beta_{n-k} + \beta_k\alpha_{n-k})W_t]\int_0^T (T-s)^{2n+2}ds$$

$$= \frac{2(T-t)^{2n+3}}{2n+3}\sum_{k=0}^{n}[-(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_t^{\perp} + (\alpha_k\beta_{n-k} + \beta_k\alpha_{n-k})W_t].$$

Equating the coefficients at $W, W^{\perp}$, we obtain the desired formula. One can also check that (see Appendix)

$$\limsup_{n\to\infty} \sqrt[n]{|\alpha_n|} = \frac{2\sqrt{2}}{\pi}, \quad \limsup_{n\to\infty} \sqrt[n]{|\beta_n|} = \frac{2\sqrt{2}}{\pi}.$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{E}^{\frac{1}{2}} |L_T^{(n)} + L_T^{(n)\perp}|^2 = \sum_{n=0}^{\infty} \left(4(\alpha_n^2 + \beta_n^2) \int_0^T (T-s)^{4n+2} s ds \right)^{\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{\alpha_n^2 + \beta_n^2}}{\sqrt{(n+1)(4n+3)}} T^{2n+2},$$

which means that series

$$\sum_{n=0}^{\infty} (L^{(n)} + L^{(n)\perp})$$

is convergent as quadratic integrable martingales, when $T < d\dfrac{\pi}{2\sqrt{2}}$.

Introducing $\alpha(s) = \sum_{n=0}^{\infty} \alpha_n$ and s^{2n+1}, $\beta(s) = \sum_{n=0}^{\infty} \beta_n s^{2n+1}$, one obtains

$$\begin{aligned} L_t &= \int_0^t (\alpha(T-s)W_s + \beta(T-s)W_s^{\perp}) dW_s, \\ L_t^{\perp} &= \int_0^t (\beta(T-s)W_s - \alpha(T-s)W_s^{\perp}) dW_s^{\perp}. \end{aligned} \tag{9}$$

On the other hand, we can derive ODE for the pair (α, β):

$$\begin{aligned} \dot{\alpha}(s) &= 2\alpha^2(s) - 2\beta^2(s), \ \alpha(0) = 0, \\ \dot{\beta}(s) &= 1 + 4\alpha(s)\beta(s), \ \beta(0) = 0. \end{aligned} \tag{10}$$

Indeed,

$$\dot{\alpha}(s) = \alpha_0 + \sum_{n=0}^{\infty} (2n+3)\alpha_{n+1} s^{2n+2}$$

$$= 2\sum_{n=0}^{\infty} \sum_{k=0}^{n} (\alpha_k \alpha_{n-k} - \beta_k \beta_{n-k}) s^{2n+2} = 2\alpha^2(s) - 2\beta^2(s),$$

$$\dot{\beta}(s) = \beta_0 + \sum_{n=0}^{\infty} (2n+3)\beta_{n+1} s^{2n+2}$$

$$= 1 + 4\sum_{n=0}^{\infty} \sum_{k=0}^{n} \alpha_k \beta_{n-k} s^{2n+2} = 1 + 4\alpha(s)\beta(s).$$

Equation (10) is easy to solve, if we pass to the equation for complex-variable function $z(s) = \alpha(s) + i\beta(s)$:

$$\dot{z}(s) = i + 2z^2(s),\ z(0) = 0.$$

It is obvious that $z(s) = \dfrac{1}{1-i}\tan((1+i)s)$ is a solution. We have

$$\begin{aligned} z(s) &= \frac{1}{2}(1+i)\frac{\sin((1+i)s)\cos((1-i)s)}{|\cos((1+i)s)|^2} \\ &= \frac{1}{4}(1+i)\frac{\sin(2s) + i\sinh(2s)}{|\cos((1+i)s)|^2} \\ &= \frac{1}{4}\frac{\sin(2s) - \sinh(2s) + i(\sin(2s) + \sinh(2s))}{\cos^2(s)\cosh^2(s) + \sin^2(s)\sinh^2(s)}. \end{aligned}$$

Hence, we can write the explicit solution

$$\begin{aligned} \alpha(s) &= \frac{1}{4}\frac{\sin(2s) - \sinh(2s)}{\cos^2(s)\cosh^2(s) + \sin^2(s)\sinh^2(s)}, \\ \beta(s) &= \frac{1}{4}\frac{\sin(2s) + \sinh(2s)}{\cos^2(s)\cosh^2(s) + \sin^2(s)\sinh^2(s)} \end{aligned}$$

of (10) and conclude that it exists on whole $[0, \infty)$, since the denominator does not vanish. Despite of convergence of series only for $T < \dfrac{\pi}{2\sqrt{2}}$, the pair (9) is a solution to (8) for each T.

Finally, we consider exponential equation of the form (2) with $\lambda_t = \sqrt{2}(W_t - W_t^{\perp})$ defining the variance-optimal martingale measure

$$\mathcal{E}_T(-\int_0^{\cdot} \lambda_s dW_s + m^{\perp})$$

for the price process $X_t = X_0 + \int_0^t \lambda_s ds + W_t$. Inserting $L = m/2$ and $L^{\perp} = -m^{\perp}/2$, one obtains

$$L_T + L_T^{\perp} = c + \langle L\rangle_T - \langle L^{\perp}\rangle_T + \int_0^T (W_s - W_s^{\perp})^2 ds.$$

As in previous example, the solution can be found in the form

$$\begin{aligned} L_t &= \int_0^t (\alpha(T-s)W_s + \beta(T-s)W_s^{\perp})dW_s, \\ L_t^{\perp} &= \int_0^t (\beta(T-s)W_s - \alpha(T-s)W_s^{\perp})dW_s^{\perp}, \end{aligned}$$

where pair (α, β) satisfy ODE

$$\begin{aligned} \dot{\alpha} &= 2(\alpha^2 - \beta^2 + 1), \qquad \alpha(0) = 0, \\ \dot{\beta} &= 4\alpha\beta - 2, \qquad \beta(0) = 0. \end{aligned}$$

For $z = \alpha + i\beta$, we get ODE

$$\dot{z} = 2z^2 + 2 - 2i = 2z^2 + 2^{\frac{3}{2}} e^{-i\frac{\pi}{4}}$$

with solution

$$z(t) = 2^{\frac{1}{4}} e^{-i\frac{\pi}{8}} \tan(2^{\frac{5}{4}} e^{-i\frac{\pi}{8}} t).$$

Therefore,

$$\alpha(t) = Re(2^{\frac{1}{4}} e^{-i\frac{\pi}{8}} \tan(2^{\frac{5}{4}} e^{-i\frac{\pi}{8}} t)), \qquad \beta(t) = Im(2^{\frac{1}{4}} e^{-i\frac{\pi}{8}} \tan(2^{\frac{5}{4}} e^{-i\frac{\pi}{8}} t)),$$

and

$$\mathcal{E}_T \left(-\sqrt{2} \int_0^{\cdot} (W_s - W_s^{\perp}) dW_s + 2 \int_0^{\cdot} (\beta(T-s) W_s - \alpha(T-s) W_s^{\perp}) dW_s^{\perp} \right)$$

is the variance-optimal martingale measure.

A Appendix

The formula $\mathbb{E} e^{-T^2 \int_0^1 W_t^2 dt} = 1/\sqrt{\cosh(\sqrt{2}T)}$ is derived in [10]. Similarly, we can prove

Proposition 3 *One has*

$$\mathbb{E} e^{\int_0^T W_t^2 dt} = \begin{cases} \dfrac{1}{\sqrt{\cos(\sqrt{2}T)}}, & \text{if } T < \dfrac{\pi}{2\sqrt{2}}, \\ \infty, & \text{if } T \geq \dfrac{\pi}{2\sqrt{2}}. \end{cases}$$

Proof. Let $e_n(t)$ be orthonormal basis in $L^2[0,1]$. Then

$$\mathbb{E} e^{\int_0^T W_t^2 dt} = \mathbb{E} e^{T^2 \int_0^1 W_t^2 dt} = \mathbb{E} e^{T^2 \sum_{n=1}^{\infty} (\int_0^1 e_n(t) W_t dt)^2}$$

Since

$$\mathbb{E}(\int_0^1 e_n(t) W_t dt)(\int_0^1 e_m(t) W_t dt) = \int_0^T e_n(t) \int_0^T (t \wedge s) e_m(s) ds dt,$$

it is convenient to use the orthonormal basis of eigenvectors of the operator $\int_0^T (t \wedge s) f(s) ds$ in $L^2[0,1]$. From $\lambda f(t) = \int_0^T (t \wedge s) f(s) ds$ follows that

$\lambda f''(t) = -f(t)$, $f(0) = 0$, $f'(1) = 0$. Function $\sin \mu\pi t$ satisfies these conditions iff $\mu^2 = 1/\lambda$ and $\mu = -1/2 + n$. Thus,

$$\lambda_n = \frac{1}{(n-1/2)^2\pi^2}, \qquad e_n(t) = \sqrt{2}\sin((n-1/2)\pi t), \quad n \geq 1,$$

and

$$E(\int_0^1 e_n(t)W_t dt)(\int_0^1 e_m(t)W_t dt) = \lambda_n \int_0^1 e_n(t)e_m(t)dt = 0, \; n \neq m.$$

Since random variables $(\int_0^1 e_n(t)W_t dt)$ are orthogonal and normal, they are also independent. Hence, taking into account the Parseval identity and infinite product decomposition of $\cos(\sqrt{2}t)$, one gets

$$\mathbb{E}e^{T^2\lambda_n W_1^2} = \prod_{n=1}^{\infty} \frac{1}{\sqrt{1-\frac{2T^2}{(n-1/2)^2\pi^2}}} \sqrt{\prod_{n=1}^{\infty} \frac{1}{1-\frac{8T^2}{(2n-1)^2\pi^2}}} = \frac{1}{\sqrt{\cos(\sqrt{2}T)}},$$

if $\sqrt{2}T < \pi/2$.

One can easily check that

$$\mathbb{E}\exp\left(\int_0^{\frac{\pi}{2\sqrt{2}}} W_t^2 dt\right) = \lim_{T\uparrow\frac{\pi}{2\sqrt{2}}} \mathbb{E}\exp\left(\int_0^T W_t^2 dt\right) = \lim_{T\uparrow\frac{\pi}{2\sqrt{2}}} \frac{1}{\sqrt{\cos(\sqrt{2}T)}} = \infty.$$

If $T > \dfrac{\pi}{2\sqrt{2}}$, then $\mathbb{E}e^{\int_0^T W_t^2 dt} > \mathbb{E}e^{\int_0^{\frac{\pi}{2\sqrt{2}}} W_t^2 dt} = \infty$. $\square$

Lemma 3 *Let $(a_n)_{n\geq 0}$ be a solution to the system of recurrent equations*

$$a_0 = 1, \qquad a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}. \tag{11}$$

Then $a_n = \dfrac{1}{4n+2}\binom{2n+2}{n+1}$.

Proof. For the series $u(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ from (11), we get equation $u(\lambda) = 1 + \lambda u^2(\lambda)$ with the roots $u(\lambda) = \dfrac{1}{2\lambda}(1 \pm \sqrt{1-4\lambda})$. The equality $u(\lambda) = \dfrac{1}{2\lambda}(1+\sqrt{1-4\lambda})$ is impossible since the decomposition of the right hand side is starting from the term $\dfrac{1}{\lambda}$. Therefore, equality $a_n = \dfrac{1}{4n+2}\binom{2n+2}{n+1}$ follows from the Taylor expansion of $1-\sqrt{1-4\lambda}$, since

$$\begin{aligned} u(\lambda) &= \frac{1}{2\lambda}(1-\sqrt{1-4\lambda}) = -\frac{1}{2}\sum_{n\geq 1}\frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4)^n\lambda^{n-1} \\ &= \frac{1}{2}\sum_{n\geq 1}\frac{(2-1)\cdots(2n-2-1)}{2^n n!}4^n\lambda^{n-1} = \frac{1}{2}\sum_{n\geq 1}\frac{(2n-3)!!}{n!}2^n\lambda^{n-1} \\ &= \frac{1}{2}\sum_{n\geq 1}\frac{1}{2n-1}\binom{2n}{n}\lambda^{n-1}. \end{aligned}$$

□

Lemma 4 *Let* $(\alpha_n, \beta_n)_{n\geq 0}$ *be a solution of the system*

$$\begin{aligned} \alpha_0 &= 0, \qquad \beta_0 = 1, \\ \alpha_{n+1} &= \frac{2}{2n+3}\sum_{k=0}^{n}(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k}), \\ \beta_{n+1} &= \frac{4}{2n+3}\sum_{k=0}^{n}\alpha_k\beta_{n-k}, n\geq 0. \end{aligned}$$

Then

$$\begin{aligned} \alpha_{n-1} = 2^n\frac{2^{2n}-1}{\pi^{2n}}\zeta(2n)\cos(\frac{\pi}{2}n), \\ \beta_{n-1} = 2^n\frac{2^{2n}-1}{\pi^{2n}}\zeta(2n)\sin(\frac{\pi}{2}n), \qquad n\geq 1, \end{aligned} \tag{12}$$

where $\zeta(s)$ *is the zeta function.*

Proof. The tangent function admits the Taylor series expansion:

$$\tan x = \sum_{n=1}^{\infty} 2^{2n+1}(2^{2n}-1)\zeta(2n)\frac{x^{2n-1}}{(2\pi)^{2n}}.$$

It is clear that

$$1 \leq \zeta(n) = 1 + \sum_{k=1}^{\infty}\frac{1}{(k+1)^n} \leq 1 + \int_1^{\infty}\frac{1}{x^n}dx = 1 + \frac{1}{n-1} \overset{n\to\infty}{\to} 1.$$

For the series

$$\tan((1+i)x) = \sum_{n=1}^{\infty} 2^{2n+1}(2^{2n}-1)\zeta(2n)\frac{2^{n-1/2}e^{i\frac{\pi}{4}(2n-1)}x^{2n-1}}{(2\pi)^{2n}}$$

radius of convergence is $\frac{\pi}{2\sqrt{2}}$ since

$$\sqrt[2n]{2^{2n+1}(2^{2n}-1)\zeta(2n)\frac{2^{n-1/2}}{(2\pi)^{2n}}} \to \frac{2\sqrt{2}}{\pi}.$$

From

$$\frac{1}{1-i}\tan((1+i)s) = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\sum_{n=1}^{\infty}2^{2n+1}(2^{2n}-1)\zeta(2n)\frac{2^{n-1/2}e^{i\frac{\pi}{4}(2n-1)}s^{2n-1}}{(2\pi)^{2n}}$$
$$= \sum_{n=1}^{\infty}2^{3n}(2^{2n}-1)\zeta(2n)\frac{e^{i\frac{\pi}{2}n}}{(2\pi)^{2n}}s^{2n-1}$$
$$= \sum_{n=0}^{\infty}(2^{2n+2}-1)\frac{2^{n+1}}{\pi^{2n+2}}\zeta(2n+2)e^{i\frac{\pi}{2}(n+1)}s^{2n+1},$$

equations (12) follows. □

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Revaz Tevzadze
Georgian American University,
8 Aleksidze Str., Tbilisi 0193, Georgia;
Georgian Technical University,
77 Kostava str., Tbilisi 0175, Georgia;
Institute of Cybernetics,
5 Euli str., 0186, Tbilisi, Georgia.
rtevzadze@gmail.com

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