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The Newton Polyhedron, Spaces of Differentiable Functions and General Theory of Differential Equations

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Abstract. In the paper we investigate the role of the Newton polyhedron $\Re$, which generates a multianisotropic Sobolev space $W_p^{\Re}$ and Gevrey space $G^{\Re}$, and the role of the Newton polyhedron $\Re(P)$ of a polynomial $P(\xi)$ (of a linear differential operator $P(D)$) in the behavior of $P(\xi)$ at infinity and in the smoothness of solutions of the equation $P(D)u = f$. The paper is partly of an overview nature. However, some of the results are new and not published anywhere (see, for instance, theorems 2.4, 2.5 and 4.2). Some results are proved in a new way (see, for instance, theorems 3.1, 4.3 and others).

Key Words: Newton polyhedron, non-degenerate operator, hypoelliptic operator, almost hypoelliptic operator, multi-anisotropic Sobolev and Gevrey spaces.
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1 Introduction

We use the following standard notations: $\mathbb{N}$ is the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_0^n = \mathbb{N}_0 \times ... \times \mathbb{N}_0$ is the set of all n-dimensional multi-indices, $\mathbb{E}^n$ and $\mathbb{R}^n$ are the n-dimensional Euclidean spaces of points (vectors) $x = (x_1, ..., x_n)$ and $\xi = (\xi_1, ..., \xi_n)$ respectively, $\mathbb{R}^{n,+} = \{\xi : \xi \in \mathbb{R}^n, \xi_j \geq 0 \ (j = 1, ..., n)\}$, $\mathbb{R}^{n,0} = \{\xi : \xi \in \mathbb{R}^n, \xi_1 ... \xi_n \neq 0\}$.

For $\xi \in \mathbb{R}^n, x \in \mathbb{E}^n$ and $\alpha \in \mathbb{R}^{n,0}$ we put $|\xi| = \sqrt{\xi_1^2 + ... + \xi_n^2}$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$ and for $\alpha \in \mathbb{N}_0^n$ we put $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$, where $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$ $(j = 1, ..., n)$.

Let $\mathcal{A} = \{\alpha^j = (\alpha_1^j, ..., \alpha_n^j)\}_1^M$ be a finite set of points in $\mathbb{R}^{n,+}$. By the **Newton polyhedron** of the set $\mathcal{A}$ we mean the minimal convex hull (which is a polyhedron) $\Re = \Re(A)$ in $\mathbb{R}^{n,+}$ containing all points of $\mathcal{A}$.

A polyhedron $\Re$ with vertices in $R^{n,+}$ is said to be **complete** (see [45], or [24]), if $\Re$ has a vertex at the origin and one vertex (distinct from the origin) on each coordinate axis of $\mathbb{R}^{n,+}$. The k-dimensional faces of a polyhedron $\Re$ are denoted by $\Re_i^k$ $(i = 1, ..., M_k', k = 0, 1, ..., n-1)$. The set of 0-dimensional faces (vertices) of $\Re$ we denote by $\Re^0$.

In the sequel, an outward (with respect to $\Re$) normal to a supporting hyperplane of a complete polyhedron $\Re$ containing some face $\Re_i^k$ and not containing any other face of dimension greater than k will be called simply an **outward normal** to the face $\Re_i^k$. Thus, a given vector λ can serve as an outward normal to one and only one face of a convex complete polyhedron $\Re$.

The face $\Re_i^k$ $(1 \le i \le M_k', 0 \le k \le n-1)$ of a polyhedron $\Re$ is said to be **principal** (see [45]) if among the outward normals of this face there is one with at least one positive component. If among the outward normals of the principal face $\Re_i^k$ there is one whose components are all nonnegative (positive), then the face $\Re_i^k$ is said to be **regular** (**completely regular**). A complete polyhedron $\Re$ is said to be **regular** (**completely regular**), if all its non-coordinate $(n-1)$-dimensional faces are regular (completely regular) (see [29], [3] or [11] and [51]).

Let $\Re$ be a complete polyhedron with vertices in $\mathbb{N}_0^n$, $\Re^0$ be the set of its vertices, Ω be a domain in $\mathbb{E}^n$, and $1 < p < \infty$. Denote by $W_p^{\Re}(\Omega)$ (respectively $W_p^{\Re^0}(\Omega)$) the set of functions u with bounded norms (see [29] or [3], paragraph 13)

$$||u||_{W_p^{\Re}(\Omega)} = \sum_{\alpha \in \Re} ||D^\alpha u||_{L_p(\Omega)} \tag{1.1}$$

respectively

$$||u||_{W_p^{\Re^0}(\Omega)} = \sum_{\alpha \in \Re^0} ||D^\alpha u||_{L_p(\Omega)}. \tag{1.2}$$

For the collections $\mathcal{A}_1 = \{(0, ..., 0) \cup [\alpha : \alpha \in \mathbb{N}_0^n, |\alpha| \le m]\}$ and $\mathcal{A}_2 = \{(0, ..., 0) \cup [\alpha : \alpha \in N_0^n, |\alpha| = m]\}$ the sets $\Re^0(\mathcal{A}_1)$ and $\Re^0(\mathcal{A}_2)$ coincide, where the sets $W_p^{\Re(\mathcal{A}_1)}(\Omega)$ (respectively $W_p^{\Re(\mathcal{A}_2)}(\Omega)$) are coinciding with the Sobolev space $W_p^m(\Omega)$ (respectively $\tilde{W}_p^m(\Omega)$) with the norm

$$||u||_{W_p^m} = \sum_{|\alpha| \le m} ||D^\alpha u||_{L_p(\Omega)} \;\; (||u||_{\tilde{W}_p^m} = \sum_{|\alpha| = m} ||D^\alpha u||_{L_p(\Omega)} + ||u||_{L_p(\Omega)}).$$

Therefore the sets $W_p^{\Re}(\Omega)$ with the suitable norms we will call multianisotropic Sobolev spaces.

Sobolev spaces play an outstanding role in the modern Analysis. In particular, many fields of mathematics are interested in weighted Sobolev spaces, and first of all they arise in various issues of the theory of partial differential equations. A lot of monographs and papers have already been devoted to this concepts. We mention only some of such works which are closely related to the present paper.

In [7] it is proved the density of finite functions in the Sobolev space $W_p^l(\Omega)$ for any open set Ω. In [8] it is proved the density of the infinitely differentiable functions in the same class of functions.

In [5] O.V. Besov proved the density of the infinitely differentiable finite functions in a weighted Sobolev space. In book [38] A. Kufner deals with properties of weighted Sobolev spaces, the weight function being dependent on the distance of a point of the domain from the boundary of the domain or from its part. In the book [44, section 17] V.Maz'ya discusses possibility of approximation of functions from weighted Sobolev spaces by smooth functions. The paper [17] discusses the density of polynomials in Sobolev-type function spaces. The problems considered are motivated by consideration of the spectral representation of certain Jacobi-type orthogonal polynomials. In [56] V. Zhikov considered the case when smooth functions are not dense in a weighted Sobolev space. In [37] duality and complex interpolation are investigated for weighted Sobolev spaces. In [12] necessary and sufficient conditions for approximation by test functions in a type of weighted Sobolev spaces are given for a weight μ which is a nontrivial positive Radon measure.

All of these works are devoted to isotropic or anisotropic (weighted) Sobolev spaces, i.e. the spaces which are generated by a homogeneous or nonhomogeneous vector $m = (m_1, ..., m_n)$. Their Newton polyhedrons are $(n + 1)$-simplices (geometrically, for example in the case $n = 2$, they are right triangles with a vertex in origin, isosceles or not). Here we consider a general case, when the Sobolev space generates a Newton polyhedron of any kind.

It turned out that for arbitrary collections $\mathcal{A}$ (polyhedrons $\Re$) the character of multianisotropic Sobolev spaces can be essentially different from usual (isotropic or anisotropic) Sobolev spaces. Therefore a natural problem arose to find some conditions on a polyhedron $\Re$ of a set $\mathcal{A}$ (and on a domain Ω) under which

1) the norms (1.1) and (1.2) are equivalent, i.e. the spaces $W_p^{\Re}(\Omega)$ and $W_p^{\Re^0}(\Omega)$ coincide

2) the set of infinitely differentiable functions with compact supports in Ω is dense in the multianisotropic Sobolev space $W_p^{\Re}(\Omega)$.

3) the set $W_p^{\Re}(\Omega)$ is a semilocal space. Recall that a Banach space $B(\Omega)$ is called semilocal if $u \in B(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ leads $\varphi u \in B(\Omega)$ (see, for instance, [28, Definition 10.1.18]).

We will discuss these problems in 2 of the present paper. It turns out that there is a very direct connection between the geometric (regularity) properties of a Newton polyhedron $\Re$ and the answers to these questions. We note in this connection that for a Sobolev space $W_p^m(\Omega)$ with a large set of domains Ω all these questions have there answers (see, for instance, [3] or [9]). In particular, the spaces $W_p^m(\Omega)$ and $\tilde{W}_p^m(\Omega)$ are isometrically isomorphic.

Let $P(D) = P(D_1, ..., D_n) = \sum_\alpha \gamma_\alpha D^\alpha$ be a linear differential operator with constant coefficients and let $P(\xi) = P(\xi_1, ..., \xi_n) = \sum_\alpha \gamma_\alpha \xi^\alpha$ be its characteristic polynomial (the complete symbol). Here the sum is taken over a finite set of multi-indices $(P) = \{\alpha \in \mathbb{N}_0^n; \gamma_\alpha \neq 0\}$.

The Newton polyhedron of the set $(P) \cup \{0\}$ is called **the Newton or characteristic polyhedron of the operator** $P(D)$ **(the polynomial** $P(\xi)$**)** (see [48], [45] or [24]) and is denoted by $\Re(P)$.

Newton polyhedron generalizes the notion of the degree of a polynomial in n variables and the notion of degree of partial differential equations. There are great many applications of Newton polyhedron's concept to different fields of mathematics (see, for instance, [35], [36] [54], [16], [46] [49] and others) but in this work we will only deal with multianisotropic Sobolev spaces generated by some Newton polyhedron, the behavior at infinity of polynomials, and the regularity properties of solutions of linear partial differential equation.

An operator $P(D)$ (a polynomial $P(\xi)$) is called **hypoelliptic** (see [28], Definition 11.1.2 and Theorem 11.1.1) if the following equivalent conditions are satisfied:

a) if $u \in D'(\Omega)$ (Ω is an open set in $\mathbb{E}^n$, $D'(\Omega)$ is the set of distributions defined in Ω) is a solution of the equation $P(D)u = 0$ then $u \in C^\infty(\Omega)$,

b) all solutions $u \in D' = D'(\mathbb{E}^n)$ of the equation $P(D)u = f$ are infinitely differentiable (belong to $C^\infty = C^\infty(\mathbb{E}^n)$) for all $f \in C^\infty$.

c) if $|\xi| \to \infty$, and $0 \neq \alpha \in \mathbb{N}_0^n$ then

$$P^{(\alpha)}(\xi)/P(\xi) \equiv D^\alpha P(\xi)/P(\xi) \to 0.$$

An operator $P(D)$ is called **partally hypoelliptic** with respect to the hyperplane $x'' := (x_2, ..., x_n) = 0$ (see [28, Definition 11.1.4]) when $P^{(\alpha)}(\xi)/P(\xi) \to 0$ if $0 \neq \alpha \in N_0^n$ and $|\xi''| \to \infty$ while $\xi' := \xi_1$ remain bounded.

A polynomial $P(\xi)$ is called **almost hypoelliptic** (see [30]) if for a constant $C > 0$

$$|P^{(\alpha)}(\xi)|/[1 + |P(\xi)|\,] \leq C \ \ \forall \xi \in R^n, \ \forall \alpha \in \mathbb{N}_0^n.$$

In [19] the following statement was proved: let f and its derivatives be square integrable on E^n with a certain exponential weight, then all solutions of the equation $P(D)u = f$, which are square integrable with the same weight, are also such that all their derivatives are square integrable with this weight, if and only if the operator $P(D)$ is almost hypoelliptic.

It turnes out that there is a strong connection between the (almost) hypoelliptisity of operator $P(D)$ (with the simbol $P(\xi)$) and the behavior at infinity of the polynomial P. Denote by $\mathbb{I}_n$ the set of polynomials $P(\xi) = P(\xi_1, ..., \xi_n)$ such that $|P(\xi)| \to \infty$, as $|\xi| \to \infty$. It is easy to verify that $P \in \mathbb{I}_n$ when $P(D)$ is elliptic or hypoelliptic.

In 3 we present some necessary conditions and sufficient conditions on a polyhedron $\Re(P)$ of a polynomial P under which $P \in \mathbb{I}_n$.

In 4 we present conditions on a polyhedron $\Re(P)$ differential operator $P(D)$ with the symbol $P \in \mathbb{I}_n$ under which operator $P(D)$ is hypoelliptic or almost hypoelliptic.

Since partialy hypoelliptic and almost hypoelliptic differential equations have solutions which are not infinitely differentiable, a natural problem arose of finding additional assumptions on solutions of those equations ensuring there infinitely differentiablity, or what is the same, a problem of selection of infinitely differentiable solutions of those equations from the set of there distributional solutions. In 4 we discuss this problem too.

We note that this is not first attempt to reveal the role of the Newton polyhedron in the theory of partial differential equations. In addition to the papers [35] - [36] of Khovanskii, which are devoted to applications of the Newton polyhedron to algebraic and geometric problems, it is worth noting the monograph [24] of Gindikin and Volevich. In this monograph the method of the Newton polyhedron is applied to various problems of mathematical physics. However, the issues that are discussed in this paper are not adressed there.

As already noted in the annotation, work is partally of an overview nature. However, a number of results, such as theorems 2.4, 2.5 and 4.2, are new and nowhere published. At the same time, the proofs of several results, such as theorems 3.1 and 4.3, are carried out here in a new way. We hope that these proofs are more compact and easily perceived.

2 Some properties of multianisotropic Sobolev and Gevrey spaces

2.1 Multianisotropic Sobolev spaces without weights

Definition 2.1. (See, for example, [43] or [3, section 11]) A measurable function $\Phi(\xi)$ is called L_p-multiplicator ($\phi \in M_p^p$), if the transformation $T_\Phi : L_p \to L_p$ defined by the equality

$$T_\Phi f = \frac{1}{(2\,\pi)^{n/2}} \int_{E^n} \Phi(\xi)\, F[f](\xi)\, e^{i(x,\xi)} d\xi \equiv F^{-1}[\Phi\, F[f]]$$

is bounded for all functions $f \in C_0^\infty$, i.e. there exists a constant $C > 0$ such that $||T_\Phi f||_p \le C\,||f||_p$ for all $f \in C_0^\infty$.

We mention two statements on which we shall rely in the sequel.

Theorem L (P.I. Lizorkin, see, for example, [43] or [3, section 11]) *Let* $\Phi \in C^{(n)}(R^{n,0})$. *Then* $\phi \in M_p^p$ *if there exists a number* $M > 0$ *such thet* $|\xi_1^{k_1} ... \xi_n^{k_n} D^k \phi(\xi)| \le M$ *for all* $\xi \in R^{n,0}$, *where* $k = (k_1, ..., k_n)$, k_j $(j = 1, ..., n)$ *takes only values 0 and 1.*

Theorem M (V.P.Mikhailov, see [45]) *For any set* $\mathcal{A}$ *of* $e^1, ..., e^{N_0} \in R^{n,0}$ *with the Newton polyhedron* $\Re = \Re(A)$ *there exists a constant* $C > 0$ *such thet for all* $\xi \in R^n$

$$\sum_{\alpha \in \Re} |\xi^\alpha| \le C \sum_{i=1}^{N_0} |\xi^{e^i}|.$$

Lemma 2.1 *Let* $1 < p < \infty$ *and* $\Re = \Re(\mathcal{A})$ *be the Newton polyhedron of a collection of multi - indices* $\mathcal{A} = \{ e^1, ..., e^{N_0}\}$. *Then there exists a constant* $C > 0$ *such that for all* $u \in C_0^\infty$

$$\sum_{\nu \in \Re} ||D^\nu u||_{L_p} \le C \sum_{i=1}^{N_0} ||D^{e^i} u||_{L_p}. \tag{2.1}$$

Proof. Perform the Fourier transformation for functions $u \in C_0^\infty$. By applying Theorem M and Parseval's equality we obtain inequality (2.1) for $p = 2$. To prove inequality (2.1) for $p \neq 2$ note that by properties of Fourier transformation we have

$$F[D^\nu u] = \xi^\nu F[u]; \;\; F[D^{e^j} u] = \xi^{e^j} F[u] \;\; (j = 1, ..., N_0).$$

A simple computation gives

$$F[D^\nu u] = \sum_{j=1}^{N_0} \phi_j(\xi)\, F[D^{e^j} u],$$

where

$$\phi_j(\xi) = \frac{\xi^{\nu + e^j}}{\sum_{k=1}^{N_0} \xi^{2\, e^j}} \equiv \frac{\xi^{\nu + e^j}}{Q(\xi)} \;\; (j = 1, ..., N_0).$$

To prove inequality (2.1) for any $p \in (1, \infty)$ it is sufficient to show that $\phi_j \in M_p^p$ $(j = 1, ..., N_0)$. For this purpose we apply Theorem L.

The boundedness of $\{\phi_j\}$ follows immediately from Theorem M. Let us show the boundedness of (for example) $\{\xi_1 \frac{\partial \phi_j}{\partial \xi_1}\}$.

Again, a simple computation gives for each $j = 1, ..., N_0$

$$\xi_1 \frac{\partial \phi_j}{\partial \xi_1} = \phi_j(\xi)\,[(e_1^j + \nu_1) - 2\sum_{k=1}^{N_0} e_1^k \frac{\xi^{2\,e^k}}{Q(\xi)}].$$

Since $|\xi^{2\,e^k}/Q(\xi)| \leq 1$ $(k = 1,...,N_0)$ for all $\xi \in \mathbb{R}^n$ this implies for $\{\xi_1 \frac{\partial \phi_j}{\partial \xi_1}\}$. In the same way one can prove the boundeness of the other derivatives. Lemma 2.1 is proved. □

Remark 2.1 It follows from results of V.P. Il'in (see [29] or [3], Theorem 13.3.2$'$).

Theorem (V.P. Il'in) *Let* $1 < p < \infty$, *the domain* Ω *satisfy the rectangle condition (see [2, section 13.1]) and let the Newton polyhedron* $\Re(\mathcal{A})$ *of a collection* $\mathcal{A} = \{\,e^1,...,e^{N_0}\}$ *be completely regular. Then there exists a constant* $C > 0$ *such that for all* $u \in W_p^{\Re}(\Omega)$

$$||u||_{W_p^{\Re}(\Omega)} \equiv \sum_{\nu\in\Re(\mathcal{A})} ||D^\nu u||_{L_p(\Omega)} \leq C\sum_{i=1}^{N_0} ||D^{e^i} u||_{L_p(\Omega)}. \tag{2.2}$$

Since the reverse inequality is obvious, this implies that for completely regular polyhedron and the domain Ω satisfying the rectangle condition, the norms (1.1) and (1.2) are equivalent, i.e. the spaces $W_p^{\Re}(\Omega)$ and $W_p^{\Re^0}(\Omega)$ coincide.

V.P. Il'in proved also that for a collection $\mathcal{A}$ with nonregular Newton polyhedron $\Re(\mathcal{A})$ (even, for example, complete) the estimate (2.2) **can not be valid.**

Lemma 2.2 *Let the Newton polyhedron* $\Re = \Re(\mathcal{A})$ *of a collection of multiindices* $\mathcal{A} = \{\,e^1,...,e^{N_0}\}$ *be regular. The set of infinitely differentiable finite (in* $\mathbb{E}^n$*) functions is dense in* $W_p^{\Re} = W_p^{\Re}(\mathbb{E}^n)$ *if and only if the inequality (2.1) is valid for all functions* $u \in W_p^{\Re}$.

Proof. Sufficiency. Inequality (2.1) be valid and let $\omega(t) \in C_0^\infty$ be a function of one variable such that $\omega(t) = 0$ outside of $(0,1)$, $\int_0^1 \omega(t)dt = 1$.

Let a function $u \in W_p^{\Re}$ be fixed and put

$$u_h(x) = \frac{1}{h^n}\int \prod_{i=1}^n \omega(\frac{y_i}{h})\, u(x+y)\, dy.$$

Then it is easily seen that (see, for example [3]) 1) $u_h \in C^\infty$, 2) $||u - u_h||_{W_p^{\Re}} \to 0$ as $h \to 0$. The set of infinitely differentiable functions is dense in $W_p^{\Re}$ and it remains to proof that every infinitely differentiable function $u \in W_p^{\Re}$ one can approximate in $W_p^{\Re}$ by $C_0^\infty-$functions.

Let for any $k \in \mathbb{N}$ $\chi_k(x) \in C_0^\infty$, $0 \leq \chi_k(x) \leq 1$ for all $x \in \mathbb{E}^n$, $\chi_k(x) = 1$ for $|x| \leq k$, $\chi_k(x) = 0$ for $|x| > k+1$, $|D^\alpha \chi_k(x)| \leq M$, where the constant $M > 0$ does not depend on $\alpha \in \mathbb{N}_0^n$ and k. Denote $\varphi_k(x) = \chi_k(x)u(x)$. Since $u \in C^\infty$ it is clear that $\varphi_k \in C_0^\infty$. On the other hand it follows by the Leibnitz' formula and the regularity of the polyhedron $\Re$ that $\varphi_k \in W_p^{\Re}$. Then we get for a number $C > 0$

$$\sum_{j=1}^{N_0} ||D^{e^j} u - D^{e^j} \varphi_k||_{L_p} = \sum_{j=1}^{N_0} ||D^{e^j} [u - \varphi_k]||_{L_p} =$$

$$\sum_{j=1}^{N_0} D^{e^j} [u\,(1-\chi_k)]||_{L_p}(|x| > k) \leq \sup_{x,k,\alpha\in\Re} |D^\alpha[1-\chi_k(x)] \sum_{\nu\in\Re} ||D^\nu u||_{L_p}(|x| > k)$$

$$\leq C \sum_{\nu\in\Re} ||D^\nu u||_{L_p}(|x| > k).$$

Since $u \in W_p^{\Re}$, hence $D^\nu u \in L_p$ and, consequently $||D^\nu u||_{L_p}(|x| > k) \to 0$ as $k \to \infty$, i.e. $||u - \varphi_k||_{W_p^{\Re}} = \sum_{j=1}^{N_0} ||D^{e^j} u - D^{e^j} \varphi_k||_{L_p} \to 0$ as $k \to \infty$. Sufficiency is proved.

Necessity. By Lemma 2.1 the inequality (2.1) is valid for functions from C_0^∞. If the set C_0^∞ is dense in $W_p^{\Re}$ then the inequality (2.1) is valid for all functions $u \in W_p^{\Re}$. Lemma 2.2 is proved. □

Combining Lemmas 2.1 and 2.2 we obtain

Theorem 2.1 *Let the Newton polyhedron $\Re$ of a set $e^1, ..., e^{N_0}$ be regular. Then the set C_0^∞ is dense in $W_p^{\Re}$.* □

In this section we define the notion of Newton polyhedron for the set of differential operators (polynomials) too. Let

$$Q_j(D) = \sum_{\alpha\in(Q_j)} q_\alpha^j D^\alpha \ \ (j = 1, ..., M)$$

be a set of linear differential operators with constant coefficients. The Newton polyhedron of the set $\bigcup(Q_j) \cup \{0\}$ is called the Newton polyhedron of the set of operators $\{Q_j(D)\}$ (the polynomials $\{Q_j(\xi)\}$) and is denoted by $\Re(\{Q_j\})$.

For given $\Omega \subset E^n$ and $1 < p < \infty$ we denote by $w_p^{\Re(\{Q_j\})}(\Omega)$ the set of functions $\{u\}$ with the bounded norm

$$||u||_{w_p^{\Re(\{Q_j\})}(\Omega)} = \sum_{j=1}^{M} ||Q_j(D)u||_{L_p(\Omega)} + ||u||_{L_p(\Omega)}.$$

For differential operators with homogeneous principal parts of the same degree K.T. Smith in [52] and J. Necas in [47] proved that under a certain assumptions on the domain Ω the following estimate is fulfilled with some constant $C > 0$

$$\sum_{\nu\in w_p^{\Re(\{Q_j\})}(\Omega)} ||D^\nu u||_{L_p(\Omega)} \leq C\,||u||_{w_p^{\Re(\{Q_j\})}(\Omega)} \quad \forall u\in w_p^{\Re(\{Q_j\})}(\Omega)$$

if homogeneous principal parts of polynomials (symbols) $Q_j(\xi)$ have not any common complex zero except of $\xi=(0,..,0)$.

O.V.Besov in [4] obtained a similar result for operators with generalized homogeneous principal parts of the same degree and in [34] a similar inequality was proved for more general operators with a completely regular Newton's polyhedron $\bigcup \Re(Q_j)\cdot$ □

2.2 **Multianisotropic Gevrey spaces**

The Gevrey classes $G^s(\Omega)$ (see [18], [28, Definition 11.4.1], or [51, Def. 1.4.1]) are intermediate spaces between the spaces $C^\infty(\Omega)$ and $A(\Omega)$ and play the important role in the theory of linear partial differential equations.

The question of finding the largest functional class in which the Cauchy problem is well-posed is an important example of the role of Gevrey functions as intermediate classes between the analytic and C^∞ classes. In fact, the Cauchy - Kowalevsky Theorem ensures the well posedness in the frame of analytic functions for any operator with analytic coefficients. However, the hypothesis of weak hyperbolicity is necessary whenever we require a common domain of existence of the solution not depending on the initial data. On the other hand, whereas strict hyperbolicity implies well posedness in C^∞, the Cauchy problem is not well posed in general for weakly hyperbolic operators, which can be seen immediately for the heat operator in R^2. In the case of weakly hyperbolic operators with constant coefficients, necessary and sufficient conditions of well - posedness in C^∞ were given by [41], [25], [40], [53], in the case of $s-$hyperbolic operators in [39], [14] and others.

Let Ω be an open subset of E^n and let $s\geq 1$ be a fixed real number. Denote by $G^s(\Omega)$ the set of functions $f\in C^\infty(\Omega)$ such that for every compact subset $K\subset\Omega$ there exists a constant $C>0$ such that for all multi - indices $\alpha\in N_0^n$ and $x\in K$

$$|D^\alpha f(x)|\leq C^{|\alpha|+1}(\alpha!)^s.$$

In particular $G^1(\Omega)$ is the space $A(\Omega)$ of all analytic functions.

Let $\Re$ be a completely regular polyhedron, $\partial\Re$ be the exterior boundary of $\Re$, i.e the set of points $x\in\Re$ for which outward (with respect to $\Re$) normal $\lambda(x)$ has only positive components and (see, for example [11])

$$\mu=\max_{x\in\partial\Re,\,1\leq j\leq n}\frac{1}{\lambda_j(x)};\;\; k(\alpha,\Re)=inf\{t>0:\alpha\in t^{-1}\Re\}.$$

Following Corli [15] and Zanghirati [55] we define a multianisotropic Gevrey space $G^{s,\Re}(\Omega)$ as the set of all $f \in C^\infty(\Omega)$ such that for every compact $K \subset \Omega$ there exists a constant $C > 0$ such that for all $\alpha \in N_0^n$ and $x \in K$

$$|D^\alpha f(x)| \le C^{|\alpha|+1}(\mu\, k(\alpha, \Re))^{s\,\mu\, k(\alpha,\Re)}.$$

These spaces include as particular cases the standard Gevrey classes (sf. [18] and [51] for a comprehensive exposition) and the anisotropic Gevrey classes (cf. [24], [50], [2], [55]).

The standard Gevrey classes (as well as the standard Sobolev classes) can be seen as an example of the multianisotropic Gevrey (Sobolev) spaces associated to the Newton polyhedron of an elliptic operator having vertices $(0, ..., 0)$, $(m, 0, ..., 0)$, ..., $(0, ..., 0, m)$.

Let us consider partial differential operators with constant coefficients in $\mathbb{E}^{n+1} = \mathbb{E}_t \times \mathbb{E}_x^n$ non characteristic with respect to the $t-$hyperplane, i.e. operators that can be written in the form:

$$P(D) := P(D_t, D_x) = D_t^m + \sum_{|\nu|+j\le m,\, j\ne m} a_{\nu\, j}\, D_x^\nu D_t^j.$$

We say that $P(D)$ is $s-$hyperbolic (with respect to the variable t, $1 < s < \infty$) if for some constant $C > 0$ its symbol satisfies the condition:

$$\lambda^m + \sum_{|\nu|+j\le m,\, j\ne m} a_{\nu\, j}\, \xi^\nu \lambda^j \ne 0$$

for any $(\lambda, \xi) \in \mathbb{C}_t \times E_x^n$, $\mathrm{Im}\,\lambda < -C\,(1 + |\xi|^{\frac{1}{s}})$.

In the case $\mathrm{Im}\,\lambda < -C$ we say that $P(D)$ is hyperbolic (by Petrovskii - Gȯrding).

Following Daniela Calvo we say that operator $P(D)$ is multiquasihyperbolic of order s $(1 < s < \infty)$ with respect to a completely regular polyhedron $\Re$ if $P(\lambda, \xi) \ne 0$ for any $(\lambda, \xi) \in C_t \times \mathbb{E}_x^n$, $\mathrm{Im}\,\lambda < -C\,|\xi|_{\Re}^{\frac{1}{s}}$, where $|\xi|_{\Re}$ is weight associated to $\Re$ as follows

$$|\xi|_{\Re} = (\sum_{\alpha\in\Re^0} \xi^{2\alpha})^{\frac{1}{2\mu}}.$$

In [11] it is proved the following result of well posedness of the Cauchy problem stating the multianisotropic Gevrey regularity with respect to the space variables (see [11, Theorem 8])

Theorem *Let $P(D)$ be a differential operator in $\mathbb{E}_t \times \mathbb{E}_x^n$, multiquasi-hyperbolic of order s $(1 < s < \infty)$ with respect to a completely regular polyhedron $\Re$. Let $1 < r < s$, and assume $f_k \in G_0^{r,\Re}(E_x^n) \equiv G^{r,\Re}(\mathbb{E}_x^n) \cap C_0^\infty(\mathbb{E}_x^n)$ for $k = 0, 1, ..., m-1$. Then the Cauchy problem:*

$$P(D)u = 0;\ \ D_t^k u(0,x) = f_k(x)\ \ \forall x \in \mathbb{E}^n,\ \ \forall k = 0,1,...,m-1$$

admits a uniqe solution $u \in C^\infty([-T,T], G^{r,\Re})$ *for any* $T > 0$.

This theorem generalizes analogous results for $s-$hyperbolic operators, earlier obtained by Larsson, Cattabriga and others (see, for instans, [14], [39]).

In monograph [24] S. Gindikin and L. Volevich introduced a notion of dominantly correct operator, gave an algebraic description of dominantly correct polynomials and proved the correctness of Cauchy's problem for such operators with variable coefficients. They presented sufficient conditions for correctness of Cauchy's problem for general differential operators with variable coefficients of constant strength as well.

In connection with the study of Fourier L_p- multiplicators V.I. Burenkov and M.Sh. Tuyakbaev introduced in [10] a Gevrey type class $J_{\gamma,\,p}(\mathbb{E}^n)$ which elements are defined by the norms of L_p- multiplicators. □

2.3 Weighted multianisotropic Sobolev spaces

In this point we consider three kind of weighted multianisotropic Sobolev spaces connected with the boundedness of a domain Ω, i.e. when A) $\Omega = \mathbb{E}^n$, B) $\Omega \neq \mathbb{E}^n$, but Ω is unbounded, C) Ω is bounded. In the cases A) - C) a particular polyhedron and a weight correspond to each Ω.

A) Let $\alpha \in \mathbb{N}_0^n$ be an arbitrary multi-index and $g \in C^\infty := C^\infty(\mathbb{E}^n)$ be any positive function such that a) for some positive constants κ and κ_α

$$\kappa^{-1} e^{-\delta\,|x|} \leq g_\delta(x) \leq \kappa\, e^{-\delta\,|x|};\ \ |D^\alpha g_\delta(x)| \leq \kappa_\alpha\, \delta^{|\alpha|}\, g_\delta(x)\ \ \forall x \in \mathbb{R}^n, \qquad (2.3)$$

where $g_\delta(x) = g(\delta\, x)$ for any $\delta > 0$.

b) Let $T > 0$, $S_T := \{x \in R^n : |x| < T\}$ and $G \subset S_T$. Then there exist positive numbers σ_1 and σ_2 such that for any $\delta > 0$ and $x \in \mathbb{R}^n$

$$\sup_{y \in G} g_\delta(x+y) \leq \sigma_1\, g_\delta(x),\ \ \sup_{y \in G} |g_\delta(x+y) - g_\delta(x)| \leq \sigma_2\, T\, g_\delta(x). \qquad (2.4)$$

In [19] it is proved the existence of such a function. Note that the regularization (averaging) of the function $H(x) = e^{-|x|}$ for $|x| > 1$ and $H(x) = e^{-1}$ for $|x| \leq 1$ can be taken as a function g (see, for instance [3, section 5]).

Let $1 < p < \infty$ and $\delta > 0$. Denote by $L_{p,\,\delta} := L_{p,\,g\,\delta}(\mathbb{E}^n)$ the set of functions locally integrable in $\mathbb{E}^n$ with a bounded norm

$$||u||_{L_{p,\,\delta}} := ||u\,g_\delta||_{L_p} = [\int_{\mathbb{E}^n} |u(x)|^p\, g_\delta^p(x)\,dx\,]^{\frac{1}{p}} \tag{2.5}$$

and for any regular polyhedron $\Re$ with vertices in $\mathbb{N}_0^n$ by $W^{\Re}_{p,\,\delta}$ the set of functions $u \in L_{p,\,\delta}$ with a bounded norm

$$||u||_{W^{\Re}_{p,\,\delta}} := \sum_{\alpha\in\,\Re} ||(D^\alpha u)g_\delta||_{L_p} = \sum_{\alpha\in\,\Re} ||D^\alpha u||_{L_{p,\,\delta}}. \tag{2.6}$$

By the dimensional reasons one can proof (for a nonweighted Sobolev spaces see [3, section 9.4] or [29, Theorem 2])

Lemma 2.3 *Let $\Re$ be a completely regular polyhedron, and $\nu \in N_0^n$ be any interior point of $\Re$. Then, for any $\varepsilon > 0$ there exists a number $C(\varepsilon) > 0$ such that*

$$||D^\nu u||_{L_{p,\,\delta}} \le \varepsilon ||u||_{W^{\Re}_{p,\,\delta}} + C(\varepsilon)||u||_{L_{p,\,\delta}} \;\; \forall u \in W^{\Re}_{p,\,\delta}. \tag{2.7}$$

□

Lemma 2.4 *Let $\Re$ be a completely regular polyhedron. Then, one can introduce in $W^{\Re}_{p,\,\delta}$ a norm*

$$||u||'_{W^{\Re}_{p,\,\delta}} := \sum_{\alpha\in\,\Re} ||D^\alpha(ug_\delta)||_{L_p} \tag{2.6$'$}$$

which is equivalent to the norm (2.6).

Proof. By the Leibnitz' formula

$$\sum_{\alpha\in\Re} D^\alpha(ug_\delta) = \sum_{\alpha\in\Re}[D^\alpha u]g_\delta + \sum_{\alpha\in\Re}\sum_{|\beta|=1}^{|\alpha|} C_{\alpha,\beta} D^{\alpha-\beta}uD^\beta g_\delta. \tag{2.8}$$

Applying property (2.4) of function g_δ we obtain then

$$||u||'_{W^{\Re}_{p,\,\delta}} \le C_1 ||u||_{W^{\Re}_{p,\,\delta}} \;\; \forall u \in W^{\Re}_{p,\,\delta} \tag{2.9}$$

with a positive constant $C_1 = C_1(\delta)$. To prove the reverse inequality write Leibnitz' formula (2.8) in the form

$$\sum_{\alpha\in\Re}[D^\alpha u]g_\delta = \sum_{\alpha\in\Re} D^\alpha(ug_\delta) - \sum_{\alpha\in\Re}\sum_{|\beta|=1}^{|\alpha|} C_{\alpha,\beta} D^{\alpha-\beta}uD^\beta g_\delta. \tag{2.8$'$}$$

Since $|\beta| > 0$ and polyhedron $\Re$ is completely regular, all multiindices $\alpha - \beta$ in the right hand side of (2.8') are interior points of $\Re$. Then for any $\varepsilon > 0$ we can use the inequality (2.7) for the second sum in the right hand side of (2.8'), i.e. independent on ε there exist some positive constants C_2 and C_3, such that for all $u \in W^{\Re}_{p,\,\delta}$

$$||\sum_{\alpha\in\Re}\sum_{|\beta|=1}^{|\alpha|} C_{\alpha,\beta}D^{\alpha-\beta}uD^{\beta}g_{\delta}||_{L_{p,\,\delta}} \le \varepsilon C_2||u||_{W^{\Re}_{p,\,\delta}} + C(\varepsilon)C_3||u||_{L_{p,\,\delta}},$$

which together with (2.8') this implies the following inequality

$$||u||_{W^{\Re}_{p,\,\delta}} \le ||u||'_{W^{\Re}_{p,\,\delta}} + \varepsilon C_2||u||_{W^{\Re}_{p,\,\delta}} + C(\varepsilon)C_3||u||_{L_{p,\,\delta}}. \tag{2.10}$$

Choose the number $\varepsilon > 0$ such that $1-\varepsilon C_2 > 0$, move the second term from the right hand side of the relation (2.10) to the left hand side and divide both parts of the received inequality by $1-\varepsilon C_2 > 0$. We get with some positive constants C_4 and C_5

$$||u||_{W^{\Re}_{p,\,\delta}} \le C_4||u||'_{W^{\Re}_{p,\,\delta}} + C_5||u||_{L_{p,\,\delta}} \;\; \forall u \in W^{\Re}_{p,\,\delta}.$$

Together with inequality (2.9) this proves the lemma. □

Remark 2.2 Let a polyhedron $\Re$ be not completely regular. Then for some multi-index $\alpha \in \Re$ and some $0 \neq \beta \in \mathbb{N}_0^n$ the multi-index $\alpha-\beta$ is not an interior point of $\Re$. Then we can not apply Lemma 2.3 in the proof of Lemma 2.4. In [19] an analogue of Lemma 2.4 is proved for the regular polyhedron $\Re$ when $p = 2$. □

Theorem 2.2 *Let $\Re$ be any completely regular polyhedron. The set $C_0^\infty = C_0^\infty(E^n)$ is dense in $W^{\Re}_{p,\delta}$.*

Proof. Let $u \in W^{\Re}_{p,\delta}$, $S_1 := \{x \in E^n : |x| < 1\}$, $\varphi \in C_0^\infty(S_1)$, $\varphi(x) \ge 0$, $\int \varphi(x)dx = 1$, $\varepsilon > 0$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Now we put

$$u_\varepsilon(x) := u * \varphi_\varepsilon = \int u(x-y)\varphi_\varepsilon(y)dy = \varepsilon^{-n}\int u(x-y)\varphi(y/\varepsilon)dy.$$

It is well known (see, for instance, [3, section 5]) that $u_\varepsilon \in C_0^\infty$ and $||u-u_\varepsilon||_{L_p} \to 0$ as $\varepsilon \to 0$. To complete the proof of the theorem we shall prove that

$$||u-u_\varepsilon||_{W^{\Re}_{p,\,\delta}} \to 0 \;\; as \;\; \varepsilon \to 0. \tag{2.11}$$

Since $D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$ we have

$$||u-u_\varepsilon||_{W^{\Re}_{p,\,\delta}} = \sum_{\alpha\in\Re}||D^\alpha(u-u_\varepsilon)||_{L_{p,\,\delta}} = \sum_{\alpha\in\Re}||[D^\alpha u-(D^\alpha u)_\varepsilon]\,g_\delta||_{L_p} \le$$

$$\le \sum_{\alpha\in\Re}||(D^\alpha u)\,g_\delta-((D^\alpha u)\,g_\delta)_\varepsilon||_{L_p} + \sum_{\alpha\in\Re}||((D^\alpha u)\,g_\delta)_\varepsilon-(D^\alpha u)_\varepsilon\,g_\delta||_{L_p}. \tag{2.12}$$

Since $(D^\alpha u)\, g_\delta \in L_p$ for $u \in L_p$, and $\alpha \in \Re$, and a function in L_p is mean continuous (see, for instance, [3]), we get

$$\sum_{\alpha\in\Re} ||(D^\alpha u)\, g_\delta - ((D^\alpha u)\, g_\delta)_\varepsilon||_{L_p} \to 0 \;\; as \;\; \varepsilon \to +0. \tag{2.13}$$

The proof is completed by showing that

$$A_\varepsilon := \sum_{\alpha\in\Re} ||((D^\alpha u)\, g_\delta)_\varepsilon - (D^\alpha u)_\varepsilon\, g_\delta||_{L_p} \to 0 \;\; as \;\; \varepsilon \to +0. \tag{2.14}$$

Since $\varphi_\varepsilon \in C_0^\infty(S_\varepsilon)$ for any $\varepsilon > 0$ hence

$$A_\varepsilon = \sum_{\alpha\in\Re} || \int (D^\alpha u)(x-y)[g_\delta(x-y) - g_\delta(x)]\varphi_\varepsilon(y)dy||_{L_p}.$$

In view of the inequality (2.4) it follows when $T = \varepsilon$

$$A_\varepsilon \le \sigma_2 \varepsilon \sum_{\alpha\in\Re} || \int (D^\alpha u)(x-y)\, g_\delta(x-y)\, \varphi_\varepsilon(y)dy||_{L_p}.$$

Applying here Young's inequality, we get

$$A_\varepsilon \le \sigma_2 \varepsilon \sum_{\alpha\in\Re} ||\, (D^\alpha u)\, g_\delta||_{L_p}\, ||\, \varphi_\varepsilon||_{L_1}.$$

Since $u \in W^{\Re}_{p,\,\delta}$ and $||\,\varphi_\varepsilon||_{L_1} = 1$ for any $\varepsilon > 0$, it follows that $A_\varepsilon \to 0$ as $\varepsilon \to 0$, i.e the relation (2.14) is proved. Besides since (2.13) and (2.14), our theorem is proved. □

B) In the present subsection even numbers m and m_2 $(m > m_2)$ are fixed and we denote by $\Re = \Re(m, m_2) \subset R^{n,+}$ the Newton polyhedron with the vertices $(0, ..., 0)$, $(m, 0, ..., 0)$, ..., $(0, ..., 0, m)$ and $(m, m_2, ..., 0)$. It is easy to verify that $\Re$ is a regular (but not completely regular) polyhedron.

Let us introduce some weight functions and weighted multianisotropic Sobolev spaces connected with the polyhedron $\Re = \Re(m, m_2)$ and the domain $\Omega_\kappa := \{x = (x_1, x'') = (x_1, x_2, ..., x_n) \in \mathbb{E}^n;\ |x_1| < \kappa\}$ for a given $\kappa > 0$. Namely, as a weight function we consider a function $g \in C^\infty(-1, 1)$ of one variable $t \in \mathbb{E}^1$ such that

1) $0 \le g(t) \le 1$, $g(-t) = g(t)$ for $t \in \mathbb{E}^1$ and $g(t) = 0$ for $|t| \ge 1$.

2) Let $\kappa > 0$ and $g_\kappa(t) = g(t/\kappa)$. Then

$$g_\kappa^{(l)}(t) := D^l[g_\kappa(t)] = \kappa^{-1}(D^l g)_\kappa(t)$$

for $t \in (-\kappa, \kappa)$ and for all $l = 0, 1, ...$ It is easy to verify that such a function is the following one (for any $k \in \mathbb{N}$) $g(t) = 1/(2k)!(1 - t^{2k})$ for $t \in (-1, 1)$ and $g(t) = 0$ for $|t| \geq 1$.

Let $\Re'$ be the set of multi indices $\alpha \in \Re$ such that $(\alpha_1, \alpha'') \in \Re$, $(\alpha_1 + 1, \alpha'') \notin \Re$.

We introduce an integer valued function $d(\alpha)$ with the domain $\Re \cap \mathbb{N}_0^n$, which satisfies the following conditions:

1) $d(\alpha_1 \pm l) = d(\alpha) \pm l$ for any $l \in \mathbb{N}$ such that $\alpha_1 - l \in \mathbb{N}_0$,

2) $d(\alpha) < m$ for $\alpha \in \Re \setminus \Re'$ and 3) $d(\alpha) = m$ for $\alpha \in \Re'$.

Such a function for the polyhedron $\Re(m, m_2)$ were constructed in [22].

Note that for a polyhedron $\Re$, which corresponds to an isotropic Sobolev space W_p^m (for example, when $m_2 = 0$ in the polyhedron $\Re(m, m_2)$) the set $\Re'$ coincides with the set $\{\alpha \in \Re, |\alpha| = m\}$, and $d(\alpha) = |\alpha|$ for any $\alpha \in \Re$.

Let $1 < p < \infty$, $\kappa > 0$, polyhedron $\Re = \Re(m, m_2)$, domain Ω_κ and functions g and d be defined as just above. Denote by $W_{p,\,g}^{\Re,\,d}(\Omega_\kappa)$ the set of functions u locally integrable on Ω_κ with a finite norm

$$||u||_{W_{p,\,g}^{\Re,\,d}(\Omega_\kappa)} = \sum_{\alpha \in \Re} ||D^\alpha u\, g_\kappa^{d(\alpha)}||_{L_2(\Omega_\kappa)}. \tag{2.15}$$

Theorem 2.3 *The set* $C_0^\infty(\Omega_\kappa)$ *is dense in* $W_{p,\,g}^{\Re,\,d}(\Omega_\kappa)$.

Proof. Let a function $u \in W_{p,\,g}^{\Re,\,d}(\Omega_\kappa)$ be fixed. By the definition of the improper Lebesgue integral, for each $\varepsilon > 0$ there exist numbers $\delta \in (0, \kappa)$ and $M \geq 1$ such that

$$||u||_{W_{p,\,g}^{\Re,\,d}(\Omega_\kappa \setminus \Omega_{\kappa-\delta}^M)} < \varepsilon, \tag{2.16}$$

where $\Omega_{\kappa-\delta}^M = \{x \in E^n, |x_1| < \kappa - \delta, |x_j| < M, j = 2, ..., n\}$.

For the fixed κ, δ and M we construct nonnegative functions $\varphi_\delta \in C_0^\infty(\mathbb{E}^1)$ of one variable $x_1 \in \mathbb{E}^1$, and $\psi \in C_0^\infty(\mathbb{E}^{n-1})$ of variables $x'' = (x_2, ..., x_n) \in \mathbb{E}^{n-1}$ such that

1) $\varphi_\delta(x_1) = 1$ for $|x_1| < \kappa - \delta$ and $\varphi_\delta(x_1) = 0$ for $|x_1| > \kappa - \delta/2$,

2) $\psi(x'') = 1$ for $|x_j| < M$ and $\psi(x'') = 0$ for $|x_j| \geq M + 1$ $(j = 2, ..., n)$,

3) for a number $b \geq 1$ and for all $x = (x_1, x'') \in \mathbb{E}^n$

$$\varphi_\delta^{(j)} \leq b\, \delta^{-j} \ (j = 0, 1, ..., m); \ |D^{\alpha''} \psi(x'')| \leq b \ (|\,\alpha''| \leq m).$$

The existence of such function ψ is obvious. To construct the function φ_δ we denote by χ_A the characteristic function of the set $\mathcal{A} = \mathcal{A}(\kappa, \delta) = \{|x_1| \leq \kappa - \frac{3}{4}\,\delta\}$ and put for $\omega:\ 0 \leq \omega \in C_0^\infty(-1, 1)$, $\int \omega(x)dx = 1$, $\omega_\varepsilon(x) = \varepsilon^{-1}\omega(\frac{x}{\varepsilon})$

$$\varphi_\delta(x_1) = (\chi_{\mathcal{A}} * \omega_{\delta/4})(x_1) = \int\limits_{\mathbb{E}^1} \chi_{\mathcal{A}}(x_1 - t)\omega_{\delta/4}(t)dt =$$

$$= \int_{-\infty}^{\infty} \chi_{\mathcal{A}}(z)\, \omega_{\delta/4}(x_1 - z) dz. \qquad (2.17)$$

It is obvious that $\varphi_\delta \in C_0^\infty(\mathbb{E}^1)$. Firstly we show that φ_δ satisfies the condition 1). Let $|x_1| \leq \kappa - \delta$. Since $|t| \leq \delta/4$ and $|x_1 - t| \leq |x_1| + |t| \leq \kappa - \delta + \delta/4 = \kappa - \frac{3}{4}\,\delta$ hence $\chi_A(x_1 - t) = 1$, and from (2.17) we have for $|x_1| \leq \kappa - \delta$

$$\varphi_\delta(x_1) = \int_{-\delta/4}^{\delta/4} \omega_{\delta/4}(t) dt = \int_{-\delta/4}^{\delta/4} (\delta/4)^{-1}\, \omega(\frac{t}{\delta/4})\, dt = 1.$$

Let now $|x_1| \geq \kappa - \delta$. Then $|x_1 - t| \geq |x_1| - t > \kappa - \delta/2 - \delta/4 = \kappa - \kappa - \frac{3}{4}\,\delta$, therefore $\chi_A(x_1 - t) = 0$ and it follows from (2.17) that $\varphi_\delta(x_1) = 0$. Thus, condition 1) is proved.

Let us prove the property 3) of function φ_δ. From (2.17) and from the definition of the function χ_A we have

$$\varphi_\delta(x_1) = \int_{-(\kappa - \frac{3}{4}\,\delta)}^{\kappa - \frac{3}{4}\,\delta} \omega_{\delta/4}(x_1 - z)\, dz = \frac{4}{\delta} \int_{-(\kappa - \frac{3}{4}\,\delta)}^{\kappa - \frac{3}{4}\,\delta} \omega(\frac{x_1 - z}{\delta/4})\, dz.$$

Therefore

$$\varphi_\delta^{(j)}(x_1) = \frac{4}{\delta} \int_{-(\kappa - \frac{3}{4}\,\delta)}^{\kappa - \frac{3}{4}\,\delta} D_{x_1}^j \omega(\frac{x_1 - z}{\delta/4})\, dz =$$

$$= (\frac{4}{\delta})^{j+1} \int_{-(\kappa - \frac{3}{4}\,\delta)}^{\kappa - \frac{3}{4}\,\delta} (D_{x_1}^j \omega)(\frac{x_1 - z}{\delta/4})\, dz = (\frac{4}{\delta})^j \int_{-(\kappa - \frac{3}{4}\,\delta)}^{\kappa - \frac{3}{4}\,\delta} \omega^{(j)}(t)\, dt.$$

Then

$$|\varphi_\delta^{(j)}(x_1)| \leq (\frac{4}{\delta})^j \int_{-\infty}^{\infty} |\,\omega^{(j)}(t)|\, dt \equiv C_j \delta^{-j} \;\; (j = 0, 1, ..., m).$$

Denoting by b the maximum of the numbers $\{C_j\}$, we get the property 3) of the function φ_δ. After the construction of functions φ_δ and ψ we put $v(x) = u(x)\varphi_\delta(x_1)\psi(x'')$. Then $supp\, v \subset \Omega^M_{\kappa - \delta/2}$.

It is assumed henceforth that for all $\alpha \in \Re$ the functions $D^\alpha u$ are continued by zero outside of Ω_κ. We denote by $D^\alpha u$ the continued functions too.

Since $v(x) = u(x)$ for $x \in \Omega^M_{\kappa-\delta/2}$ and $D^\alpha u \in L_p$ for $\alpha \in \Re$, we obtain by (2.16)

$$\sum_{\alpha\in\Re} ||(D^\alpha v - D^\alpha u)\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n)} = \sum_{\alpha\in\Re} ||(D^\alpha v - D^\alpha u)\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n\setminus\Omega^M_{\kappa-\delta/2})} \leq$$

$$\leq \sum_{\alpha\in\Re}[||D^\alpha v\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n\setminus\Omega^M_{\kappa-\delta/2})} + ||D^\alpha u\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n\setminus\Omega^M_{\kappa-\delta/2})} \leq$$

$$\leq \sum_{\alpha\in\Re} ||D^\alpha[u(x)\,\varphi_\delta^{(j)}(x_1)\,\psi(x'')]\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n\setminus\Omega^M_{\kappa-\delta/2})} + \varepsilon. \tag{2.18}$$

Since $g_\kappa(x_1) \leq (2\delta)/\kappa$ for $x \in \mathrm{supp}(\varphi_\delta\,\psi) \cap (\Omega_\kappa \setminus \Omega^M_{\kappa-\delta/2})$ and $g_\kappa^{d(\alpha)} \leq g_\kappa^{d(\beta)}$ for $\beta \leq \alpha$, applying the Leibnitz' formula and properties 1) - 3) of the functions φ_δ and ψ, we obtain for the first part in the right - hand side of (2.18) with a constant $A_1 = A_1(\kappa) > 0$

$$\sum_{\alpha\in\Re} ||D^\alpha[u(x)\,\varphi_\delta^{(j)}(x_1)\,\psi(x'')]\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n\setminus\Omega^M_{\kappa-\delta/2})} \leq$$

$$\leq \sum_{\alpha\in\Re}\sum_{\beta\leq\alpha} C_\alpha^\beta ||D^\beta u(D^{\alpha_1-\beta_1}\varphi_\delta)(D_2^{\alpha_2-\beta_2}...D_n^{\alpha_n-\beta_n}\psi)\, g_\kappa^{d(\alpha)}||_{L_p(E^n\setminus\Omega^M_{\kappa-\delta/2})} \leq$$

$$\leq \sum_{\alpha\in\Re}\sum_{\beta\leq\alpha} C_\alpha^\beta\, b^{|\alpha-\beta|}\, \delta^{-(\alpha_1-\beta_1)}\, (\frac{\delta}{\kappa})^{\alpha_1-\beta_1}\, ||D^\beta u\, g_\kappa^{d(\alpha)}||_{L_p(E^n\setminus\Omega^M_{\kappa-\delta/2})} \leq$$

$$\leq A_1 \sum_{\alpha\in\Re} ||D^\beta u\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n\setminus\Omega^M_{\kappa-\delta/2})} \leq A_1\,\varepsilon.$$

From here and (2.18) we get

$$\sum_{\alpha\in\Re} ||(D^\alpha v - D^\alpha u)\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n)} \leq (A_1 + 1)\,\varepsilon. \tag{2.19}$$

Let $h > 0,\ S_h = \{x \in R^n, |x| < h\},\ \theta \in C_0^\infty(S),\ \theta(x) \geq 0,\ \int \theta(x)dx = 1,$ $\theta_h(x) = h^{-2}\,\theta(x/h)$ and $v_h = v * \theta_h$.

It is easy to see that $v_h \in C^\infty(\mathbb{E}^n)$ for $h > 0$, where $v_h(x) = 0$ for $x \neq \mathrm{supp}\ v \cup \bar{S}_h$. On the other hand since $\mathrm{supp}\, v \cup \bar{S}_h \subset \Omega_\kappa$ for $h \in (0, \delta/4)$ we have $v_h \in C_0^\infty(\Omega_\kappa)$ for $h \in (0, \delta/4)$.

Since $g_\kappa(x_1) \leq 1$ and $u \in W^{\Re,\, d}_{p,\, g}(\Omega_\kappa)$, we obtain $D^\alpha v \in L_p(\mathbb{E}^n)$ for all $\alpha \in \Re$, where (see, for instance, [3] 6.3. (2)) $D^\alpha(v_h) = (D^\alpha v)_h$. Then by Yung's inequality and by the continuity in the mean of functions from L_p we get

$$\sum_{\alpha\in\Re} ||D^\alpha (v_h - v)\, g_\kappa^{d(\alpha)}||_{L_p(\mathbb{E}^n)} \leq \sum_{\alpha\in\Re} D^\alpha (v_h - v)\, ||_{L_p(\mathbb{E}^n)} =$$

$$= \sum_{\alpha\in\Re} ||(D^\alpha v)_h - D^\alpha v\, ||_{L_p(\mathbb{E}^n)} \leq \sum_{\alpha\in\Re} \sup_{|x|<h} ||D^\alpha v(x-y) - D\alpha v(x)\, ||_{L_p(\mathbb{E}^n)} \to 0$$

as $h \to 0$.

Since $\varepsilon > 0$ is arbitrary, from (2.16) - (2.19) we complete the proof of the theorem. □

2.4 **Semilocallity of weighted multianisotropic Sobolev spaces**

It is easy to verify that classical Sobolev spaces (isotropic or anisotropic) are semilocal. It turnes out that for the multianisotropic Sobolev space $W_p^\Re := W_p^\Re(\mathbb{E}^n)$ generated by a regular polyhedron $\Re$, weighted spaces $W^\Re_{p,\,\delta} := W^\Re_{p,\,\delta}(\mathbb{E}^n)$ and $W^{\Re,\, d}_{p,\, g}(\Omega_\kappa)$ considered in the section 2 are also semilocal. Indeed, by applying Leibnitz' formula one can easily prove

Theorem 2.4 *Let $\Re$ be a regular polyhedron forming spaces $W_p^\Re$ and $W^\Re_{p,\,\delta}$, and let $\Re = \Re(m, m_2)$ (see point 2.2.B) be the polyhedron forming the space $W^{\Re,\, d}_{p,\, g}(\Omega_\kappa)$. Then these spaces are semilocal, i.e. if u belongs to one of those spaces and $\varphi \in C_0^\infty(\mathbb{E}^n)$ ($\varphi \in C_0^\infty(\Omega_\kappa)$ in the last case) then $\varphi\, u$ also belongs to the corresponding space.* □

C) In this point we consider a two dimensional multianisotropic weighted Sobolev space with bounded domain $\Omega \subset \mathbb{E}^2$. We prove the semilocality in a limited sense of such space.

At this point, the natural numbers l and m ($l < m$) and, as a consequence, the polyhedron $\Re = \Re(l, m) = \{v \in \mathbb{N}_0^2, v_1 \leq l,\ v_1 + v_2 \leq m\}$ will be fixed. The polyhedron $\Re$ is a regular quadrangle in $\mathbb{R}^{2,+}$ with vertices $(0, 0)$, $(l, 0)$, $(l, m-l)$, and $(0, m)$. Note that the polyhedron $\Re$ is not completely regular since the one dimensional side $[(l, 0) - (l, m-l)]$ of quadrangle $\Re$ is perpendicular to semi axis $(0, \alpha_1)$, or what is the same, the second coordinate of outward normal of this side is zero.

Let a and b be positive numbers and $\prod = \prod(a, b) := \{(x, y) \in \mathbb{E}^2,$ $x \in (-a, a),\ y \in (-b, b)\}$. It is easy to see that a linear transformation of coordinates does not change the type of Newton polyhedron. Therefore by linear transformation $x = x,\ y = b\,y_1$ one can pass from the rectangle $\prod(a,b)$ to the rectangle $\prod = \prod(a) := \prod(a,1)$.

Finally, as a weighted function we consider the function $g(t) = 1 - t^2$ for $|t| \le 1$ and $g(t) = 0$ for $|t| > 1$ and for $p \in (1,\infty)$ denote by $W^{\Re}_{p,\,g}(\prod) := W^{\Re(l,m)}_{p,\,g}(\prod(a))$ the set of functions $u \in L_p(\prod)$ with a bounded norm

$$||u||_{W^{\Re}_{p,\,g}(\prod)} = \sum_{\alpha\in\Re} ||D^\alpha u\, g^{|\alpha|}(y)||_{L_p(\prod)}. \tag{2.20}$$

By regularity of the polyhdron $\Re$ and definition of the weight function g it follows that $W^{\Re}_{p,\,g}$ is a weighted Sobolev type Banach space.

By $\dot{W}^{\Re}_{p,\,g}(\prod)$ we denote the supplement of the set $C^\infty_0(\prod)$ by the norm (2.20) and by $W^{\Re}_{p,\,g,\,loc}(\prod)$ the set of functions $u \in W^{\Re}_{p,\,g}(\prod')$ for any $a' \in (0,a)$, where $\prod' = \prod(a') := \{(x,y) \in \prod;\ -a' < x < a'\} \subset \prod$.

Remark 2.3 It is easy to see that $u \in W^{\Re}_{p,\,g,\,loc}(\prod)$ if and only if $u(x,y)\,\psi(x) \in W^{\Re}_{p,\,g}(\prod)$ for any $\psi \in C^\infty_0(-a,a)$. □

Below, in the proof of Theorem 2.4 we will use following simple assertions (for proofs see [23]).

Lemma 2.5 *Let $\delta \in (0,1)$, $m \in \mathbb{N}$ and $k, n \in \mathbb{N}_0$, $k + n \le m$. Then*
1) $g^n(t) \le (2\,\delta)^n$ for any $t : 1 - \delta \le |t| < 1$
2) $g^{m-k}(t) \le (2\,\delta)^n g^{m-k-n}(t)$ for any $t : 1 - \delta \le |t| \le 1$
3) There exists a number $\sigma = \sigma(m) > 0$ such that

$$|\frac{d^k}{dt^k}\, g^m(t)| \le \sigma\, g^{m-k}(t)\ \ \forall t \in (-1,1),\ \ (k = 0,1,...,m).$$

Lemma 2.6 *Let $\delta > 0$. There exists a function $\upsilon_\delta(t)$ such that*
1) $\upsilon_\delta \in C^\infty_0(-1+\delta/2, 1-\delta/2)$; 2) $0 \le \upsilon_\delta(t) \le 1\ \forall t \in \mathbb{E}^1$;
3) $\upsilon_\delta(t) = 1$; for $|t| \le 1-\delta$; 4) for each $k \in \mathbb{E}_0$ there is a number $C_k > 0$, independent of δ, such that $|\upsilon^{(k)}_\delta(t)| \le C_k\,\delta^{-k}\ \forall t \in \mathbb{E}^1$, $(k = 0,1,...)$. □

Theorem 2.5 *Let $u \in W^{\Re}_{p,\,g,\,loc}(\prod)$ and $\psi \in C^\infty_0(-a,a)$. Then*

$$u(x,y)\,\psi(x)\,g^m(y) \in \dot{W}^{\Re}_{p,\,g}\left(\prod\right).$$

Proof. Let $\delta \in (0,1)$, and the function $\upsilon_\delta(y)$ be chosen by Lemma 2.6, $d_1 = \max\,\{C_k;\ 1 \le k \le m\}$ and $\prod_\delta = \prod(a,1) \setminus \prod(a, 1-\delta)$. Then, by the property 3) of the function υ_δ it follows

$$I := ||u(x,y)\,\psi(x)\,g^m(y)[1 - \upsilon_\delta(y)]||_{W^{\Re}_{p,\,g}(\prod)} =$$

$$= ||u(x,y)\,\psi(x)\,g^m(y)[1 - \upsilon_\delta(y)]||_{W^{\Re}_{p,\,g}(\prod_\delta)} =$$

$$= \sum_{\alpha\in\Re} ||D^{\alpha}[u(x,y)\,\psi(x)\,g^{m}(y)(1-\upsilon_{\delta}(y))]||_{L_p(\prod_\delta)}.$$

Let us apply the properties 2) and 4) of the function υ_δ and Leibnitz' formula to estimate the expression I. We get (below $||\cdot|| = ||\cdot||_{L_p(\prod_\delta)}$)

$$I \le \sum_{\alpha\in\Re} ||D^{\alpha}[u(x,y)\,\psi(x)\,g^{m}(y)]||+$$

$$+d_1 \sum_{\alpha\in\Re,\,\alpha_2>0}\sum_{\beta_2=1}^{\alpha_2} C_{\alpha_2}^{\beta_2}\,\delta^{-\beta_2}\,||D^{\alpha}-(0,\beta_2)[u(x,y)\,\psi(x)\,g^{m}(y)]||,$$

where $\{C_i^j\}$ are binomial coefficients. By applying once more Leibnitz' formula we obtain

$$I \le \{\sum_{\alpha\in\Re} ||D^{\alpha}[u(x,y)\,\psi(x)\,g^{m}(y)]||+$$

$$+\sum_{\alpha\in\Re,\,\alpha_2>0}\sum_{\beta_2=1}^{\alpha_2} C_{\alpha_2}^{\beta_2}\,\delta^{-\beta_2}\,||D^{\alpha-(0,\beta_2)}[u(x,y)\,\psi(x)\,[D^{\beta_2}g^{m}(y)]||\}+$$

$$+d_1\{\sum_{\alpha\in\Re,\,\alpha_2>0}\sum_{\beta_2=1}^{\alpha_2} C_{\alpha_2}^{\beta_2}\,\delta^{-\beta_2}\,||D^{\alpha-(0,\beta_2)}[u(x,y)\,\psi(x)\,g^{m}(y)]||+$$

$$+\sum_{\alpha\in\Re,\,\alpha_2>0}\sum_{\beta_2=1}^{\alpha_2} C_{\alpha_2}^{\beta_2}\,\delta^{-\beta_2}\sum_{\gamma_2=1}^{\alpha_2-\beta_2} C_{\alpha_2-\beta_2}^{\gamma}||D^{\alpha-(0,\gamma_2)}[u(x,y)\,\psi(x)\,D_2^{\gamma_2}g^{m}(y)]||\}$$

$$=: I_1 + d_1\, I_2. \tag{2.21}$$

By Lemma 2.5 we get for the term I_1

$$I_1 \le \sum_{\alpha\in\Re} ||D^{\alpha}[u(x,y)\,\psi(x)\,g^{m}(y)]||+$$

$$+\sigma \sum_{\alpha\in\Re,\,\alpha_2>0}\sum_{\beta_2=1}^{\alpha_2} C_{\alpha_2}^{\beta_2}\,\delta^{-\beta_2}\,||D^{\alpha-(0,\beta_2)}[u(x,y)\,\psi(x)\,[D^{\beta_2}g^{m}(y)]||.$$

Analogously, for the term I_2 we have with a constant $\sigma_1 > 0$

$$I_2 \le \sum_{\alpha\in\Re,\,\alpha_2>0}\sum_{\beta_2=1}^{\alpha_2} C_{\alpha_2}^{\beta_2}\,\delta^{-\beta_2}\,(2\delta)^{\beta_2}||D^{\alpha-(0,\beta_2)}[u(x,y)\,\psi(x)]\,g^{m-\beta_2}(y)]||+$$

$$+\sigma_1 \sum_{\alpha\in\Re,\,\alpha_2>0} \sum_{\beta_2=1}^{\alpha_2} \delta^{-\beta_2} \sum_{\gamma_2=1}^{\alpha_2-\beta_2} (2\delta)^{-\beta_2} ||D^{\alpha-(0,\beta_2+\gamma_2)}[u(x,y)\,\psi(x)]g^{m-\beta_2-\gamma_2}(y)]||.$$

Grouping corresponding terms in the estimates for I_1 and I_2 we get from (2.21) with a constant $d_2 > 0$

$$I \leq d_2 \sum_{\alpha\in\Re} \sum_{\beta_2=0}^{\alpha_2} ||D^{\alpha-(0,\beta_2)}[u(x,y)\,\psi(x)]\,g^{m-\beta_2}(y)]||. \qquad (2.22)$$

Since $m - \beta_2 \geq |\alpha - (0,\beta_2)|$ for $\alpha \in \Re : 0 \leq \beta_2 \leq \alpha_2$ and $g(y) \leq 1$ for $|y| \leq 1$ by the definition of function g we have

$$g^{m-\beta_2}(y) \leq g^{|\alpha-(0,\beta_2)|}(y) \;\; \forall y \in (-1,1). \qquad (2.23)$$

On the other hand, since the polyhedron $\Re$ is regular, the set $\{\alpha-(0,\beta_2) : \alpha \in \Re,\ 0 \leq \beta_2 \leq \alpha_2\} \subset \Re$. Therefore it follows from (2.22), (2.23) with a constant $d_3 > 0$

$$I \leq d_3 \sum_{\alpha\in\Re} ||D^{\alpha}[u(x,y)\,\psi(x)]\,g^{|\alpha|}(y)]||_{L_p(\prod_\delta)}. \qquad (2.24)$$

According to the Remark 2.3 $D^{\alpha}[u(x,y)\,\psi(x)]\,g^{|\alpha|}(y) \in L_p(\prod)$ for all $\alpha \in \Re$. Besides, $\mathrm{mes}\prod_\delta \to 0$ as $\delta \to +0$. Therefore the right - hand side of (2.24) tends to zero as $\delta \to +0$ i.e.

$$I \equiv ||u(x,y)\,\psi(x)\,g^m(y)[1 - \upsilon_\delta(y)]||_{W^{\Re}_{p,\,g}(\prod_\delta)} \to 0 \;\; as \;\; \delta \to +0. \qquad (2.25)$$

Now we show that for every $\varepsilon > 0$ there is a function $\phi_\varepsilon \in C_0^\infty$ such that

$$||u(x,y)\,\psi(x)\,g^m(y)\upsilon_\delta(y) - \phi_\varepsilon(x,y)||_{W^{\Re}_{p,\,g}(\prod)} < \varepsilon. \qquad (2.26)$$

For the fixed functions u and ψ we denote by $\tilde{u\psi}$ the extension by zero of the function $u\,\psi$ in hole space E^2. Then the function $\tilde{u\,\psi}\,g^m\,\upsilon_\delta$ will be extended by zero in E^2 too. Further we will suppose that all these functions are extended by zero in E^2 and omit in the notation the tilde sign.

Let $\mathrm{supp}\ \psi \subset [-a+\Delta, a-\Delta];\ \Delta > 0$ and (see Lemma 2.6) $\mathrm{supp}\ \upsilon_\delta \subset [-1+\delta/4, 1-\delta/4]$. Then

$$\mathrm{supp}\ (u\,\psi\,g^m\,\upsilon_\delta) \subset [-a+\Delta, a-\Delta] \times [-1+\delta/4, 1-\delta/4].$$

Let $S_1 := \{(x,y) \in E^2; |x|^2 + |y|^2 < 1\}$ be an open circle in E^2, $0 \leq \omega \in C_0^\infty(S_1)$, $\int \omega(x)dx = 1$, $h > 0$ and $\omega_h(x) = h^{-2}\omega(x/h)$. Then (see, for example [6])

$$\text{supp}\ \ [u\,\psi\,g^m\,\upsilon_\delta)*\omega_h]\subset\text{supp}\ \ [u\,\psi\,\upsilon_\delta]+\text{supp}\ \ \omega_h$$

and putting $h_0=\min\,\{\Delta,\delta/4\}$ we get for any $h\in(0,h_0)$

$$\text{supp}\ \ [u\,\psi\,g^m\,\upsilon_\delta)]*\omega_h\subset\text{supp}\ \ [u\,\psi\,\upsilon_\delta]+\text{supp}\ \ \omega_h\subset\prod. \tag{2.27}$$

Since (see [3, section 2] or [6, section 3]) $[u\,\psi\,g^m\,\upsilon_\delta)]*\omega_h\subset C^\infty$, in view of (2.27) we have

$$[u\,\psi\,g^m\,\upsilon_\delta)]*\omega_h\in C_0^\infty(\prod). \tag{2.28}$$

Let us apply Minkowski's inequality and properties of ω to estimate the following expression

$$\sum_{\alpha\in\Re}||D^\alpha[u\,\psi\,g^m\,\upsilon_\delta*\omega_h]-D^\alpha[u\,\psi\,g^m\,\upsilon_\delta)]||_{L_p(\prod)}=$$

$$\sum_{\alpha\in\Re}||\int_{|y|<h}\{D^\alpha[u\,\psi\,g^m\,\upsilon_\delta](x-z_1,y-z_2)-D^\alpha[u\,\psi\,g^m\,\upsilon_\delta](x,y)\}\,\omega_h(z)\,dz||_{L_p(\prod)}$$

$$\le\sum_{\alpha\in\Re}\sup_{|z|<h}D^\alpha[u\,\psi\,g^m\,\upsilon_\delta](\cdot-z_1,\cdot-z_2)||_{L_p(\prod)}\cdot||\omega_h||_{L_1(\prod)}=$$

$$=\sum_{\alpha\in\Re}\sup_{|z|<h}D^\alpha[u\,\psi\,g^m\,\upsilon_\delta](\cdot-z_1,\cdot-z_2)||_{L_p(\prod)}. \tag{2.29}$$

Since a function in L_p is mean continuous, this implies that

$$\sum_{\alpha\in\Re}||D^\alpha[u\,\psi\,g^m\,\upsilon_\delta*\omega_h]-D^\alpha[u\,\psi\,g^m\,\upsilon_\delta)]||_{L_p(\prod)}\to 0\ \ as\ \ h\to 0. \tag{2.30}$$

Let $h\in(0,h_0)$ and $\phi_h=[u(x,y)\,\psi(x)\,g^m(y)\,\upsilon_\delta(y)]*\omega_h(x,y)$, then

$$||u\,\psi\,g^m-\phi_h||_{W^{\Re}_{p,g}(\prod)}=||u\,\psi\,g^m(1-\upsilon_\delta)+u\,\psi\,g^m\,\upsilon_\delta-\phi_h||_{W^{\Re}_{p,g}(\prod)}\le$$

$$\le||u\,\psi\,g^m(1-\upsilon_\delta)||_{W^{\Re}_{p,g}(\prod)}+||u\,\psi\,g^m\,\upsilon_\delta-\phi_h||_{W^{\Re}_{p,g}(\prod)}. \tag{2.31}$$

In view of (2.25) one can choose the number $\delta_0>0$ such that for $\delta\le\delta_0$

$$||u\,\psi\,g^m(1-\upsilon_\delta)||_{W^{\Re}_{p,g}(\prod)}\le\frac{\varepsilon}{2}. \tag{2.32}$$

On the other hand in view of (2.30) for the given number δ_0 one can choose the number $h\in(0,h_0)$ such that

$$||u\,\psi\,g^m\,\upsilon_\delta - \phi_h||_{W^{\Re}_{p,g}(\prod)} \leq \frac{\varepsilon}{2}. \tag{2.33}$$

The relations (2.31) - (2.33) implies (2.26), which completes the proof of Theorem 2.5. □

Remark 2.4 One can consider $n-$ dimensional polyhedron

$$\Re = \Re(l, m, ..., m) = \{\upsilon \in N_0^n, \upsilon_1 \leq l,\ |\upsilon| = \upsilon_2 + ... + \upsilon_n \leq m\},$$

$n-$ dimensional right parallelepiped

$$\prod = \prod(a, b_2, ..., b_n) = \{(x, y) = (x, y_2, ..., y_n) \in R^n,$$
$$x \in (-a, a),\ y_j \in (-b_j, b_j)\ (j = 2, ..., n)\}.$$

$n-$ dimensional space $W^{\Re}_{p,\,g}(\prod) = W^{\Re(l,m,m,...,m)}_{p,\,g}(\prod(a, b_2, ..., b_n))$ and prove an analogue of Theorem 2.5. But here we prefer 2 dimensional case by technical reasons. □

We present two examples showing that for nonregular Newton polyhedron $\Re$ the multianisotropic Sobolev space $W^{\Re}_p(\Omega)$ in general is not semilocal. The first example refers to a bounded domain, and the second one to an unbounded domain Ω. In preparing these examples, the author consulted with Professor V.N. Margaryan, for which he expresses his deep gratitude to him.

Example 1. Let $n = 2$ and $\Re$ be the Newton polyhedron of multi - indices $(0, 0), (1, 0), (0, 1), (2, 1) \in \mathbb{N}_0^2$. It is easily seen that $\Re$ is a nonregular quadrangle (the projection (2,0) of the vertex (2,1) of $\Re$ on the axis $0\alpha_1$ does not belong to $\Re$).

Let $u(x) = u(x_1, x_2) = x_1^{4/3} + x_2$, and $\Delta_1 = \{-1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$. Then a simple computation shows that $u, D^{(1,\,0)}u, D^{(0,1)}u, D^{(2,1)}u$ belong to $L_2(\Delta_1)$, and $D^{(2,\,0)}u = \frac{4}{9}\,x_1^{-2/3} \notin L_2(\Delta_1)$.

Let $\psi \in C_0^\infty(\Delta_1)$, $\psi(x) = \psi(x_1, x_2) = x_2$ for $x \in \Delta_{1/2}$. Since $D^{(0,\,1)}\psi(x) = 1$ for $x \in \Delta_{1/2}$, it follows that

$$D^{(2,1)}(\psi(x)\,u(x)) = \frac{4}{9}D^{(0,1)}\psi(x)\,x_1^{-2/3} = \frac{4}{9}\,x_1^{-2/3} \notin L_2(\Delta_1),$$

i.e. $\psi\,u \notin W^{\Re}_2(\Delta_1)$, which means that $W^{\Re}_2(\Delta_1)$ is not semilocal. □

Example 2. Let a polyhedron $\Re$ be as in Example 1, and the function $f \in C_0^\infty(-1, 1)$ be chosen such that

$$A(f) \equiv \int_{-1}^{1} [f(t) + 5tf'(t) + 2t^2 f''(t)]^2 dt \neq 0, \tag{2.34}$$

$u(x, y) = x_1^2\,f(x_1^2\,x_2)$; $\Omega = \{(x_1, x_2) \in \mathbb{E}^2,\ |\,x_1| < 1,\ -\infty < x_2 < \infty\}$. Then

$$||u||^2_{L_2(\Omega)} = \int\int\limits_{\Omega} x_1^4 f^2(x_1^2 x_2)\, dx_1 dx_2 = \int\limits_{|x_1|<1} x_1^4\, [\int\limits_{-\infty}^{\infty} f^2(x_1^2 x_2)\, dx_2]dx_1 =$$

$$= \int\limits_{|x_1|<1} x_1^2\, [\int\limits_{-\infty}^{\infty} f^2(x_1^2 x_2) d(x_1^2 x_2)]dx_1 = \int\limits_{|x_1|<1} x_1^2\, [\int\limits_{|x_1^2 x_2|<1} f^2(x_1^2 x_2)\, dx_2]dx_1 =$$

$$= \int\limits_{-1}^{1} x_1^2\, [\int\limits_{-1}^{1} f^2(r) dr]dx_1 < \infty.$$

For $D^{1,0}u$ we have $D^{1,0}u = 2\, x_1\, f(x_1^2 x_2) + x_1^2\, (2\, x_1\, x_2) f'(x_1^2 x_2)$, where

$$\int\int\limits_{\Omega} |x_1\, f(x_1^2 x_2)|^2 dx_1\, dx_2 = \int\limits_{|x_1|<1} [\int\limits_{|x_1^2 x_2|<1} |f(x_1^2 x_2)|^2 d(x_1^2 x_2)]dx_1 < \infty,$$

$$\int\int\limits_{\Omega} x_1^2\, |x_1^2 x_2|^2 |f'(x_1^2 x_2)|^2 dx_1\, dx_2 = \int\limits_{|x_1|<1} [\int\limits_{|r|<1} r^2\, |f'(r)|^2 dr]dx_1 < \infty.$$

Therefore $D^{1,0}u \in L_2(\Omega)$. It is obvious that $D^{(0,1)}u = x_1^4\, f'(x_1^2 x_2) \in L_2(\Omega)$.

For $D^{2,0}u$ and $D^{(2,1)}$ we have respectively

$$D^{2,0}u = 2\, f(x_1^2 x_2) + 10(x_1^2 x_2)\, f'(x_1^2 x_2) + 4\, (x_1^2 x_2)^2\, f''(x_1^2 x_2);$$
$$D^{(2,1)} = x_1^2\, [12\, f'(x_1^2 x_2) + 18\, (x_1^2 x_2)\, f''(x_1^2 x_1) + 4\, (x_1^2 x_2)^2\, f'''(x_1^2 x_2)].$$

Denoting by $x_1 = x_1,\ x_1^2 x_2 = r,\ h_k(r) = r^{k-1} f^{(k-1)}(r)\ (k = 1, 2, 3)$ we have for each $k = 1, 2, 3$

$$\int\int\limits_{\Omega} x_1^2\, |x_1^2 x_2|^{2(k-1)}\, |h_k(x_1^2 x_2)|^2 dx_1\, dx_2 = \int\limits_{-1}^{1} x_1^2\, [\int\limits_{-1}^{1} |h_k(r)|^2 dr]dx_1 < \infty,$$

i.e $D^{(2,1)}u \in L_2(\Omega)$.

As for $D^{2,0}u$, we get by the condition (2.34) that

$$\int\int\limits_{\Omega} |D^{2,0}u|^2 dx_1\, dx_2 = A(f) \int\limits_{-1}^{1} \frac{dx_1}{x_1^2} = \infty,$$

i.e. $D^{2,0}u \notin L_2(\Omega)$.

Taking a function $\psi \in C_0^\infty(\Omega)$, as in Example 1, where $\Delta_{1/2} \subset \Delta_1 \subset \Omega$, we get $D^{2,1}(\psi\, u) = \psi'_{x_2}\, D^{2,0}u + \psi\, D^{2,1}u = D^{2,0}u + \psi\, D^{2,1}u$ for $x \in \Delta_{1/2}$. As above one can see that $\psi\, D^{2,1}u \in L_2(\Omega)$ and since $D^{2,0}u \notin L_2(\Omega)$ hence $D^{2,1}(\psi\, u) \notin L_2(\Omega)$, i.e. $W_2^{\Re}(\Omega)$ is not semilocal. □

3 Behavior at infinity of polynomials of many variables

Let $\Re = \Re(P)$ be the Newton polyhedron of the polynomial $P(\xi) = P(\xi_1, ..., \xi_n)$, $\Re_i^k$ $(i = 1, ..., M_k, k = 0, ..., n-1)$ be its principal faces, and

$$P^{i,k}(\xi) = \sum_{\alpha \in \Re_i^k} \gamma_\alpha \xi^\alpha$$

be subpolynomials corresponding to the principal faces $\Re_i^k$ $(i = 1, ..., M_k, k = 0, ..., n-1)$. A face $\Re_i^k$ $(1 \le i \le M_k, 0 \le k \le n-1)$ is called **nondegenerate** if $P^{i,k}(\xi) \ne 0$ for all $\xi \in R^{n,0}$. If all principal faces of $\Re(P)$ are nondegenerate, P is called nondegenerate. In [45] it is proved

Theorem (V. P. Mikhailov) *Let a polynomial P with the complete Newton polyhedron $\Re = \Re(P)$ be nondegenerate. Then a) $P \in \mathbb{I}_n$, b) there is a constant $C > 0$ such that*

$$\sum_{\alpha \in \Re} |\xi|^\alpha \le C[1 + |P(\xi)|] \;\; \forall \xi \in R^n.$$

First we prove two elementary statements, which give necessary conditions for $P \in \mathbb{I}_n$.

Lemma 3.1 *If the Newton polyhedron $\Re = \Re(P)$ of a polynomial P is not complete then $P \notin \mathbb{I}_n$.*

Proof. Since $\Re$ is not complete, $\Re$ has no vertex on one of coordinate axes of $\mathbb{R}^n$, for example on axis $(0, \xi_1)$. Then $P(\xi^s) = P(s, 0, ..., 0) =$ const $(s = 1, 2, ...)$ while $|\xi|^s = s \to \infty$, as $s \to \infty$. □

Mikhailov's theorem together with this lemma solves a problem of $P \in \mathbb{I}_n$ for nondegenerate polynomials P. Namely: *a nondegenerate polynomial is in $\mathbb{I}_n$ if and only if the Newton polyhedron $\Re(P)$ is complete.* □

Let $0 \ne \lambda \in \mathbb{R}^n$. A polynomial $R(\xi)$ is said to be $\lambda-$homogeneous of degree $d(\lambda)$ if for any $\xi \in \mathbb{R}^n$ and $t > 0$, $R(t^\lambda\, \xi) := R(t^{\lambda_1}\, \xi_1, ..., t^{\lambda_n}\, \xi_n)$ $= t^{d(\lambda)} R(\xi)$. For $\lambda-$homogeneous polynomial $R(\xi)$ we put $\Sigma(R) = \{\eta \in \mathbb{R}^{n,0}, |\eta| = 1, R(\eta) = 0\}$ and for $\eta \in \Sigma(R)$ denote $\mathcal{A}(\eta) = \mathcal{A}(\eta, R) = \{\nu \in \mathbb{N}_0^n; D^\nu R(\eta) \ne 0\}$, $\Delta(\eta) = \Delta(\eta, R) = \max_{\nu \in \mathcal{A}(\eta)} (\lambda, \nu)$.

In [45] it is also proved that a subpolynomial $P^{i,k}(\xi)$ of a polynomial $P(\xi)$ is $\lambda-$homogeneous for any $\lambda \in \Lambda(\Re_i^k)$, where $\Lambda(\Re_i^k)$ $(1 \le i \le$

$M_k, 1 \le k \le n-1)$ is the set of outward normals of the face $\Re_i^k$ of Newton polyhedron $\Re(P)$.

It is obvious that for any principal face $\Re_i^k$ $(1 \le i \le M_k, 1 \le k \le n-1)$ of the complete polyhedron $\Re(P)$ of the polynomial $P(\xi)$ and for any $\lambda \in \Lambda(\Re_i^k)$ the polynomial P can be represented in the form of sum of $\lambda-$homogeneous polynomials

$$P(\xi) = \sum_{j=0}^{N(\lambda)} P_j(\xi) = \sum_{j=0}^{N(\lambda)} \sum_{(\lambda,\,\alpha)=d_k(\lambda)} \gamma_\alpha \xi^\alpha, \tag{3.1}$$

where $P_j(\xi) = P_j(\xi,\lambda)$ $(j = 0, 1, ..., N(\lambda))$, $P_0(\xi) \equiv P^{i_0,k_0}(\xi)$, $d_0(\lambda) > d_1(\lambda) > ... > d_{N_1-1}(\lambda) > d_{N_1}(\lambda) = 0 > d_{N_1+1}(\lambda) > ... > d_N(\lambda)$ and $P_j(\xi) \equiv 0$ $(j = N_1 + 1, ..., N)$ when the face $\Re_i^k$ is regular or completely regular.

Remark 3.1 One can verify that a) for any polynomial $P \in \mathbb{I}_n$ with real coefficients there is a number $r \ge 0$ such that either $P(\xi) > 0$ for all $|\xi| > r$, or $P(\xi) < 0$ for all $|\xi| > r$, b) polynomials $P(\xi)$ and $|P(\xi)|^2$ simultaneously belong or not to $\mathbb{I}_n$. Therefore without loss of generality in the sequel we assume that $\mathbb{I}_n$ consists of polynomials with real coefficients such that $P(\xi) > 0$ for all $\xi \in R^n$.

Let $\Gamma := \Re_{i_0}^{k_0}$ $(1 \le i_0 \le M_k, 1 \le k_0 \le n-1)$ be a degenerate principal face of the complete Newton polyhedron $\Re = \Re(P)$ of the polynomial P and $\eta \in \Sigma(\Gamma)$. Let for $\lambda \in \Lambda(\Gamma)$ polynomial P be represented in the form (3.1). Define the number $J = J(\Gamma, \eta, \lambda)$ as follows: $0 < J \le N(\lambda)$, $P_0(\eta) = ...P_{J-1}(\eta) = 0$ and $P_J(\eta) \ne 0$.

Lemma 3.2 *Let $\Re = \Re(P)$ be the complete Newton polyhedron of a polynomial $P \in \mathbb{I}_n$ with principal faces $\Re_i^k$ $(i = 1, ..., M_k, k = 0, 1, ..., n-1)$ and $\Gamma := \Re_{i_0}^{k_0}$ $(1 \le i_0 \le M_k, 1 \le k_0 \le n-1)$ be a degenerate principal face. Then a) $P^{i,k}(\xi) \ge 0$ $(i = 1, ..., M_k, k = 0, 1, ..., n-1)$ $\forall \xi \in \mathbb{R}^n$, b) let for each $\lambda \in \Lambda(\Gamma)$ polynomial P be represented in the form (3.1), then b.1) $d_{J(\Gamma,\eta,\lambda)} > 0$ (or, what is the same, $J(\Gamma, \eta, \lambda) < N_1(\lambda)$) and b.2) $P_{J(\Gamma,\eta,\lambda)} > 0$ for all $\eta \in \Sigma(\Gamma)$.*

Proof of point a). Suppose to the contrary that $P^{i_1,k_1}(\eta) < 0$ for some pair (i_1, k_1) and some point $0 \ne \eta \in \mathbb{R}^n$. Let $\lambda \in \Lambda(\Re^{i_1,k_1})$ and $\xi^s = s^\lambda \eta$ $(s = 1, 2, ...)$. Represent polynomial P in the form (3.1), then by $\lambda-$homogeneity of polynomials $\{P_j\}$ and conditions $d_0(\lambda) > d_1(\lambda) > ... > d_N(\lambda)$ we get $P(\xi^s) = P^{i_1,k_1}(\eta)\, s^{d_0(\lambda)}[1 + o(1)] \to -\infty$ as $s \to \infty$, which contradicts the assumption $P \in \mathbb{I}_n$.

To prove the point b.1) suppose that $d_{J(\Gamma,\eta^0,\lambda^0)} \le 0$ for a pair (λ^0, η^0) $(\lambda^0 \in \Lambda(\Gamma), \eta^0 \in \Sigma(\Gamma))$. Acting as above we obtain $P(\xi^s) = P(s^{\lambda^0} \eta^0) = P^{i_1,k_1}(\eta^0)\, s^{d_{(\Gamma,\eta^0,\lambda^0)}}[1 + o(1)]$ as $s \to \infty$, which means that the sequence $P(\xi^s)$ is bounded, whereas $|\xi^s| \to \infty$ as $s \to \infty$. In the same manner we can prove the point b.2). □

So, let P be a degenerate polynomial, i.e. the Newton polyhedron $\Re = \Re(P)$ of P have degenerate principal faces. Without loss of generality one can assume that all of the principal faces of $\Re(P)$ are nondegenerate with the exception of one principal face $\Gamma := \Re^{k_0}_{i_0}$

When a polynomial P is degenerate, the partial solution of problem $P \in \mathbb{I}_n$ is given by the following statement

Theorem 3.1 *Let $\Re = \Re(P)$ be the complete Newton polyhedron of the polynomial $P : P(\xi) > 0\ \forall \xi \in R^n$. Let all the principal faces $\Re^k_i$ $(i = 1, ..., M_k, k = 0, 1, ..., n-1)$ of $\Re$ except of the principal face $\Gamma := \Re^{k_0}_{i_0}$ $(1 \le i_0 \le M_k, 1 \le k_0 \le n-1)$ be nondegenerate, and the face Γ be degenerate. Let $J(\Gamma, \eta, \lambda) = 1$ for all $\lambda \in \Lambda(\Gamma)$ and $\eta \in \Sigma(\Gamma)$. Then $P \in \mathbb{I}_n$ if and only if $d_1(\Gamma, \eta, \lambda) > 0$ and $P_1(\eta) > 0$ for all $\lambda \in \Lambda(\Gamma)$ and $\eta \in \Sigma(\Gamma)$.*

Proof. The necessity is contained in Lemma 3.2. To prove the sufficiency we apply the method of V.P. Mikhailov, applied to the nondegenerate case, and modified by us for degenerate case (see [45] and [33]). Suppose, on the contrary, that there exist a sequence $\{\xi^s\}$ and a number $C > 0$ such that $|\xi^s| \to \infty$ as $s \to \infty$ and $|P(\xi^s)| \le C$ $(s = 1, 2, ...)$.

It can be assumed without loss of generality that all of the coordinates of the vectors $\{\xi^s\}$ are positive. Let

$$\rho_s = exp \sqrt{\sum_{j=1}^{n} (ln\ \xi^s_j)^2}, \ \ \lambda^s_i = \frac{ln\ \xi^s_i}{ln\ \rho_s}, \ \ i = 1, ..., n, \tag{3.2}$$

then $\xi^s = \rho_s^{\lambda^s}$ $(\xi^s_i = \rho_s^{\lambda^s_i},\ i = 1, ..., n)$, λ^s being the unit vector $(s = 1, 2, ...)$. Clearly $\rho_s \to \infty$ if $|\xi^s| \to \infty$, or if a coordinate of ξ^s tends to zero.

Sins the vectors λ^s lie on the unite sphere, then the sequence $\{\lambda^s\}$ has a limit point λ^∞. It can be assumed that $\lambda^s \to \lambda^\infty$. From the convexity of $\Re(P)$ it follows that λ^∞ is an outword normal to one and only one face of $\Re(P)$.

We take in $\mathbb{R}^n$ a basis $(e^{1,1}, e^{1,2}, ..., e^{1,n})$ in which $e^{1,1} = \lambda^\infty$. Then $\lambda^s = \Sigma^n_{i=1} \kappa^s_{1,\,i}\, e^{1,\,i}$, and because of $\lambda^s \to \lambda^\infty = e^{1,1}$ as $s \to \infty$, it follows $\kappa^s_{1,\,i} = o(\kappa^s_{1,1}) = o(1)$ for $i = 2, 3, ..., n$.

If $\Sigma^n_{i=2} \kappa^s_{1,\,i}\, e^{1,\,i} = 0$ for sufficiently large s then we denote by $(e^1, ..., e^n)$ the basis $(e^{1,1}, e^{1,2}, ..., e^{1,n})$. Otherwise it can be assumed that

$$\sum_{i=2}^{n} \kappa^s_{1,\,i}\, e^{1,\,i} \neq 0$$

for all $s \in N$ and that, as $s \to \infty$,

$$[\Sigma^n_{i=2} \kappa^s_{1,\,i}\, e^{1,\,i}] / |\Sigma^n_{i=2} \kappa^s_{1,\,i}\, e^{1,\,i}| \to e^{2,2}; \ \ |e^{2,2}| = 1.$$

We go over in the subspace spanned by $(e^{1,2},...,e^{1,n})$ to a new basis $(e^{2,2},...,e^{2,n})$. Then

$$\lambda^s = \kappa^s_{1,\,1}\, e^{1,1} + \sum_{i=2}^{n} \kappa^s_{2,\,i}\, e^{2,\,i},$$

where it is clear that $\kappa^s_{2,\,2} = o(\kappa^s_{1,\,1})$, $\kappa^s_{2,\,i} = o(\kappa^s_{2,\,1})$, as $s \to \infty$, $i = 3,...,n$.

Proceeding analogously in a subspace with basis $(e^{2,3},...,e^{2,n})$ etc..., we obtain (after the corresponding re - notation) $\lambda^s = \sum_{i=1}^{n} \kappa^s_i\, e^i$, where $\kappa^s_1 \to 1$, $\kappa^s_{i+1} = o(\kappa^s_i)$ $(i = 1,...,n-1)$, as $s \to \infty$.

We note in this connection that there exist natural numbers s_0 and m such that for all $s \geq s_0$ we have $\kappa^s_i > 0$ $(i = 1,...,m)$ and $\kappa^s_i = 0$ $(i = m+1,...,n)$ $m \leq n$.

Let $\Re^{k_1}_{i_1}, \Re^{k_2}_{i_2}, ..., \Re^{k_m}_{i_m}$ denote the faces of $\Re(P)$ satisfying the condition that $\Re^{k_1}_{i_1}$ lies in the hyperplane of support of $\Re(P)$ with outward normal e^1, $\Re^{k_2}_{i_2}$ lies in the hyperplane of support of $\Re^{k_1}_{i_1}$ (treating as an isolated object) and either coincides with $\Re^{k_1}_{i_1}$, or is a subspace of $\Re^{k_1}_{i_1}$. If there is more than one subface of $\Re^{k_1}_{i_1}$ with normal e^2, we take as $\Re^{k_2}_{i_2}$ the subface containing the point α at which the value of (e^2, α) is maximal, and so on.

From the construction of the faces $\Re^{k_1}_{i_1}, \Re^{k_2}_{i_2}, ..., \Re^{k_m}_{i_m}$ we see that their dimensions are nonincreasing: $k_1 \geq k_2 \geq ... \geq k_m$ and (see notation (3.2))

$$\xi^s = \rho_s^{\sum_{j=1}^{n} \kappa^s_j e^j} \quad (s = 1, 2, ...), \tag{3.3}$$

where $\rho_s \to \infty$, as $s \to \infty$, and for a r $(1 \leq r \leq m)$ and $b \geq 1$

$$\rho_s^{\kappa^s_j} \to \infty\ (j = 1,...,r),\ \rho_s^{\kappa^s_{r+1}} \to b,\ (s = 1, 2, ...).$$

For $r = m = n$ we put $\kappa^s_{r+1} = 0$ $(s = 1, 2, ...)$.

Let, as above, $P^{i_j,k_j}(\xi)$ denote the part of the polynomial P whose multi indices belong to $\Re^{k_j}_{i_j}$, and let α denote an arbitrary point belonging to all of the $\Re^{i_j}_{k_j}$ $(j = 1,...,m)$, i.e. $\alpha \in \Re^{k_m}_{i_m}$. We will study the behavior of the polynomial P, as $\rho_s \to \infty$,, and $\{\xi^s\}$ which is defined by the formula (3.3). The index s will be omitted for the sake of simplicity in notation.

Then from the e^jhomogeneity of the polynomials $\{P^{i_j,k_j}(\xi)\}$ and from the convexity of $\Re(P)$ and its faces, we get for certain positive $\sigma_1,...,\sigma_r$ and $1 \leq r \leq m \leq n$ (e^{n+1} is a unite vector)

$$P(\xi) = \rho^{(\alpha,\, \kappa_1\, e^1)}\, [P^{i_1,\, k_1}(\rho^{\sum_{j=2}^{n+1} \kappa_j\, e^j}) + o(\rho^{-\sigma_1\, \kappa_1})] =$$

$$= \rho^{(\alpha,\, \kappa_1\, e^1 + \kappa_2\, e^2)}\, [P^{i_2,\, k_2}(\rho^{\sum_{j=3}^{n+1} \kappa_j\, e^j}) + o(\rho^{-\sigma_2\, \kappa_2})] =$$

$$... = \rho^{(\alpha, \sum\limits_{j=1}^{r} \kappa_j e^j)} [P^{i_r, k_r}(\rho^{\sum\limits_{j=r+1}^{n+1} \kappa_j e^j}) + o(\rho^{-\sigma_r \kappa_r})]\cdot \tag{3.4}$$

Since $\rho^{\kappa_{r+1}} \to b \geq 1$, it follows that $\rho^{\sum\limits_{j=r+1}^{n+1} \kappa_j e^j} \to b^{e^{r+1}} \equiv \eta$. Clearly, $0 < \eta_i < \infty$ $(i = 1, ..., n)$ (in accordance with the definition of η_i $(i = 1 \leq i \leq n))$ as finite powers of a positive number).

We consider two cases: a) $(e^1, \alpha) > 0$ and b) $(e^1, \alpha) = 0$. The case $(e^1, \alpha) < 0$ is impossible, as the equation for the hyperplane of support with outward unit normal λ of a polyhedron $\Re$ can be written in the form $(\lambda, \alpha) = d$, where $d \geq 0$ is the distance from the origin to the given hyperplane and α is a roving point of the hyperplane (see, for example, [1]).

Case a). Firstly, suppose $P^{i_r, k_r}(\eta) \neq 0$. Since $(e^1, \alpha) > 0$ and $\kappa_i = o(\kappa_1)$ for $i = 2, ..., n$ as $s \to \infty$, we eventually have $(\alpha, \Sigma_1^r \kappa_j e^j) > 0$ beginning at some number s_0. Therefore (3.4) implies

$$P(\xi) = \rho^{(\alpha, \sum\limits_{j=1}^{r} \kappa_j e^j)} [P^{i_r, k_r}(\eta) + o(1)],$$

which means that $|P(\xi^s)| \to \infty$ as $s \to \infty$, and contradicts our assumption on the boundedness of $\{|P(\xi^s)|\}$.

Suppose now $P^{i_r, k_r}(\eta) = 0$. Since $(e^1, \alpha) > 0$, the face $\Re_{i_r}^{k_r}$ is principal, hence $\Re_{i_r}^{k_r}$ coincides with the degenerate face Γ and $\eta \in \Sigma(\Gamma)$. In this case we represent P in the form (see (3.1))

$$P(\xi) = P_0(\xi) + P_1(\xi) + r(\xi). \tag{3.5}$$

Since $P_0(\xi) \geq 0$ for all $\xi \in R^n$ and $P_1(\eta) \equiv P_J(\eta) > 0$ for all $\eta \in \Sigma(\Gamma)$ it follows from (3.4) that for sufficiently large $|\xi|$

$$P_0((\rho^{\sum\limits_{j=r+1}^{n+1} \kappa_j e^j})) \equiv P^{i_r, k_r}(\rho^{\sum\limits_{j=r+1}^{n+1} \kappa_j e^j}) \geq 0, \;\; P_1((\rho^{\sum\limits_{j=r+1}^{n+1} \kappa_j e^j})) > 0. \tag{3.6}$$

On the other hand since $d_i(\Gamma, \lambda) < d_1 \equiv d_1(\Gamma, \lambda)$ $(i = 2, 3, ...)$ and $d_1 > 0$ hence 7

$$|r(\xi)| = o(\rho^{d_1}) \;\; as \;\; |\xi| \to \infty. \tag{3.7}$$

From (3.4) - (3.7) we get $|P(\xi^s)| \geq C\, \rho_s^{d_1}\, |P_1(\eta)|$ for sufficiently large $|\xi^s|$ with a constant $C > 0$, i.e. $|P(\xi^s)| \to \infty$, as $|\xi^s| \to \infty$ which contradicts our assumption on the boundedness of $\{|P(\xi^s)|\}$.

Case b) $(e^1, \alpha) = 0$. In this case the face whose outward normal is e^1 clearly passes through the origin and hence is not a principal face of $\Re(P)$. Consequently $e_i^1 \leq 0$ $(i = 1, ..., n)$. In this connection, if a nonprincipal face with outward normal e^1 has the dimension l, then l among the numbers $e_1^1, ..., e_n^1$ are equal to zero the remaining numbers being negative. It can clearly be assumed without loss of generality that $e_1^1 = ... = e_l^1 = 0,\ e_{l+1}^1 < 0, ..., e_n^1 < 0$. Since

$$e_j^1 = \lim_{s\to\infty} ln\ \xi_j^s/[\sum_{k=1}^n (ln\ \xi_k^s)^2]^{1/2} < 0\ \ (j = l+1, ..., n),$$

we have $\xi_j < 1$ $(j = l+1, ..., n)$ beginning at some point. On the other hand, since $|\xi| \to \infty$, we have $\xi_i \to \infty$ for a certain $i \in [1, l]$. But since $e_i^1 = 0$ for those i, it follows (at least for a subsequence of sequence $\{\xi^s\}$) that $\xi_j \to 0$ for at least one $j \in (l, n]$.

Suppose (after modifying the notations) $\xi_1 \to \infty, ..., \xi_{l_0} \to \infty$ $(l_0 \leq l)$ and $\xi_{l_0+1} \to 0, ..., \xi_{l_0+l_1} \to 0$ $(l_0 + l_1 \leq n)$. Let $\psi(\xi) = \max_{1\leq j\leq l_0} \xi_j$, then, clearly,

$$\ln\psi(\xi)/[\sum_{k=1}^n (\ln\ \xi_k)^2]^{1/2} \to 0\ \ (as\ \ |\xi^s| \to \infty). \tag{3.8}$$

On the other hand, there exist positive constants M_1 and M_2 such that

$$M_1 \leq \sum_{k=1}^{l_0} (ln\ \xi_k)^2/[ln\ \psi(\xi)]^2 \leq M_2. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\sum_{k=l_0+1}^{n} (\ln\xi_k)^2/[\ln\psi(\xi)]^2 \to \infty\ \ (as\ \ |\xi^s| \to \infty).$$

From this, by going over to a subsequence if necessary, we get that for some $j \in [l_0+1, n]$

$$|\ln\xi_j|/\ln\psi(\xi) \to \infty\ \ (as\ \ |\xi^s| \to \infty),$$

i.e. $|\ln\xi_j| \to \infty$ "faster" than $\ln\psi(\xi) \to \infty$. Hence for a $\sigma > 0$ $\xi_j = o([\psi(\xi)]^{-\sigma})$ $(as\ \ |\xi^s| \to \infty)$, or, equivalently,

$$\xi_j^{\alpha_1}\,[\psi(\xi)]^{\alpha_2} \to 0\ \ (as\ \ |\xi^s| \to \infty) \tag{3.10}$$

for any $\alpha_1 > 0$ and $\alpha_2 \geq 0$.

Let $\breve{\xi} = (\breve{\xi}_1, ..., \breve{\xi}_n)$, where $\breve{\xi}_j = 0$ if j satisfy the condition (3. 10) and $\breve{\xi}_j = \xi_j$ otherwise. As a result, the polynomial $P(\xi) = P(\xi_1, ..., \xi_n)$ turns to a polynomial $P(\breve{\xi})$ of less than n variables, the dimension of $\breve{\Re}(P)$ is less than the dimension of $\Re(P)$, and the (generate or nongenerate) principal faces of $\breve{\Re}(P)$ are those and only those of $\breve{\Re}(P)$ that are principal faces of $\Re(P)$.

Thus, in the course of the proof of the theorem, the assumption of boundedness of the polynomial P leades either to a contradiction, or to a polynomial $P(\breve{\xi})$ and polyhedron $\breve{\Re}(P)$ in a space of the dimension less than n. Repeating the arguments presented above in the proof of this theorem with respect to the polynomial $P(\breve{\xi})$ and polyhedron $\breve{\Re}(P)$, and so on, we clearly arrive after a finite number of steps at either a contradiction, or the assumption of boundedness of a polynomial of one variable with nonzero main coefficient (for a one dimensional polyhedron $\breve{\Re}(P)$). In this case the contradiction is obvious. □

Remark 3.2 In the case $J(\Gamma, \eta, \lambda) > 1$ for a (η, λ), the necessary and sufficient conditions for $P \in I_2$ (for polynomials of two variables) are obtained in [20].

Remark 3.3 In the general case ($n \geq 2$) **when** $J(\Gamma, \eta, \lambda) > 1$ for a (η, λ), by repeating the arguments presented above in the proof of this theorem one can prove the following

Theorem 3.1$'$ *Let (as in Theorem 3.1)* Γ *be a (unique) degenerate face of the complete Newton polyhedron of a polynomial* $P : P(\xi) > 0 \ \forall \xi \in R^n$. *Let* $J(\Gamma, \eta, \lambda) \geq 1$, *and for any* $\eta \in \Sigma(P_0)$ *there exists a neighborhood* $U(\eta)$ *of* η *such that* $P_j(\xi) \geq 0$ $(j = 1, ..., J(\Gamma, \eta, \lambda) - 1)$ *for all* $\lambda \in \Lambda(\Gamma)$ *and* $\xi \in U(\eta)$.

Then $P \in I_n$ *if and only if* $d_J(\Gamma, \eta, \lambda) > 0$ *and* $P_J(\eta) > 0$ *for all* $\lambda \in \Lambda(\Gamma)$ *and* $\eta \in \Sigma(\Gamma)$.

Remark 3.4 Let the polynomial P satisfies the hypotheses of Theorem 3.1' and let $\Upsilon(\Gamma) = \{\mu : \mu \in \Re, \ (\lambda, \mu) \leq J(\Gamma, \eta, \lambda) \ \forall \lambda \in \Lambda(\Gamma)\}$. By applying the same method one can prove that there exists a constant $C > 0$ such that

$$\sum_{\nu \in \Upsilon(\Gamma)} |\, \xi^\nu| \leq C\,[1 + |P(\xi)|] \ \ \forall \xi \in R^n.$$

□

4 Properties of the smoothness of solutions of general partial differential equations depending on the Newton Polyhedrons.

Let $P(D) = P(D_1, ..., D_n) = \sum_\alpha \gamma_\alpha \, D^\alpha$ be a nondegenerate linear differential operator with constant coefficients and let $P(\xi) = P(\xi_1, ..., \xi_n) = \sum_\alpha \gamma_\alpha \, \xi^\alpha$ be its characteristic polynomial (the complete symbol). Let $\Re = \Re(P)$ be the complete Newton polyhedron of P.

Theorem 4.1 *Let $\Re = \Re(P)$ be the Newton polyhedron of a nondegenerate operator $P(D)$ (polynomial $P(\xi)$). Then 1) P is hypoelliptic if and only if $\Re$ is completely regular, 2) P is almost hypoelliptic if and only if $\Re$ is regular.*

Proof of the point 1. Let $\Re = \Re(P)$ be the comletely regular Newton polyhedron of a nondegenerate operator $P(D)$ (polynomial $P(\xi)$). Prove that P is hypoelliptic. According Hörmander's theorem (see point C)) it is sufficient to prove that for any $0 \neq \nu \in N_0^n$

$$|D^\nu P(\xi)|/|P(\xi)| \to 0, \text{ as } |\xi| \to \infty. \tag{4.1}$$

Since the polyhedron $\Re$ is comletely regular for obvious geometric reasons, it follows that for any $0 \neq \nu \in N_0^n$ the points (multi indices) $\alpha \in (D^\nu P)$ are nonprincipal points of the polyhedron $\Re$. It is proved in [45] that for any nonprincipal point $\beta \in \Re$

$$|\xi^\beta|/\sum_{\alpha\in\Re}|\xi^\alpha| \to 0, \text{ as } |\xi| \to \infty.$$

This, together with Mikhailov's Theorem implies (4.1).

Let us prove that a polynomial with the noncompletely regular Newton's polyhedron (regardless of nondegenerateness) can not be hypoelliptic.

Note that noncompletely regularity of the polyhedron $\Re$ means geometrically that the projection of some principal vertex of $\Re$ on some coordinate hyperplane either a) falls outside the limits of $\Re$ (if $\Re$ is nonregular), or b) coincides with some principal vertex of $\Re$ (if $\Re$ is regular).

Case a). Let $e = (e^1, ..., e^n)$ be a vertex of $\Re$ such that its projection $e' = (0, ..., 0, e^{k+1}, ..., e^n)$ on the coordinate hyperplane $\alpha_1 = ... = \alpha_k = 0$ $(1 \leq k \leq n-1)$ falls outside the limits of $\Re$. Let us construct the $(n-1)$dimensional hyperplane which passes through the point e', does not passes through the origin, and does not have any common point with $\Re$. Let λ be the outward normal, and $(\lambda, \alpha) = d$ be the equation of this hyperplane. Then $d > 0$, $(\lambda, e') = d$, and $(\lambda, \alpha) < d$ for all $\alpha \in \Re$.

Let $\nu = (e^1, ..., e^k, 0, ..., 0)$, for the vector λ, the polynomials P and $D^\nu P$ is represented in the form (3.1) of sum of $\lambda-$homogeneous polynomials, a point $\eta \in R^n$ be chosen such that $D^\nu P(\eta) \neq 0$, and $\xi^s = s^\lambda \eta$ $(s = 1, 2, ...)$. Then

$$D^\nu P(\xi^s) = D^\nu P(\eta)\, s^d; \;\; P(\xi^s) = o(s^d), \text{ as } s \to \infty, \tag{4.2}$$

i.e. the polynomial is nonhypoelliptic.

Case b). Let $e' = (0, ..., 0, e^{k+1}, ..., e^n)$ coincide with some principal vertex of the polyhedron $\Re$. Then, constructing the $(n-1)$dimensional hyperplane supporting to $\Re$ which passes through the point e', does not passes through the origin, and does not have a common point with the polyhedron $\Re$, and repeating the previos arguments, we obtain the following estimate

$$|D^\nu P(\xi^s)| \geq C_1 s^d; \ \ |P(\xi^s)| \leq C_2 s^d \ \ (s = 1, 2, ...),$$

we obtain that hypoelliptic of the polynomial P is not hypoelliptic.

Proof of the point 2) is completely analogous to the proof of the point 1) so we omit the details. □

The following proposition shows that, in contrast to hypoelliptic polynomials, which Newton polyhedron can have degenerate faces of any kind, a Newton polyhedron of an almost hypoelliptic polynomial can have only completely regular degenerate faces.

Theorem 4.2 *Let $\Re = \Re(P)$ be a regular Newton polyhedron of a polynomial $P(\xi)$. Let all completely regular faces of $\Re$ be nondegenerate. Then the polynomial P is almost hypoelliptic if and only if P is nondegenerate, i.e. all noncompletly regular principal faces of $\Re$ are also nondegenerate.*

Proof. We prove that under the conditions of the theorem, all principal faces of $\Re$ are nondegenerate. Since 0 dimensional faces (vertices) of $\Re$ are nondegenerate, we begin our proof with one dimensional faces. Let Γ be an one dimensional principal (but not completly regular) degenerate face of $\Re(P)$. We prove that P can not be almost hypoelliptic.

Let $\lambda \in \Lambda(\Gamma)$, and $(\lambda, \alpha) = d_0$ be an equation of the $(n-1)$ dimensional support hyperplane of the polyhedron $\Re$, containing Γ, and not containing the points of $\Re \setminus \Gamma$. Since the face Γ is not completely regular, λ has at least one nonpositive coordinate. Let $\lambda_1 \leq 0$.

Put $m_1 = \max\{\alpha_1; \, \alpha \in \Gamma\}$, $\Gamma_1 = \{\alpha \in \Gamma; \, \alpha_1 = m_1\}$ and show that the set Γ_1 consists of a unique point, and hence is a 0 dimensional subface of Γ, i.e. a vertex of $\Re$. Really, if Γ contains two different points α^1 and α^2 with $\alpha_1^1 = \alpha_1^2$, hence $\alpha^1 = m_1$ for all $\alpha \in \Gamma$ (since an $1-$dimensional face is uniquely determined by its two points). Then Γ is perpendicular to the $0\alpha_1$ axis, i.e. $\lambda_1 > 0$, which contradicts our assumption.

Thus, Γ_1 coincides with a principal vertex $e = (m_1, e_2, ..., e_n)$ of $\Re$.

With the help of a vector λ, represent the polynomials P and $D_1^{m_1} P$ in the form (3.1) of sums of $\lambda-$homogeneous polynomials and consider behaviors of these polynomials on the sequence $\xi^s = s^\lambda \eta = (s^{\lambda_1}, ..., s^{\lambda_n})$ $(s = 1, 2, ...)$, where $\eta \in \Sigma(\Gamma)$. We obtain

$$P(\xi^s) = P_0(\eta) s^{d_0} + \sum_{j=1}^{M} P_j(\eta) s^{d_j} = \sum_{j=1}^{M} P_j(\eta) s^{d_j}, \tag{4.3}$$

$$D_1^{m_1} P(\xi^s) = D_1^{m_1}[\gamma_e (\xi^s)^e + \sum_{\alpha \in \Gamma, \alpha_1 < m_1} \gamma_\alpha (\xi^s)^\alpha] + \sum_{j=1}^{M} D_1^{m_1} P_j(\xi^s) =$$

$$= \gamma_e\,(m_1!)\,|\eta_2^{e_2}...\eta_n^{e_n}|\,s^{d_0-\lambda_1 m_1} + \sum_{j=1}^{M} D_1^{m_1} P_j(\eta) s^{d_j-\lambda_1 m_1}.$$

Since $\lambda_1 \leq 0$, hence $d_0 - \lambda_1 m_1 > d_1 - \lambda_1 m_1$ and as $s \to \infty$, these relations lead to

$$|P(\xi^s)| = o(s^{d_0}),\ |D_1^{m_1} P(\xi^s)| = \gamma_e\,(m_1!)\,|\eta_2^{e_2}...\eta_n^{e_n}|\,s^{d_0-\lambda_1 m_1}\,(1+o(1)).$$

Since $\eta_2^{e_2}...\eta_n^{e_n} \neq 0$, the last relations show that P is not almost hypoelliptic.

Let now Γ be a $2-$dimensional principal (but not completely regular) degenerate face of $\Re(P)$, $\lambda \in \Lambda(\Gamma)$ and let, for definiteness, $\lambda_1 \leq 0$. Introduce notation m_1 and Γ_1 as above and prove that in this case either Γ_0 consists of a unique point (i.e. is a vertex), or is a $1-$dimensional subcase of $\Re(P)$.

Let us show that in the case when Γ_1 contains more than one point, all of them lie on one straight line. Suppose. to the contrary that there are three points $\alpha^j \in \Gamma, \alpha_1^j = m_1$ $(j = 1, 2, 3)$, not lying on a straight line. Since these three points uniquely determine a $2-$dimensional face Γ, that means the plane passing through Γ is perpendicular to the axis $0\alpha_1$, i.e. $\lambda_1 > 0$, which contradicts our assumption.

Since all subfaces of a principal face are principal, Γ_0 is either $0-$ dimensional or $1-$dimensional principal face of $\Re$ and in both cases the subpolynomial P_0 has a form

$$P_0(\xi) = [\sum_{\alpha\in\Gamma,\alpha_1=m_1} + \sum_{\alpha\in\Gamma,\alpha_1<m_1}]\gamma_\alpha \xi^\alpha =$$

$$= \xi_1^{m_1} q(\xi_2, ..., \xi_n) + \sum_{\alpha\in\Gamma,\alpha_1<m_1} \gamma_\alpha \xi^\alpha, \tag{4.4}$$

were in the $0-$dimensional case $q(\eta_2, ..., \eta_n) \neq 0$. If Γ_1 is one-dimensional completely regular face then $q(\eta_2, ..., \eta_n) \neq 0$ by the assumption of our theorem. If Γ_1 is one-dimensional principal but not completely regular face, then $q(\eta_2, ..., \eta_n) \neq 0$ since the part of the theorem already proved.

Considering behaviors of polynomials P and $D_1^{m_1} P$ on the sequence $\xi^s = s^\lambda\,\eta = (s^{\lambda_1}, ..., s^{\lambda_n})$ $(s = 1, 2, ...)$, where $\eta \in \Sigma(\Gamma)$, for the polynomial P we obtain the representation (4.3). For the polynomial $D_1^{m_1} P$, according to (4.4) we get

$$D_1^{m_1} P(\xi^s) = (m!) q(\eta_2, ..., \eta_n) s^{d_0-\lambda_1 m_1} + \sum_{j=1}^{M} D_1^{m_1} P_j(\eta) s^{d_j-\lambda_1 m_1}. \tag{4.5}$$

Since $q(\eta_2, ..., \eta_n) \neq 0$, arguing as above one can show that (4.3) and (4.5) leads that the polynomial P is not almost hypoelliptic.

If $n \geq 4$, arguing as above we prove that the polynomial P is nondegenerate.

Almost hypoellipticity of a nondegenerate polynomial with regular Newton's polyhedron follows from Theorem 4.1. □

Corollary 4.1 *Let the Newton polyhedron $\Re(P)$ of a polynomial P be an $n-$dimensional rectangular parallelepiped with the vertex in the origin. Then P is almost hypoelliptic if and only if P is nondegenerate.*

The proof is a corollary of a geometrical obvious fact: completely regular faces of an $n-$dimensional rectangular parallelepiped with the vertex in the origin can be only his vertices. □

Let now $\Gamma \equiv \Re_i^k$ be some degenerate face of the Newton's polyhedron $\Re(P)$ of a polynomial P. For a $\lambda \in \Lambda(\Gamma)$ and an $\eta \in \Sigma(\Gamma)$ represent the polynomial P in the form (3.1) and introduce the notations $J(\Gamma, \lambda, \eta)$, $\mathcal{A}(P_j, \lambda, \eta)$ and $\Delta(P_j, \lambda, \eta)$ $(j = 0, 1, ..., J = J(\Gamma, \lambda, \eta))$ as in the point 3.

As a supplement of Lemma 3.1 let us prove

Lemma 4.1 *Let $\Gamma \equiv \Re_i^k$ be some principal degenerate face of the regular Newton's polyhedron $\Re(P)$ of an almost hypoelliptic polynomial P. Then for all $\lambda \in \Lambda(\Gamma)$ and $\eta \in \Sigma(\Gamma)$*

$$d_j(\lambda) - \Delta(P_j, \lambda, \eta) \leq d_{J(\Gamma,\lambda,\eta)}\, (j = 0, 1, ..., J - 1). \tag{4.6}$$

Proof. We argue by contradiction. Suppose that for some values of $\lambda, \eta, J = J(\Gamma, \lambda, \eta)$, and $j \in [0, J - 1]$ the inequality (4.6) is violated. Assuming the pair (λ, η) is fixed, omit (λ, η) in the notation and denote by j_0 the least of such j. Thus, let

$$d_j - \Delta(P_j) \leq d_J \ (j = 0, 1, ..., j_0 - 1), \ \ d_{j_0} - \Delta(P_{j_0}) > d_J. \tag{4.7}$$

Choose a multiindex $\beta \in \mathbb{N}_0^n$ in such a way that $D^\beta P_{j_0}(\eta) \neq 0$ and $(\lambda, \beta) = \Delta(P_{j_0})$. Consider polynomials P and $D^\beta P$ on the sequence $\xi^s = s^\lambda \eta \ (s = 1, 2, ...)$.

Since $d_j > d_{j_0}$ $(j = 0, 1, ..., j_0 - 1)$, it follows from (4.7) that $\Delta(P_{j_0}) < \Delta(P_j)$ $(j = 0, 1, ..., j_0 - 1)$. Then $P_j(\eta) = D^\beta P_j(\eta) = 0$ $(j = 0, 1, ..., j_0 - 1)$, $D^\beta P_{j_0}(\eta) \neq 0$ and according to the representation (3.1) and the inequality (4.7) we obtain

$$P(\xi^s) = P_J(\eta)\, s^{d_J} + o(s^{d_J}), \ \text{as} \ s \to \infty. \tag{4.8}$$

For the polynomial $D^\beta P$ we get for all $s \in \mathbb{N}$

$$D^\beta P(\xi^s) = s^{d_{j_0} - \Delta(P_{J_0})}\, D^\beta P_{J_0}(\eta) + \sum_{j=j_0+1}^{M} s^{d_j - \Delta(P_{J_0})}\, D^\beta P_J(\eta).$$

Since $d_j < d_{j_0}$ $(j = j_0 + 1, ..., M)$, it follows from here that

$$|D^\beta P(\xi^s)| = |D^\beta P_{J_0}(\eta)|\, s^{d_{j_0} - \Delta(P_{J_0})}\,(1 + o(1)), \text{ as } s \to \infty. \tag{4.9}$$

Since $D^\beta P_{J_0}(\eta) \neq 0$, the relations (4.8) - (4.9) together with (4.7) contradict the almost hypoellipticity of P. □

Below, we get necessary and sufficient condition for almost hypoellipticity of degenerate polynomials with regular Newton's polyhedron. To simplify our presentation we will consider only two-dimensional polynomials with real coefficients. Firstly we present a simple statement needed below.

Lemma 4.2 *Let $\lambda = (\lambda_1, \lambda_2) \in R^2$, $Q(\xi) = Q(\xi_1, \xi_2) \in I_2$ be a $\lambda-$ homogenous polynomial and $\eta \in \Sigma(Q)$. Then, there exist a neighborhood $U(\eta)$ of η, a natural number $m = m(\eta)$, a pair of $\lambda-$ homogeneous smooth functions $r(\xi)$ and $q(\xi)$ such that $q(\eta) = 0$, $D_1 q(\eta)\, D_2 q(\eta) \neq 0$, $r(\eta) \neq 0$* and

$$Q(\xi) = r(\xi)\,[q(\xi)]^m \;\; \forall \xi \in U(\eta). \tag{4.10}$$

Proof. Since $\Sigma(Q)$ consists of bounded number of (isolated) points $\{\eta^1, ..., \eta^{N_Q}\}$, one can represent the polynomial Q in the form

$$Q(\xi) = r(\xi) \prod_{j=1}^{N_Q} (\xi_1 - \kappa_j\, \xi_2^{\lambda_1/\lambda_2})^{m_j} \equiv r(\xi) \prod_{j=1}^{N_Q} q_j(\xi), \tag{4.11}$$

where m_j are natural numbers, $\kappa_j \neq 0$ are pairwise distinct real numbers, $q_j(\eta^j) = 0$ $(j = 1, ..., N_Q)$, $r \in C^\infty(R^2)$, $r(\xi) \neq 0$ for $\xi \in R^{n,0}$. □

We present an elementary statement which we will use in the proof of Theorem 4.3.

Lemma 4.3 *Let a, b, c, d, e be some positive numbers such that $a < c$, $e < c$, and $(a - e)/(c - e) \leq b/d$. Then for all $x \geq 1$ and $y \in [0, 1]$ $(x^a y^b)/(x^c y^d + x^e) \leq 1$.*

Proof. It is sufficient to prove this inequality for such a, b, c, d, e for which $b < d$ and $(a-e)/(c-e) = b/d$. Dividing both parts of this inequality by x^e and replacing x^{c-e} by x and y^d by y we get an equivalent inequality $(xy)^{b/d} \leq xy + 1$, which is obvious since $b < d$. Lemma 4.3 is proved.

Theorem 4.3 *Let all principal faces of the regular Newton's polygon $\Re(P)$ of polynomial $P \in \mathbb{I}_2$ be nondegenerate, except of a (unique) completely regular one - dimensional face $\Gamma \equiv \Re^1_{i_0}$ which is degenerate. Let λ (which is defined uniquely) be the outward normal of Γ, and $J = J(\Gamma, \lambda, \eta^j) = 1$ $(j = 1, ..., N_P)$.*

Then P is almost hypoellitic if and only if

$$d_0 - \Delta(\eta^j, \Gamma) \leq d_1 \;\; (j = 1, ..., N_P). \tag{4.12}$$

Proof. The necessity is contained in Lemma 4.1. To prove the sufficiency, we assume that for some multiindex $\nu \in \mathbb{N}_0^2$ and a sequence $\{\xi^s\}$: $|\xi^s| \to \infty$ and $|D^\nu P(\xi^s)|/|P(\xi^s)| \to \infty$, as $s \to \infty$. In this connection it is sufficiently to assume that $|\nu| = 1$ (see [20]). For definiteness one can assume that $\nu = (1, 0)$. Thus, let

$$|D_1 P(\xi^s)|/|P(\xi^s)| \to \infty \ \ as \ \ s \to \infty. \tag{4.13}$$

Proceeding as in the proof of Theorem 3.1, we get $\xi^s = \rho_s^{\kappa_1^s e^1 + \kappa_2^s e^2} = \rho_s^{\kappa_1^s \lambda} \rho_s^{\kappa_2^s e^2}$, where $\kappa_1^s \to 1$, $\kappa_2^s = o(\kappa_1^s)$ and $\rho_s^{\kappa_2^s e^2} \to \eta$ as $s \to \infty$.

In the case a) when $P^{i_r,k_r}(\eta) \neq 0$ we come to the contradiction as in the proof of Theorem 3.1. If $P^{i_r,k_r}(\eta) = 0$ we get the one-dimensional degenerate face Γ with outward normal $e^1 = \lambda$ and with the point $\eta \in \Sigma(\Gamma)$.

By the vector λ and point $\eta \in \Sigma(\Gamma)$ represent polynomials P and $D_1 P$ in the form (3.1) of sum of $\lambda-$homogeneous polynomials (see also Lemma 4.2 and representation (4.10))

$$P(\xi) = P_0(\xi) + P_1(\xi) + p(\xi) = r(\xi)\,[q(\xi)]^m + P_1(\xi) + p(\xi), \qquad 4.14)$$

$$D_1 P(\xi) = D_1 P_0(\xi) + D_1 P_1(\xi) + D_1 p(\xi) =$$

$$= D_1[r(\xi)\,[q(\xi)]^m] + D_1 P_1(\xi) + D_1 p(\xi). \tag{4.15}$$

First we prove that there are positive numbers C_0 and C_1 such that for all $s \in N$

$$|D_1 P_0(\xi^s)| \le C_0[|P_0(\xi^s)| + |P_1(\xi^s)|]; \ \ |D_1 P_1(\xi^s)| \le C_1 |P_1(\xi^s)|. \tag{4.16}$$

Let the number s_0 be chosen in such a way that $\tau^s \equiv \rho_s^{\kappa_2^s e^2} \in U(\eta)$ for all $s \ge s_0$. Then

$$P_0(\xi^s) = \rho_s^{\kappa_1^s d_0}\, P_0(\tau^s) = \rho_s^{\kappa_1^s d_0}\, r(\tau^s)\,[q(\tau^s)]^m; \ \ P_1(\xi^s) = \rho_s^{\kappa_1^s d_1} P_1(\tau^s). \tag{4.17}$$

For the polynomials $D_1 P_0$ and $D_1 P_1$, respectively

$$D_1 P_0(\xi^s) = \rho_s^{\kappa_1^s (d_0 - \lambda_1}\, \{m[q(\tau^s)]^{m-1} D_1 q(\tau^s)\, r(\tau^s) + [q(\tau^s)]^m D_1 r(\tau^s)\}, \tag{4.18}$$

$$D_1 P_1(\xi^s) = \rho_s^{\kappa_1^s (d_1 - \lambda_1)} D_1 P_1(\tau^s). \tag{4.19}$$

Since $q(\tau^s) \to q(\eta)$ as $s \to \infty$ and $r(\eta)\, P_1(\eta) \neq 0,$ it follows from (4.16) - (4.19) that with some positive numbers C_2 and C_3

$$|P_0(\xi^s)| \geq C_2\, \rho_s^{\kappa_1^s d_0}\, |q(\tau^s)|^m; \;\; |P_1(\xi^s)| \geq C_2\, \rho_s^{\kappa_1^s d_1}, \tag{4.20}$$

$$|D_1 P_0(\xi^s)| \leq C_3\, \rho_s^{\kappa_1^s (d_0-\lambda_1)}\, |q(\tau^s)|^{m-1}; \;\; |D_1 P_1(\xi^s)| \leq C_3\, \rho_s^{\kappa_1^s (d_1-\lambda_1)}. \tag{4.21}$$

If $m = 1,$ then $\Delta(\eta, P_0) = \min\{\lambda_1, \lambda_2\}$ and the condition (4.12) assumes the form $d_0 - \min\{\lambda_1, \lambda_2\} \leq d_1$. Therefore $d_0 - \lambda_1 \leq d_1$ and since $\kappa_1^s \to 1$ as $s \to \infty,$ the estimates (4.16) lead immediately from (4.20) - (4.21). Thus, in the sequel we assume that $m > 1$.

Reasoning as above, in this case we get $\Delta(\eta, P_0) = m\, \min\{\lambda_1, \lambda_2\} \leq m\, \lambda_1$ and the condition (4.12) assumes the form

$$d_0 - m\, \lambda_1 \leq m\, \min\{\lambda_1, \lambda_2\} \leq d_1.$$

By the definition of the polynomial $p(\xi)$ it follows that

$$|p(\xi^s)| = o(\rho_s^{\kappa_1^s\, d_1}); \;\; |D_1 p(\xi^s)| = o(\rho_s^{\kappa_1^s\, (d_1-\lambda_1)}) \;\; as \;\; s \to \infty. \tag{4.22}$$

It follows from (4.20) - (4.22) that to prove the inequality (4.16) it is sufficient to show that there is a constant $C_4 > 0$ such that for sufficiently large s

$$\rho_s^{\kappa_1^s (d_0-\lambda_1)} |q(\tau^s)|^{m-1} \leq C_4 [\rho_s^{\kappa_1^s d_0} |q(\tau^s)|^m + \rho_s^{\kappa_1^s d_1}]. \tag{4.23}$$

Applying Lemma 4.3 for $x_s = \rho_s^{\kappa_1^s},$ $y_s = |q(\tau^s)|,$ $a = (d_0 - \lambda_1),$ $b = m - 1,$ $c = d_0,$ $d = m,$ $e = d_1$ we get estimate (4.23) for sufficiently large s.

Note that the conditions of Lemma 4.3 are fulfilled since $\kappa_1^s \to 1$ $\rho_s \to \infty,$ $|q(\tau^s)| \to 0,$ as $s \to \infty,$ i.e. $x_s \to \infty,$ $y_s \to 0$ as $s \to \infty$ and

$$\frac{a-e}{c-e} = \frac{d_0 - d_1 - \lambda_1}{d_0 - d_1} \leq \frac{m-1}{m} = \frac{b}{d}.$$

Since $P \in I_2,$ $P_j(\xi^s) \geq 0$ $(j = 0, 1)$ for sufficiently larges $s,$ and the relation (4.16) lead to contradiction with (4.13). □

Let the multianisotropic weighted Sobolev space $W^{\Re}_{p,\delta} = W^{\Re}_{p,\delta}(R^n)$ be defined as in subsection 2.2.A, $k \in \mathbb{N},$ $\Re_k = \{k\,\Re = \alpha \in \mathbb{N}_0^n : \alpha/k \in \mathbb{N}_0^n \cap \Re\}$ and

$$W^{\infty}_{p,\delta} = \bigcap_{k=1}^{\infty} W^{\Re_k}_{p,\delta}, \;\; N(P, \delta) = \{u \in W^{\Re}_{2,\delta} : \langle u, P(-D)\varphi \rangle = 0 \;\; \forall \varphi \in C_0^\infty\}.$$

Note that the set $W^{\infty}_{p,\delta}$ is a Frechet space where $W^{\infty}_{p,\delta} \subset C^{\infty}$.

Applying Theorem 2.2 from [21] it is proved

Theorem 4.4 *Let* $P \in \mathbb{I}_n$ *be a differential operator with constant coefficients and with the regular Newton polyhedron* $\Re$. *Operator* P *is almost hypoelliptic if and only if there exists a number* $\delta > 0$ *such that* $N(P,\delta) \subset W^{\infty}_{2,\delta}$.

Let $P(D)$ be a nondegenerate operator with regular Newton polyhedron $\Re = \Re(m, m_2)$, defined in subsection 2.2.B. It is easy to verify that $P(D)$ is almost hypoelliptic and simultaneously partially hypoelliptic with respect to the hyperplane $x'' = 0$. Let us introduce the multianisotropic weighted space $W^{\Re}_{p,\,\delta}(\Omega_\kappa)$ as in subsection 2.2.B, the set $\Re_l = \Re_l(m, m_2)$ for any $l \in \mathbb{N}$ as above and denote by

$$N(P,\kappa) = \{u; D^{(0,\alpha'')}u \in L_2(\Omega_\kappa),\ |\alpha''| \le m,\ P(D)u = 0 \ on\ \Omega_\kappa\}.$$

The following theorem is proved in [22] by applying Theorem 2.3.

Theorem 4.5 *Let* $\Re = \Re(m, m_2)$ *be the regular Newton polyhedron of a nondegenerate operator* $P(D)$. *Then there exists a number* $\kappa_0 > 0$ *such that for any* $\kappa \ge \kappa_o$ *a)* $N(P,\kappa) \subset W^{\Re_l}_{2,\,\delta}(\Omega_\kappa)$ $(l = 0, 1, ...)$ *and b)* $N(P,\kappa) \subset C^\infty(\Omega_\kappa)$.

Let $P(D) = P(D_1, D_2)$ be a nondegenerate operator with regular Newton polyhedron $\Re = \Re(l,m) = \{\upsilon \in N_0^2, \upsilon_1 \le l,\ \upsilon_1 + \upsilon_2 \le m\}$, considered in subsection 2.4.C. Let the rectangle $\prod = \prod(a) = \prod(a,1)$ and the multianisotropic weighted space $W^{\Re}_{p,\,g}(\prod) = W^{\Re(l,m)}_{p,\,g}(\prod(a))$ be defined also as in the subsection 2.4.C.

For $j = -1, 0, 1, 2, ...$ we denote $\Re_j = \{\upsilon \in N_0^2,\ \upsilon_1 \le l,\ |\upsilon| \le m + j\}$. It is obvious that $\Re_j$ is a regular quadrangle $(j = -1, 0, 1, 2, ...)$, where $\Re_{-1} = \{\upsilon \in N_0^2,\ \upsilon_1 \le l,\ \upsilon_1 + \upsilon_2 \le m - 1\}$.

Applying Theorem 2.5 the following theorem have been proved in [23]

Theorem 4.6 *Let* $P(D)$ *be a nondegenerate operator with the regular Newton quadrangle* $\Re$ *and the function* f *satisfies the following conditions*

$$D_2^j f\, g^{m+j} \in L_2\Big(\prod(a')\Big),\ \ \forall j \in N_0,\ \ \forall a' \in (0,a).$$

Then, any generalized solution $u = u(x,y) \in W^{\Re_{-1}}_{2,\,g,\,loc}(\prod(a))$ *of the equation* $P(D)u = f$ *is an infinitely differentiable function in* $\prod(a)$ *with respect to the wariable* $y \in (0,1)$.

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