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Exploring New General Integral Lower Bounds Depending on Four Functions

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Abstract. This article focuses on the determination of appropriate lower bounds for a general term defined as the sum of two specific integrals. This term has the property of depending on four functions, one of which is associated with the two integrals involved. Two theorems are established: one with monotonicity and sign assumptions on the functions considered, and another, more technical, with special primitive-like inequality assumptions on these functions. The connections, advantages and limitations of these assumptions are discussed in detail.

Key Words: Integrals, Lower Bounds, Monotonicity, Fubini Theorem, Primitive-like Inequality Assumptions
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1 Introduction

Integrals play a central role in both the theory and application of calculus. They are used to measure areas, volumes and quantities in a wide variety of contexts. As integral evaluations often do not have exact solutions, inequalities are crucial for estimation. See [1, 2, 4, 7, 16, 17] for an overview of this topic. In particular, lower bounds on integrals provide valuable insights into minimum values. This is important for understanding worst-case scenarios, ensuring system stability, and establishing baseline performance in optimization problems, to name a few. For this reason, recent research has focused on general integral inequalities to obtain suitable lower bounds. These results combine simplicity and originality with the ability to adapt to different mathematical frameworks, see, for example, [3, 5, 6, 8–15, 18].

This article makes a new contribution to the topic of general integral lower bounds. Specifically, given four functions, say f, g, h and k, where f, g and h are defined on an interval, say $[a, b]$, the aim is to establish appropriate lower bounds for the following general term, defined as the sum of two specific integrals:

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt,$$

where $k(g)$ is the composition function of k and g. From this sum expression, we can see that g is common to the two integrals, making them functionally related. This connection adds some originality to the study. Under certain assumptions, which are made to be as less restrictive as possible, we obtain lower bounds of the following form:

$$\alpha \int_a^b f(t)dt + \beta \int_a^b h(t)dt, \tag{1}$$

where α depends on some values of g and β depends on some values of $k(g)$. In fact, we focus on the cases $\alpha \in \{g(a), g(b)\}$ and $\beta \in \{k[g(a)], k[g(b)]\}$. Two theorems are given, with extensive and detailed proofs. The first theorem makes basic assumptions on the functions involved, namely monotonicity assumptions on g and $k(g)$, and sign assumptions on f and h. The second theorem is more original and innovative, and its statement and proof are more technical. It makes monotonicity assumptions on g and primitive-like inequality assumptions of the following forms: for any $x \in [a, b]$,

$$\int_x^b f(t)dt \ (\leq \text{ or } \geq)\ k'[g(x)] \int_a^x h(t)dt,$$

or

$$\int_a^x f(t)dt \ (\leq \text{ or } \geq)\ k'[g(x)] \int_x^b h(t)dt.$$

To the best of our knowledge, considering integration intervals with a kind of dual forms, i.e., $[x, b]$ and $[a, x]$, under a common assumption is a novel concept in the literature. This contrasts with the assumptions made in some established results, such as those in [3, 13–15]. While these primitive-like inequality assumptions are less direct than those formulated in our first theorem, they have the advantage of overcoming possible restrictive sign assumptions on f and h, as well as the monotonicity assumption on $k(g)$. Consequently, f, h and k can be chosen more arbitrarily, including functions exhibiting local extremes, discontinuities, oscillations, or other complex behavior. All of these aspects will be discussed in detail in this article.

The rest of the article is divided into two sections. Section 2 presents the main results, along with detailed proofs and a discussion of the considered assumptions. Section 3 provides a conclusion.

2 Main results

Two main theorems are presented one after the other. Throughout the article, it is assumed that the introduced integrals exist (which is not guaranteed a priori, especially if $a \to -\infty$ or $b \to +\infty$).

2.1 First theorem

The first theorem, which establishes lower bounds of the form described in Equation (1), is given below.

Theorem 1 *Let $(a,b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, and let functions $f, g, h : [a,b] \mapsto \mathbb{R}$ and $k : \mathbb{R} \mapsto \mathbb{R}$ be such that f and h are integrable.*
S1. *Let f be positive and g be non-decreasing. If h is positive and $k(g)$ is non-increasing or h is negative and $k(g)$ is non-decreasing, then*

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(a)\int_a^b f(t)dt + k[g(b)]\int_a^b h(t)dt.$$

The same statement holds when f is negative and g is non-increasing.
S2. *Let f be positive and g be non-increasing. If h is positive and $k(g)$ is non-decreasing or h is negative and $k(g)$ is non-increasing, then*

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(b)\int_a^b f(t)dt + k[g(a)]\int_a^b h(t)dt.$$

The same statement is true when f is negative and g is non-decreasing.

Proof. 1. Suppose f is positive and g is non-decreasing. Then we have

$$f(t)g(t) \geq f(t)g(a)$$

for any $t \in [a,b]$, which implies that

$$\int_a^b f(t)g(t)dt \geq \int_a^b [f(t)g(a)]\, dt = g(a)\int_a^b f(t)dt.$$

If f is negative and g is non-increasing, then $-f$ is positive and $-g$ is non-decreasing, and we also have

$$\int_a^b f(t)g(t)dt = \int_a^b [-f(t)][-g(t)]dt \geq [-g(a)]\int_a^b [-f(t)]dt$$
$$= g(a)\int_a^b f(t)dt.$$

Therefore, in both cases, we have

$$\int_a^b f(t)g(t)dt \geq g(a) \int_a^b f(t)dt. \tag{2}$$

Similarly, if h is positive and $k(g)$ is non-increasing, we have

$$h(t)k[g(t)] \geq h(t)k[g(b)]$$

for any $t \in [a, b]$, which implies that

$$\int_a^b h(t)k[g(t)]dt \geq \int_a^b \{h(t)k[g(b)]\} dt = k[g(b)] \int_a^b h(t)dt.$$

If h is negative and $k(g)$ is non-decreasing, then $-h$ is positive and $-k(g)$ is non-increasing, and we also have

$$\begin{aligned}\int_a^b h(t)k[g(t)]dt &= \int_a^b [-h(t)] \{-k[g(t)]\} dt \geq \{-k[g(b)]\} \int_a^b [-h(t)]dt \\ &= k[g(b)] \int_a^b h(t)dt.\end{aligned}$$

Therefore, we have

$$\int_a^b f(t)g(t)dt \geq k[g(b)] \int_a^b h(t)dt. \tag{3}$$

Summing the lower bounds in Equations (2) and (3) gives

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(a) \int_a^b f(t)dt + k[g(b)] \int_a^b h(t)dt.$$

2. Suppose f is positive and g is non-increasing. Then we have

$$f(t)g(t) \geq f(t)g(b)$$

for any $t \in [a, b]$, which implies that

$$\int_a^b f(t)g(t)dt \geq \int_a^b [f(t)g(b)] dt = g(b) \int_a^b f(t)dt.$$

If f is negative and g is non-decreasing, then $-f$ is positive and $-g$ is non-increasing, and we also have

$$\begin{aligned}&\int_a^b f(t)g(t)dt = \int_a^b [-f(t)][-g(t)]dt \geq [-g(b)] \int_a^b [-f(t)]dt \\ &= g(b) \int_a^b f(t)dt.\end{aligned}$$

Therefore, in both cases, we have

$$\int_a^b f(t)g(t)dt \geq g(b)\int_a^b f(t)dt. \tag{4}$$

Similarly, if h is positive and $k(g)$ is non-decreasing, we have

$$h(t)k[g(t)] \geq h(t)k[g(a)]$$

for any $t \in [a, b]$, which implies that

$$\int_a^b h(t)k[g(t)]dt \geq \int_a^b \{h(t)k[g(a)]\}\, dt = k[g(a)]\int_a^b h(t)dt.$$

If h is negative and $k(g)$ is non-increasing, then $-h$ is positive and $-k(g)$ is non-decreasing, and we also have

$$\begin{aligned}\int_a^b h(t)k[g(t)]dt &= \int_a^b [-h(t)]\, \{-k[g(t)]\}\, dt \geq \{-k[g(a)]\}\int_a^b [-h(t)]dt \\ &= k[g(a)]\int_a^b h(t)dt.\end{aligned}$$

Therefore, we have

$$\int_a^b f(t)g(t)dt \geq k[g(a)]\int_a^b h(t)dt. \tag{5}$$

Summing the lower bounds in Equations (4) and (5) gives

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(b)\int_a^b f(t)dt + k[g(a)]\int_a^b h(t)dt.$$

This concludes the proof. □

In the cases $a \to -\infty$ or $b \to +\infty$, for any function $\ell : [a, b] \mapsto \mathbb{R}$, we set $\ell(a) = \lim_{t\to-\infty} \ell(t)$ and $\ell(b) = \lim_{t\to+\infty} \ell(t)$. Eventually, the following complementary cases can be obtained.

Corollary 1 *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, and consider four functions $f, g, h : [a, b] \mapsto \mathbb{R}$ and $k : \mathbb{R} \mapsto \mathbb{R}$ such that f and h are integrable.*
S3. *Let f be positive and g be non-decreasing. If h is positive and $k(g)$ is non-decreasing or h is negative and $k(g)$ is non-increasing, then*

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(a)\int_a^b f(t)dt + k[g(a)]\int_a^b h(t)dt.$$

The same statement holds when f is negative and g is non-increasing.
S4. *Let f be positive and g be non-increasing. If h is positive and $k(g)$ is non-increasing or h is negative and $k(g)$ is non-decreasing, then*

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(b)\int_a^b f(t)dt + k[g(b)]\int_a^b h(t)dt.$$

The same statement is true when f is negative and g is non-decreasing.

We have focused mainly on **S1** and **S2** because of the common form of the lower bounds in Theorem 1 and the second theorem to come, i.e., Theorem 2. In these cases, the monotonicity of the composition function $k(g)$ can only be reformulated in terms of the monotonicity of k if the monotonicity of g is known. The details are given below.

- If g is non-decreasing, then assuming that k is non-decreasing implies that $k(g)$ is non-decreasing, or assuming that k is non-increasing implies that $k(g)$ is non-increasing.

- If g non-increasing, then assuming that k is non-decreasing implies that $k(g)$ is non-increasing, or assuming that k is non-increasing implies that $k(g)$ is non-decreasing.

The assumptions **S1** and **S2** have the advantage of being simple and can be checked almost directly. However, they are not satisfied for a wide range of functions, including those that vary in sign or are non-monotonic. In the next subsection, we show that the lower bounds established in Theorem 1 are robust under general and more technical assumptions on f, g, h and k. As mentioned in the introduction, these assumptions are reduced to monotonicity assumptions on g and primitive-like inequality assumptions involving the four functions.

2.2 Second theorem

The theorem below is the analogue of Theorem 1, with the same lower bounds, but different assumptions on f, g, h and k.

Theorem 2 *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, and let functions $f, g, h : [a, b] \mapsto \mathbb{R}$ and $k : \mathbb{R} \mapsto \mathbb{R}$ be such that f is integrable, g is differentiable, h is integrable, k is differentiable, and the following integrals exist:*

$$\int_a^b |g'(u)|du, \quad \int_a^b |f(t)|dt, \quad \int_a^b |g'(u)||k'[g(u)]|du, \quad \int_a^b |h(t)|dt.$$

1. Let one of the following two conditions be satisfied:
T1. *g is non-decreasing and for any $x \in [a, b]$,*

$$\int_x^b f(t)dt \geq k'[g(x)] \int_a^x h(t)dt; \tag{6}$$

T2. *g is non-increasing and for any $x \in [a, b]$,*

$$\int_x^b f(t)dt \leq k'[g(x)] \int_a^x h(t)dt. \tag{7}$$

Then

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(a) \int_a^b f(t)dt + k[g(b)] \int_a^b h(t)dt.$$

2. Let one of the following two conditions be satisfied:
T3. *g is non-decreasing and for any $x \in [a, b]$,*

$$\int_a^x f(t)dt \leq k'[g(x)] \int_x^b h(t)dt; \tag{8}$$

T4. *g is non-increasing and for any $x \in [a, b]$,*

$$\int_a^x f(t)dt \geq k'[g(x)] \int_x^b h(t)dt. \tag{9}$$

Then

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(b) \int_a^b f(t)dt + k[g(a)] \int_a^b h(t)dt.$$

Proof. 1. Due to the differentiability of g, the following decomposition holds:

$$g(t) = [g(t) - g(a)] + g(a) = \int_a^t g'(u)du + g(a).$$

Thus, we have

$$\begin{aligned} \int_a^b f(t)g(t)dt &= \int_a^b f(t) \left[\int_a^t g'(u)du + g(a) \right] dt \\ &= \int_a^b \int_a^t f(t)g'(u)dudt + g(a) \int_a^b f(t)dt. \end{aligned} \tag{10}$$

Since the integrals $\int_a^b |f(t)|dt$ and $\int_a^b |g'(u)|du$ exist, the integral

$$\int_a^b \int_a^b |f(t)||g'(u)|dudt$$

also exists, and we can change the order of integration using the Fubini theorem. This gives

$$\begin{aligned}
&\int_a^b \int_a^t f(t)g'(u)dudt + g(a)\int_a^b f(t)dt \\
&= \int_a^b \int_u^b f(t)g'(u)dtdu + g(a)\int_a^b f(t)dt \\
&= \int_a^b g'(u)\left[\int_u^b f(t)dt\right] du + g(a)\int_a^b f(t)dt. \qquad (11)
\end{aligned}$$

If $g'(u) \geq 0$ for any $u \in [a,b]$, using Equation (6) for any $u \in [a,b]$, we get

$$g'(u)\left[\int_u^b f(t)dt\right] \geq g'(u)\left\{k'[g(u)]\int_a^u h(t)dt\right\}.$$

The same inequality can be obtained using Equation (7) in the case when $g'(u) \leq 0$ for any $u \in [a,b]$. Therefore,

$$\begin{aligned}
&\int_a^b g'(u)\left[\int_u^b f(t)dt\right] du + g(a)\int_a^b f(t)dt \\
&\geq \int_a^b g'(u)\left\{k'[g(u)]\int_a^u h(t)dt\right\} du + g(a)\int_a^b f(t)dt \\
&= \int_a^b \int_a^u g'(u)k'[g(u)]h(t)dtdu + g(a)\int_a^b f(t)dt. \qquad (12)
\end{aligned}$$

Doing a change of order of integration by the Fubini theorem, which is valid because

$$\int_a^b \int_a^b |g'(u)||k'[g(u)]||h(t)|dtdu$$

exists since the integrals $\int_a^b |g'(u)||k'[g(u)]|du$ and $\int_a^b |h(t)|dt$ exist, we get

$$\begin{aligned}
&\int_a^b \int_a^u g'(u)k'[g(u)]h(t)dtdu + g(a)\int_a^b f(t)dt \\
&= \int_a^b \int_t^b g'(u)k'[g(u)]h(t)dudt + g(a)\int_a^b f(t)dt \\
&= \int_a^b h(t)\left\{\int_t^b g'(u)k'[g(u)]du\right\} dt + g(a)\int_a^b f(t)dt \\
&= \int_a^b h(t)\left\{k[g(b)] - k[g(t)]\right\} dt + g(a)\int_a^b f(t)dt \\
&= k[g(b)]\int_a^b h(t)dt - \int_a^b h(t)k[g(t)]dt + g(a)\int_a^b f(t)dt. \qquad (13)
\end{aligned}$$

Putting Equations (10), (11), (12) and (13) together, we obtain

$$\int_a^b f(t)g(t)dt \geq k[g(b)]\int_a^b h(t)dt - \int_a^b h(t)k[g(t)]dt + g(a)\int_a^b f(t)dt,$$

which can be rearranged as

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(a)\int_a^b f(t)dt + k[g(b)]\int_a^b h(t)dt.$$

2. Due to the differentiability of g, the following decomposition holds:

$$g(t) = g(b) - [g(b) - g(t)] = g(b) - \int_t^b g'(u)du.$$

Therefore, we have

$$\begin{aligned}\int_a^b f(t)g(t)dt &= \int_a^b f(t)\left[g(b) - \int_t^b g'(u)du\right]dt \\ &= g(b)\int_a^b f(t)dt - \int_a^b\int_t^b f(t)g'(u)dudt. \end{aligned} \tag{14}$$

Since

$$\int_a^b\int_a^b |f(t)||g'(u)|dudt$$

exists because the integrals $\int_a^b |f(t)|dt$ and $\int_a^b |g'(u)|du$ exist, we can change the order of integration by the Fubini theorem. Thus, we have

$$\begin{aligned}&= g(b)\int_a^b f(t)dt - \int_a^b\int_t^b f(t)g'(u)dudt \\ &= g(b)\int_a^b f(t)dt - \int_a^b\int_a^u f(t)g'(u)dtdu \\ &= g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[\int_a^u f(t)dt\right]du. \end{aligned} \tag{15}$$

If $g'(u) \geq 0$ for any $u \in [a, b]$, using Equation (8) for any $u \in [a, b]$, we get

$$g'(u)\left[\int_a^u f(t)dt\right] \leq g'(u)\left\{k'[g(u)]\int_u^b h(t)dt\right\}.$$

The same inequality can be obtained using Equation (9) in the case when $g'(u) \leq 0$ for any $u \in [a, b]$. Thus,

$$\begin{aligned}&g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[\int_a^u f(t)dt\right]du \\ &\geq g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[k'[g(u)]\int_u^b h(t)dt\right]du \\ &= g(b)\int_a^b f(t)dt - \int_a^b\int_u^b g'(u)k'[g(u)]h(t)dtdu. \end{aligned} \tag{16}$$

Doing the change of order of integration by the Fubini theorem, which is valid because

$$\int_a^b \int_a^b |g'(u)||k'[g(u)]||h(t)|dtdu$$

exists since the integrals $\int_a^b |g'(u)||k'[g(u)]|du$ and $\int_a^b |h(t)|dt$ exist, we get

$$\begin{aligned}
& g(b)\int_a^b f(t)dt - \int_a^b \int_u^b g'(u)k'[g(u)]h(t)dtdu \\
& = g(b)\int_a^b f(t)dt - \int_a^b \int_a^t g'(u)k'[g(u)]h(t)dudt \\
& = g(b)\int_a^b f(t)dt - \int_a^b h(t)\left[\int_a^t g'(u)k'[g(u)]du\right]dt \\
& = g(b)\int_a^b f(t)dt - \int_a^b h(t)\left\{k[g(t)] - k[g(a)]\right\}dt \\
& = g(b)\int_a^b f(t)dt - \int_a^b h(t)k[g(t)]dt + k[g(a)]\int_a^b h(t)dt. \qquad (17)
\end{aligned}$$

Putting Equations (14), (15), (16) and (17) together, we obtain

$$\int_a^b f(t)g(t)dt \geq g(b)\int_a^b f(t)dt - \int_a^b h(t)k[g(t)]dt + k[g(a)]\int_a^b h(t)dt,$$

which can be rearranged as

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq g(b)\int_a^b f(t)dt + k[g(a)]\int_a^b h(t)dt.$$

This concludes the proof. □

To the best of our knowledge, Theorem 2 is a new addition to the literature. In particular, considering the integration intervals $[x, b]$ and $[a, x]$ under a common assumption differs from some similar well-established results presented, for example, in [3, 13–15].

Under specific configurations, the restrictive assumptions in Theorem 1 can imply those in Theorem 2. Some remarks on these aspects are discussed below.

- If f and h are positive and k is non-increasing, then we have

$$\int_x^b f(t)dt \geq 0, \quad k'[g(x)] \leq 0, \quad \int_a^x h(t)dt \geq 0,$$

for any $x \in [a, b]$, which implies that

$$\int_x^b f(t)dt \geq 0 \geq k'[g(x)]\int_a^x h(t)dt,$$

for any $x \in [a, b]$. Thus, inequality in Equation (6) holds. Furthermore, if we assume that g is non-decreasing, the assumption **T1** in Theorem 2 is satisfied. This also corresponds to (f positive and g non-decreasing) and (h positive and $k(g)$ non-increasing), which is a subcase of assumption **S1** in Theorem 1 (since g is non-decreasing and k is non-increasing imply that $k(g)$ is non-increasing).

The assumption **T1** in Theorem 2 is also satisfied for (f positive and g non-decreasing) and (h negative and $k(g)$ non-decreasing), which is also a subcase of **S1** in Theorem 1.

- If f is negative, h is positive and k is non-decreasing, then we have

$$\int_x^b f(t)dt \leq 0, \quad k'[g(x)] \geq 0, \quad \int_a^x h(t)dt \geq 0,$$

for any $x \in [a, b]$, which implies that

$$\int_x^b f(t)dt \leq 0 \leq k'[g(x)] \int_a^x h(t)dt,$$

for any $x \in [a, b]$. The inequality in Equation (7) holds. Furthermore, if we assume that g is non-increasing, the assumption **T2** in Theorem 2 is satisfied. This also corresponds to (f negative and g non-increasing) and (h positive and $k(g)$ non-increasing), which is a subcase of assumption **S1** in Theorem 1 (since g is non-increasing and k is non-decreasing imply that $k(g)$ is non-increasing).

The assumption **T2** in Theorem 2 is also satisfied for (f negative and g non-increasing) and (h negative and $k(g)$ non-decreasing), which is also a subcase of **S1** in Theorem 1.

- If f is negative, h is positive and k is non-decreasing, then we have

$$\int_a^x f(t)dt \leq 0, \quad k'[g(x)] \geq 0, \quad \int_x^b h(t)dt \geq 0,$$

for any $x \in [a, b]$, which implies that

$$\int_a^x f(t)dt \leq 0 \leq k'[g(x)] \int_x^b h(t)dt,$$

for any $x \in [a, b]$. The inequality in Equation (8) holds. Furthermore, if we assume that g is non-decreasing, the assumption **T3** in Theorem 2 is satisfied. This also corresponds to (f negative and g non-decreasing) and (h positive and $k(g)$ non-decreasing), which is a subcase of assumption **S2** in Theorem 1.

The assumption **T3** in Theorem 2 is also satisfied for (f negative and g non-decreasing) and (h negative and $k(g)$ non-increasing), which is also a subcase of **S2** in Theorem 1.

- If f and h are positive and k is non-increasing, then we have

$$\int_a^x f(t)dt \geq 0, \quad k'[g(x)] \leq 0, \quad \int_x^b h(t)dt \geq 0,$$

for any $x \in [a, b]$, which implies that

$$\int_a^x f(t)dt \geq 0 \geq k'[g(x)] \int_x^b h(t)dt,$$

for any $x \in [a, b]$. The inequality in Equation (9) holds. Furthermore, if we assume that g is non-increasing, the assumption **T4** in Theorem 2 is satisfied. This also corresponds to (f positive and g non-increasing) and (h positive and $k(g)$ non-decreasing), which is a subcase of assumption **S2** in Theorem 1.

The assumption **T4** in Theorem 2 is also satisfied for (f positive and g non-increasing) and (h negative and $k(g)$ non-increasing), which is also a subcase of **S2** in Theorem 1.

These are particular examples of assumptions that are common to Theorems 1 and 2.

Naturally, there are many functions that do not satisfy **S1** and **S2** in Theorem 1 but satisfy **T1**, **T2**, **T3** or **T4** in Theorem 2. In particular, Theorem 2 can be applied to cases where the behavior of the functions is complex or irregular, such as oscillatory functions or those with variable signs in the integration domain. In a sense, it extends the applicability of the integral lower bounds established in Theorem 1 to a wider class of problems. This versatility highlights the importance of Theorem 2 in advancing the integral-based techniques needed in mathematical analysis and its applications.

3 Conclusion

We have shown relevant integral inequalities, with a focus on lower bounds depending on four functions f, g, h and k. These inequalities have the following general form:

$$\int_a^b f(t)g(t)dt + \int_a^b h(t)k[g(t)]dt \geq \alpha \int_a^b f(t)dt + \beta \int_a^b h(t)dt,$$

with $\alpha \in \{g(a), g(b)\}$ and $\beta \in \{k[g(a)], k[g(b)]\}$. The main theorem (i.e., Theorem 2) makes original primitive-like inequality assumptions, allowing the consideration of complex functions beyond those satisfying the basic monotonicity and sign assumptions. This flexibility has potential applications in various fields where integral inequalities are fundamental, such as mathematical analysis, probability theory, and functional inequalities. Future work could explore refining these inequalities under alternative conditions. Additionally, the developed technique could be extended to other integral inequalities involving non-standard function classes, or even multidimensional integrals.

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