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Uniqueness theorem for sequences of piecewise polynomial functions.[1]

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Abstract. In the paper sequences of piecewise polynomial functions are considered, where each of the function is the projection of subsequent ones. A reconstruction theorem is proved for such sequences converging in measure from its limit if the majorant of the sequence satisfies some condition.

Key Words: piecewise polynomials, uniqueness, majorant, AH-integral
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Introduction

It is well known that there are trigonometric series converging a.e. to 0 and having at least one non-zero coefficient (see e.g. [13, Chapter IX, Theorem 6.14]). One can easily construct such series by Haar, Walsh and Franklin systems.

In the papers [1], [3] uniqueness questions were considered for a.e. converging or summable trigonometric series. Clearly, such questions should be considered under some restrictions.

For Haar series, G.G. Gevorkyan in [5], proved in particular the following theorem

Theorem 1 *If the Haar series*

$$\sum_{n=1}^{\infty} a_n \chi_n(x) \tag{1}$$

converges a.e. to $f(x)$ *and*

$$\lim_{\lambda\to\infty} \lambda\mu\{x \in [0,1]; S^*(x) > \lambda\} = 0, \tag{2}$$

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where $S^*(x)$ is the majorant of partial sums of the series (1), then the coefficients of series (1) are reconstructed by the following formulas

$$a_n = \lim_{\lambda\to\infty} \int_0^1 [f(x)]_\lambda \chi_n(x)dx,$$

where

$$[f(x)]_\lambda = \begin{cases} f(x), & \text{for } |f(x)| \le \lambda \\ 0, & \text{for } |f(x)| > \lambda. \end{cases}$$

Afterwards this theorem was generalized by V. Kostin in [10] for generalized Haar series and by the author in [9] for generalized Haar series under weaker conditions.

Similar results on uniqueness were also obtained for the Franklin system (see [6],[7]).

Note that partial sums of the series (1) are piecewise constant. Here we are interested in generalization Theorem 1 for piecewise polynomial sequences with (2) replaced by a weaker condition as in [9].

1 Definitions and the main result.

In order to formulate the result let us give some necessary definitions.

Let $r \in \mathbb{N}$. Denote by $\mathcal{S}_n^{(r)}$ the space of piecewise polynomial functions whose restrictions on each $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ for $0 \le k \le 2^n - 1$, are polynomials of degree not exceeding r, i.e.

$$\mathcal{S}_n^{(r)} = \left\{ f; \deg(f|_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}) \le r \text{ for } 0 \le k \le 2^n - 1 \right\}.$$

Let $\mathcal{P}_n^{(r)} : L[0,1] \to \mathcal{S}_n^{(r)}$ be the orthogonal projection, i.e.

$$(f, g) = (\mathcal{P}_n^{(r)} f, g) \text{ for all } f \in L[0,1], g \in \mathcal{S}_n^{(r)}.$$

Let the sequence of functions $(S_n)_{n\ge0}$ satisfy $S_n \in \mathcal{S}_n^{(r)}$ for $n \ge 0$ and

$$\mathcal{P}_n^{(r)}(S_m) = S_n \text{ for } m \ge n. \tag{3}$$

Set

$$S^*(x) = \sup_n |S_n(x)|.$$

We denote by $\mathcal{D}$ the set of all dyadic intervals, i.e.

$$\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n, \quad \text{where} \quad \mathcal{D}_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right); 0 \le k \le 2^n - 1 \right\}.$$

We call an interval $I \in \mathcal{D}_n$ an interval of rank n and set $r(I) := n$.

Let functions $h_m(x),\ h_m : [0,1] \to R$ satisfy the following conditions:

$$\text{(i) } 0 \le h_1(x) \le h_2(x) \le \dots \le h_m(x) \le \dots, \ \lim_{m\to\infty} h_m(x) = \infty, \tag{4}$$

(ii) there exist a constant $C > 0$ and intervals $I_1^m, \dots, I_{n_m}^m \in \mathcal{D}$, so that $I_i^m \cap I_j^m = \emptyset, i \ne j$, $\cup_{k=1}^{n_m} I_k^m = [0,1)$, and

$$\sup_{x\in I_k^m} h_m(x) \le C \inf_{x\in I_k^m} h_m(x), \tag{5}$$

for any $m \in \mathbb{N},\ 1 \le k \le n_m$, and

$$\text{(iii) } \inf_{m,k} \int_{I_k^m} h_m(x)dx > 0. \tag{6}$$

In other words, for any function h_m the interval $[0,1]$ can be split into small dyadic intervals, so that the supremum and infimum of that function on each interval are comparable and integrals over that intervals are bigger than some positive constant.

Theorem 2 *Let the functions $h_m(x)$ satisfy conditions (4),(5),(6). If the sequence (S_n) satisfying (3) converges in measure to a function S and*

$$\lim_{m\to\infty} \int_{\{x\in[0,1]; S^*(x) > h_m(x)\}} h_m(x)dx = 0 \tag{7}$$

then for any $g \in \mathcal{S}_n^{(r)}$,

$$(S_n, g) = \lim_{m\to\infty} \int_0^1 [S(x)]_{h_m(x)}\, g(x)dx.$$

This theorem actually enables to recover the sequence (S_n) from its limit S under mentioned conditions. Generally speaking the limit may be not Lebesgue integrable.

2 Proof of the main theorem.

We need the following simple lemma.

Lemma 1 *Let P be a polynomial of degree not exceeding r on $[\alpha,\beta]$ and $l = \max_{t\in[\alpha,\beta]} |P(t)|$, then*

$$\mu\left\{t \in [\alpha,\beta]; |P(t)| > \frac{l}{2}\right\} \ge \frac{\beta-\alpha}{4r^2}.$$

Proof. According to Markov's inequality $|P'(t)| \leq \frac{2r^2}{\beta-\alpha} l$ for $t \in [\alpha,\beta]$. Therefore if $|P(t_0)| = l$, for some $t_0 \in [\alpha,\beta]$, then $|P(t) - P(t_0)| \leq l/2$, for any $t \in [t_0 - \frac{\beta-\alpha}{4r^2}, t_0 + \frac{\beta-\alpha}{4r^2}] \cap [\alpha,\beta]$. This yields the desired estimate. $\square$

Proof of Theorem 2. Without loss of generality we can assume that g is a polynomial of degree no more than r. Indeed, if the theorem is true for any such polynomial, then for any $g \in \mathcal{S}_n^{(r)}$ there exist polynomials P_k, $k = 1, \ldots, 2^n$ so that $g = \sum_{k=1}^{2^n} P_k \cdot \mathbb{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}$, hence applying the theorem for each $P_k \cdot \mathbb{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}$ we get

$$(S_n, g) = \sum_{k=1}^{2^n} (S_n, P_k \cdot \mathbb{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}) = \\ = \sum_{k=1}^{2^n} \lim_{m\to\infty} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} [S(x)]_{h_m(x)} P_k(x)dx = \lim_{m\to\infty} \int_0^1 [S(x)]_{h_m(x)} g(x)dx.$$

Denote

$$\lambda_k^m = \inf_{x\in I_k^m} h_m(x) \quad \text{and } \varepsilon_0 = \inf_{m,k} \lambda_k^m \mu(I_k^m).$$

It follows from (5), (6) that $\varepsilon_0 > 0$. Take $\varepsilon < \varepsilon_0/(4r^2)$. It follows from (7) that for sufficiently large m we have

$$\int_{E_m} h_m(x)dx < \varepsilon, \text{ where } E_m = \{x \in [0,1]; S^*(x) > h_m(x)\}. \tag{8}$$

Hence we get from (5)

$$\sum_{k=1}^{n_m} \lambda_k^m \mu\{x \in I_k^m, S^*(x) > C\lambda_k^m\} < \varepsilon. \tag{9}$$

Fix m and denote by k_0 the maximal rank of the intervals I_k^m, i.e.

$$k_0 = \max_{1\leq k\leq n_m} r(I_k^m).$$

Denote $\tilde{S}_{k_0}(x) = S_{r(I_k^m)}(x)$, for $x \in I_k^m$. It is not hard to check that $|\tilde{S}_{k_0}(x)| \leq 2C\lambda_k^m$, when $x \in I_k^m$. Indeed, assume, to the contrary, that there exist a $k', 1 \leq k' \leq n_m$, and a point $x_0 \in I_{k'}^m$ such that $|\tilde{S}_{k_0}(x_0)| > 2C\lambda_{k'}^m$. Then applying Lemma 1 we get

$$\mu\left\{t \in I_{k'}^m; |S^*(t)| > C\lambda_{k'}^m\right\} \geq \mu\left\{t \in I_{k'}^m; |\tilde{S}_{k_0}(t)| > C\lambda_{k'}^m\right\} \geq \frac{\mu(I_{k'}^m)}{4r^2}.$$

Therefore it follows from (9) and the definition of ε_0 that

$$4r^2\varepsilon > 4r^2\lambda_{k'}^m \mu\left\{t \in I_{k'}^m; |S^*(t)| > C\lambda_{k'}^m\right\} > \lambda_{k'}^m \mu(I_{k'}^m) \geq \varepsilon_0,$$

which contradicts to the choice of ε.

Let I_1^m be the union of the intervals I_1, I_2 of the rank $r(I_1^m)+1$. If $|S_{r(I_1^m)+1}(x)| \le 2C\lambda_1^m$ for any $x \in I_1^m$, then we will set $\tilde{S}_{k_0+1}(x) = S_{r(I_1^m)+1}(x)$ on I_1^m and call each of the intervals I_1, I_2 the 1st class intervals for $\tilde{S}_{k_0+1}(x)$. Otherwise we will set $\tilde{S}_{k_0+1}(x) = S_{r(I_1^m)}(x)$ on I_1^m, and call I_1^m the 2nd class interval for $\tilde{S}_{k_0+1}(x)$. Similarly we can define the class of intervals $I_2^m, \ldots, I_{n_m}^m$, and determine $\tilde{S}_{k_0+1}(x)$ on each of $I_2^m, \ldots, I_{n_m}^m$ intervals.

Assuming that $\tilde{S}_{k_0+l}(x)$ is defined, determine $\tilde{S}_{k_0+l+1}(x)$ as follows. The intervals of 2nd class for $\tilde{S}_{k_0+l}(x)$ will be intervals of 2nd class for $\tilde{S}_{k_0+l+1}(x)$ as well and let us set $\tilde{S}_{k_0+l+1}(x) = \tilde{S}_{k_0+l}(x)$ on these intervals. If I is an interval of 1st class for $\tilde{S}_{k_0+l}(x)$, then we act as follows. Let I be the union of intervals I_1, I_2 of the rank $r(I)+1$. Without loss of generality we can assume that $I \subset I_1^m$. If $S_{r(I)+1}(x) \le 2C\lambda_1^m$, for $x \in I$ then we will set $\tilde{S}_{k_0+l+1}(x) = S_{r(I)+1}(x)$ on I, and each of the intervals I_1, I_2 will be called interval of 1st class for $\tilde{S}_{k_0+l+1}(x)$. Otherwise we will call the interval I the 2nd class interval for $\tilde{S}_{k_0+l+1}(x)$, and set $\tilde{S}_{k_0+l+1}(x) = S_{r(I)}(x)$ for $x \in I$.

So the function $\tilde{S}_{k_0+l}(x)$ is a polynomial of degree not exceeding r on intervals $I_1, \ldots, I_t$ (generally speaking, the ranks of the intervals $I_s, s = 1, \ldots, t$ may vary depending on s) and

$$\tilde{S}_{k_0+l}(x) = S_{r(I_j)}(x) \text{ for } x \in I_j. \tag{10}$$

It follows from the definition of $\tilde{S}_{k_0+l}(x)$ that

$$|\tilde{S}_{k_0+l}(x)| \le 2C\lambda_k^m, \text{ for } x \in I_k^m. \tag{11}$$

Denote by $A_{k_0,l}$ the union of all intervals of 2nd class for $\tilde{S}_{k_0+l}(x)$, and let $A_{k_0,l}^k = A_{k_0,l} \cap I_k^m$. Let us prove

$$\mu(A_{k_0,l}^k) \le 8r^2 \mu\{x \in I_k^m; S^*(x) > C\lambda_k^m\}. \tag{12}$$

Note that the set $A_{k_0,l}^k$ is the union of all intervals of 2nd class for $\tilde{S}_{k_0+l}(x)$, which are subsets of I_k^m. Therefore each of these intervals I contains at least one interval J_I, such that $r(J_I) = r(I)+1$ and $|S_{r(J_I)}(x_0)| > 2C\lambda_k^m$, for some $x_0 \in J_I$. Therefore applying Lemma 1 we get

$$\mu\left\{t \in J_I; |S^*(t)| > C\lambda_k^m\right\} \ge \mu\left\{t \in J_I; |S_{r(J_I)}(t)| > C\lambda_k^m\right\} \ge \frac{\mu(J_I)}{4r^2}.$$

Clearly $\mu(I) = 2\mu(J_I)$, hence we obtain

$$\mu(A_{k_0,l}^k) = \sum \mu(I) \le 2\sum \mu(J_I) \le 8r^2 \mu\{x \in I_k^m; S^*(x) > C\lambda_k^m\}.$$

Let P be a polynomial of degree not exceeding r and $M = \max_{t\in[0,1]} |P(t)|$. Notice that it follows from (3) that $(S_0, P) = \sum_{k=1}^{n_m} \int_{I_k^m} \tilde{S}_{k_0+l}(x)P(x)dx$. Now let us estimate the following expression:

$$|(S_0, P) - ([S]_{h_m}, P)| \leq \sum_{k=1}^{n_m} \left| \int_{I_k^m} \tilde{S}_{k_0+l}(x)P(x)dx - \int_{I_k^m\setminus E_m} S(x)P(x)dx \right| +$$

$$+ \int_{E_m} h_m(x)|P(x)|dx \leq \sum_{k=1}^{n_m} \left| \int_{(I_k^m\setminus E_m)\cap A^c_{k_0,l}} \left(\tilde{S}_{k_0+l}(x) - S(x)\right) P(x)dx \right| +$$

$$+ \int_{E_m} \left|\tilde{S}_{k_0+l}(x)P(x)\right| dx + M \sum_{k=1}^{n_m} \int_{(I_k^m\setminus E_m)\cap A_{k_0,l}} \left(\left|\tilde{S}_{k_0+l}(x)\right| + |S(x)|\right) dx+$$

$$+ M \int_{E_m} h_m(x)dx = I_1 + I_2 + I_3 + I_4.$$

It follows from (11) and the definition of λ_k^m that

$$I_2 \leq 2CM \int_{E_m} h_m(x)dx = 2CI_4,$$

therefore we get from (8)

$$I_2 + I_4 < (2C + 1)M\varepsilon. \tag{13}$$

Since S_n converges in measure to S, we obtain for a.a. $x \in I_k^m\setminus E_m$

$$|S(x)| \leq S^*(x) \leq h_m(x) \leq C\lambda_k^m. \tag{14}$$

Hence we get from (11) and (12)

$$I_3 \leq 4CM \sum_{k=1}^{n_m} \lambda_k^m \mu(A^k_{k_0,l}) \leq 32CMr^2 \sum_{k=1}^{n_m} \lambda_k^m \mu\{x \in I_k^m; S^*(x) > C\lambda_k^m\},$$

and applying (9) we obtain the following estimate for I_3

$$I_3 \leq 32CMr^2\varepsilon. \tag{15}$$

It remains to estimate I_1. Since S_n converges in measure to S, there exists l_0, such that for any $l' > l_0$ we have

$$\mu\{x \in I_k^m, |S_{k_0+l'}(x) - S(x)| \geq \varepsilon\} < \frac{\varepsilon}{\lambda_k^m n_m}. \tag{16}$$

Let us choose l so that $r(I_k^m) + l > k_0 + l_0$, for any $k = 1, 2, \dots n_m$. Denote by

$$B^k_{k_0,l} = \{x \in I_k^m; \left|S_{r(I_k^m)+l}(x) - S(x)\right| \geq \varepsilon\}, \quad B_{k_0,l} = \cup_{k=1}^{n_m} B^k_{k_0,l}.$$

Note that it follows from (16) that $\mu(B^k_{k_0,l}) < \varepsilon/(\lambda^m_k n_m)$, hence, taking into account $\tilde{S}_{k_0+l}(x) = S_{r(I^m_k)+l}(x)$, for $x \in I^m_k \cap A^c_{k_0,l}$, and the inequalities (11), (14), we obtain

$$I_1 \leq M \sum_{k=1}^{n_m} \int_{((I^m_k \setminus E_m)\cap A^c_{k_0,l})\setminus B_{k_0,l}} \left|\tilde{S}_{k_0+l}(x) - S(x)\right| dx +$$
$$M \sum_{k=1}^{n_m} \int_{((I^m_k \setminus E_m)\cap A^c_{k_0,l})\cap B_{k_0,l}} \left(\left|\tilde{S}_{k_0+l}(x)\right| + |S(x)|\right) dx \leq$$
$$\leq M\varepsilon + \sum_{k=1}^{n_m} 4CM\lambda^m_k \frac{\varepsilon}{\lambda^m_k n_m}.$$

So we get $I_1 \leq (4C+1)M\varepsilon$. Combining this estimate with the estimates (13), (15), we get for sufficiently large m

$$|(S_0, P) - ([S]_{h_m}, P)| < (6C + 2 + 32Cr^2)M\varepsilon.$$

Theorem 2 is proved.

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