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Asymptotic estimates for the quasi-periodic interpolations

Lusine Poghosyan* and Arnak Poghosyan**

* *Institute of mathematics,*
National Academy of Sciences of Armenia,
Bagramian ave. 24/5, 0019 Yerevan, Armenia
lusine@instmath.sci.am

** *Institute of mathematics,*
National Academy of Sciences of Armenia,
Bagramian ave. 24/5, 0019 Yerevan, Armenia
arnak@instmath.sci.am

Abstract

We investigate the convergence of the quasi-periodic interpolation on the entire interval $[-1,1]$ in the L_2-norm and at the endpoints of the interval by the limit function behavior. In both cases we derive exact constants for the main terms of the asymptotic errors. The results of numerical experiments confirm theoretical estimates and show the behavior of the quasi-periodic interpolation for specific functions.

Key Words: Trigonometric interpolation, quasi-periodic interpolation, convergence acceleration, asymptotic estimates
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Introduction

We consider the problem of function reconstruction by its values on equidistant grid. It is well known (see [2]) that for a rather smooth on the real line 2-periodic function f the classical trigonometric interpolation

$$
\begin{aligned}
I_N(f,x) &= \sum_{n=-N}^{N} \check{f}_n e^{i\pi n x}, \\
\check{f}_n &= \frac{1}{2N+1} \sum_{k=-N}^{N} f\left(\frac{2k}{2N+1}\right) e^{-\frac{2i\pi nk}{2N+1}}
\end{aligned}
\tag{1}
$$

effectively solves the problem. Otherwise, if 2-periodic extension of f on the real line is discontinuous ($f(1) \neq f(-1)$) then the quality of interpolation near the endpoints is degraded by the Gibbs phenomenon (see [3]). Another approach which does not eliminate the Gibbs phenomenon but mitigates its effect was introduced and discussed in [4] called quasi-periodic interpolation. In [5]-[7] was investigated the convergence of quasi-periodic interpolation. Here we continue investigations started there and study the convergence of quasi-periodic interpolation in the L_2-norm and explore its behavior at the endpoints of the interval by means of the limit function.

Quasi-periodic interpolation is exact for "quasi-periodic" functions

$$e^{i\pi n\alpha x},\ n=-N,\ldots,N,\ \alpha=\frac{2N}{2N+m+1} \tag{2}$$

with the period $2/\alpha$. Therefore, when $N\to\infty$ then $\alpha\to 1$. Interesting feature of such interpolations is the possibility to interpolate functions on the grid

$$x_k=\frac{k}{N},\ k=-N,\ldots,N \tag{3}$$

which includes also the endpoints $x=\pm 1$ of the interval. Such interpolations are known as the "full-interpolations" (see [4]).

First we derive an explicit formula for its realization repeating for completeness the scheme presented in paper [7], then we investigate its L_2-convergence and explore its behavior at the endpoints by the limit function technique.

1 Analysis of the case $m=0$

In this section we analyze the case $m=0$ when quasi-periodic interpolation is exact for functions

$$e^{i\pi n\frac{2N}{2N+1}x},\ n=-N,\ldots,N. \tag{4}$$

Derivation of this case essentially differs from general case $m>0$ and can be performed without much efforts.

Denote by f_n the Fourier coefficient of f

$$f_n=\frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}dx.$$

Let $f\in C^q[-1,1]$. Taking into account that in the classical interpolation $I_N(f,x)$ (see (1)) only the points from interval $\left[-\frac{2N}{2N+1},\frac{2N}{2N+1}\right]$ are used, we consider a new function f^* by the following formula

$$f^*(x)=\begin{cases} f_\ell(x), & x\in\left[-1,-\frac{2N}{2N+1}\right), \\ f\left(\frac{2N+1}{2N}x\right), & x\in\left[-\frac{2N}{2N+1},\frac{2N}{2N+1}\right], \\ f_r(x), & x\in\left(\frac{2N}{2N+1},1\right], \end{cases} \tag{5}$$

where

$$f_\ell(x) = \sum_{j=0}^{q} \frac{f^{(j)}(-1)}{j!} \left(\frac{2N+1}{2N}x + 1\right)^j,$$

and

$$f_r(x) = \sum_{j=0}^{q} \frac{f^{(j)}(1)}{j!} \left(\frac{2N+1}{2N}x - 1\right)^j.$$

As a result we have $f^* \in C^q[-1,1]$.

Then, we apply the classical interpolation to f^* with further change of variable

$$x \to \frac{2N}{2N+1}x$$

deriving the quasi-periodic interpolation $I_{N,0}(f,x)$

$$\begin{aligned} I_{N,0}(f,x) &= \sum_{n=-N}^{N} \check{f}_n^* e^{i\pi n \frac{2N}{2N+1}x}, \\ \check{f}_n^* &= \frac{1}{2N+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-i\pi n \frac{2k}{2N+1}} \end{aligned} \tag{6}$$

with the error

$$R_{N,0}(f,x) = f(x) - I_{N,0}(f,x). \tag{7}$$

It is easy to verify that $I_{N,0}(f,x)$ is exact for system (4) and interpolates f on grid (3). The next theorem just shows that.

Theorem 1 *Let $f \in C[-1,1]$. Then $I_{N,0}(f,x)$ is an interpolation of f on grid* (3) *and is exact for system* (4).

Proof. Both statements can be proved straightforwardly. For the first statement we have

$$\begin{aligned} I_{N,0}\left(f, \frac{k}{N}\right) &= \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{t=-N}^{N} f\left(\frac{t}{N}\right) e^{i\pi n \frac{2(k-t)}{2N+1}} \\ &= \frac{1}{2N+1} \sum_{t=-N}^{N} f\left(\frac{t}{N}\right) \sum_{n=-N}^{N} e^{i\pi n \frac{2(k-t)}{2N+1}}. \end{aligned}$$

This proves the first part as

$$\frac{1}{2N+1} \sum_{n=-N}^{N} e^{i\pi n \frac{2(k-t)}{2N+1}} = \delta_{t,k}.$$

For the proof of the second statement we put $f(x) = e^{i\pi \ell \frac{2N}{2N+1}x}$ and note that

$$\check{f}_n = \frac{1}{2N+1} \sum_{t=-N}^{N} e^{i\pi t \frac{2(\ell - n)}{2N+1}} = \delta_{n,\ell}.$$

Then

$$I_{N,0}\left(e^{i\pi\ell\frac{2N}{2N+1}x},x\right)=\sum_{n=-N}^{N}\delta_{n,\ell}e^{i\pi n\frac{2N}{2N+1}x}=e^{i\pi\ell\frac{2N}{2N+1}x}$$

which completes the proof. □

Now we investigate the $L_2(-1,1)$ convergence of $I_{N,0}(f,x)$. First we need some lemmas. We denote

$$A_{ks}(f)=f^{(s)}(1)-(-1)^{k+s}f^{(s)}(-1),\ s=0,\dots,q. \tag{8}$$

Lemma 1 *Let $f^{(q)}\in AC[-1,1]$ for some $q\geq 0$ and*

$$f^{(k)}(-1)=f^{(k)}(1)=0,\ k=0,\dots,q-1. \tag{9}$$

Then the following estimate holds as $N\to\infty$ and $|n|>N$

$$f_n^*=\frac{(-1)^{n+1}}{(2N+1)N^q}\mu_{q,0}\left(\frac{2n}{2N+1}\right)+o(n^{-q-1}), \tag{10}$$

where

$$\mu_{q,0}(x)=\sum_{k=0}^{q}\frac{A_{kq}(f)}{2^{q-k}(q-k)!(i\pi x)^{k+1}}. \tag{11}$$

Proof. Taking into account the smoothness of f and definition of f^* (see (5)) , we write

$$\begin{aligned}f_n^*=\frac{(-1)^{n+1}}{2}\sum_{k=0}^{q}\frac{f_r^{(k)}(1)-f_\ell^{(k)}(-1)}{(i\pi n)^{k+1}}&+\frac{1}{2(i\pi n)^{q+1}}\int_{-\frac{2N}{2N+1}}^{\frac{2N}{2N+1}}f^{*(q+1)}(x)e^{-i\pi nx}dx\\&+\frac{1}{2(i\pi n)^{q+1}}\int_{-1}^{-\frac{2N}{2N+1}}f_\ell^{(q+1)}(x)e^{-i\pi nx}dx\\&+\frac{1}{2(i\pi n)^{q+1}}\int_{\frac{2N}{2N+1}}^{1}f_r^{(q+1)}(x)e^{-i\pi nx}dx.\end{aligned}$$

Then, we have

$$\begin{aligned}f_n^*=&\frac{(-1)^{n+1}}{2}\sum_{k=0}^{q}\frac{f_r^{(k)}(1)-f_\ell^{(k)}(-1)}{(i\pi n)^{k+1}}\\&+\left(\frac{2N+1}{2N}\right)^q\frac{1}{2(i\pi n)^{q+1}}\int_{-1}^{1}f^{(q+1)}(x)e^{-i\pi n\frac{2N}{2N+1}x}dx.\end{aligned} \tag{12}$$

In view of (5)

$$f_\ell^{(k)}(-1)=\left(\frac{2N+1}{2N}\right)^k\sum_{j=k}^{q}(-1)^{j-k}f^{(j)}(-1)\frac{1}{(j-k)!(2N)^{j-k}}$$

and

$$f_r^{(k)}(1)=\left(\frac{2N+1}{2N}\right)^k\sum_{j=k}^{q}f^{(j)}(1)\frac{1}{(j-k)!(2N)^{j-k}}.$$

Substituting these into (12), in view of the generalized Riemann-Lebesgue theorem ([1]), we get

$$f_n^* = \frac{(-1)^{n+1}}{2}\sum_{k=0}^{q}\frac{1}{(i\pi n)^{k+1}}\left(\frac{2N+1}{2N}\right)^k\sum_{j=k}^{q}\frac{A_{kj}(f)}{(j-k)!(2N)^{j-k}} + o\left(n^{-q-1}\right).$$

Taking into account (9), we obtain

$$f_n^* = \frac{(-1)^{n+1}}{2N+1}\frac{1}{N^q}\sum_{k=0}^{q}\frac{A_{kq}(f)}{(q-k)!2^{q-k}(i\pi)^{k+1}\left(\frac{2n}{2N+1}\right)^{k+1}} + o\left(n^{-q-1}\right). \tag{13}$$

This completes the proof. □

Lemma 2 *Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0,\ k = 0, ..., q-1. \tag{14}$$

Then the following estimate holds as $|n| \leq N$ and $N \to \infty$

$$\check{f}_n^* - f_n^* = \frac{(-1)^{n+1}}{(2N+1)N^q}\nu_{q,0}\left(\frac{2n}{2N+1}\right) + o\left(N^{-q-1}\right), \tag{15}$$

where

$$\nu_{q,0}(x) = \sum_{k=0}^{q}\frac{A_{kq}(f)}{2^{q-k}(i\pi)^{k+1}(q-k)!}\sum_{s\neq 0}\frac{(-1)^s}{(2s+x)^{k+1}}. \tag{16}$$

Proof. We have

$$\begin{aligned}\check{f}_n^* &= \frac{1}{2N+1}\sum_{k=-N}^{N} f^*\left(\frac{2k}{2N+1}\right)e^{-i\pi n\frac{2k}{2N+1}} \\ &= \frac{1}{2N+1}\sum_{k=-N}^{N} e^{-i\pi n\frac{2k}{2N+1}}\sum_{s=-\infty}^{\infty} f_s^* e^{i\pi s\frac{2k}{2N+1}} \\ &= \sum_{s=-\infty}^{\infty} f_{n+s(2N+1)}^*.\end{aligned}$$

Then

$$\check{f}_n^* - f_n^* = \sum_{s\neq 0} f_{n+s(2N+1)}^*. \tag{17}$$

Application of Lemma 1 concludes the proof. □

The case $q = 0$ needs special attention.

Lemma 3 *Let $f' \in L_2[-1,1]$. Then the following estimate holds as $|n| \leq N$ and $N \to \infty$*

$$\check{f}_n^* - f_n^* = \frac{(-1)^{n+1}}{2N+1}\nu_{0,0}\left(\frac{2n}{2N+1}\right) + o\left(N^{-q-1}\right), \tag{18}$$

where $\nu_{0,0}$ is defined by (16).

Proof. According to Lemma 1

$$f_n^* = (-1)^{n+1}\frac{A_{00}(f)}{2i\pi n} + \frac{1}{2i\pi n}c_{n,N}, \tag{19}$$

where

$$c_{n,N} = \int_{-1}^{1} f^{*'}(x)e^{-i\pi n x}dx.$$

According to Parseval's identity and definition of f^* (see (5)) we have

$$\sum_{n=-\infty}^{\infty} |c_{n,N}|^2 = \int_{-1}^{1} \left|f^{*'}(x)\right|^2 dx = \frac{2N+1}{2N}\int_{-1}^{1} |f'(x)|^2 dx. \tag{20}$$

This completes the proof in view of (17) and (19). □

The next theorem describes the behavior of the quasi-periodic interpolation in the $L_2(-1,1)$ norm.

Theorem 2 *Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, ..., q-1.$$

Then the following estimate holds

$$\lim_{N\to\infty} N^{q+\frac{1}{2}}||R_{N,0}(f,x)||_{L_2(-1,1)} = c_{0,q}(f), \tag{21}$$

where

$$\begin{aligned} c_{0,q}(f) = \Bigg(&-\frac{1}{8}\int_{-1}^{1}\left|\int_{-1}^{1}\nu_{q,0}(h)e^{i\pi x h/2}dh - \int_{|h|>1}\mu_{q,0}(h)e^{i\pi x h/2}dh\right|^2 dx \\ &+\frac{1}{2}\int_{-1}^{1}|\nu_{q,0}(x)|^2 dx + \frac{1}{2}\int_{|x|>1}|\mu_{q,0}(x)|^2 dx\Bigg)^{\frac{1}{2}}, \end{aligned} \tag{22}$$

and functions $\mu_{q,0}$ and $\nu_{q,0}$ are defined in Lemmas 1 and 2.

Proof. We divide $R_{N,0}(f,x)$ (see (7)) into three parts

$$\begin{aligned} ||R_{N,0}(f,x)||^2_{L_2(-1,1)} &= \int_{-1}^{1}|R_{N,0}(f,x)|^2 dx = \frac{2N+1}{2N}\int_{-\frac{2N}{2N+1}}^{\frac{2N}{2N+1}}\left|R_{N,0}\left(f,\frac{2N+1}{2N}x\right)\right|^2 dx \\ &= \frac{2N+1}{2N}\int_{-1}^{1}\left|R_{N,0}\left(f,\frac{2N+1}{2N}x\right)\right|^2 dx \\ &- \frac{2N+1}{2N}\int_{\frac{2N}{2N+1}}^{1}\left|R_{N,0}\left(f,\frac{2N+1}{2N}x\right)\right|^2 dx \\ &- \frac{2N+1}{2N}\int_{-1}^{-\frac{2N}{2N+1}}\left|R_{N,0}\left(f,\frac{2N+1}{2N}x\right)\right|^2 dx \\ &= I_1 - I_2 - I_3, \end{aligned}$$

where

$$I_1 = \frac{2N+1}{2N} \int_{-1}^{1} \left| R_{N,0}\left(f, \frac{2N+1}{2N} x\right)\right|^2 dx,$$

$$I_2 = \frac{1}{2N} \int_{0}^{1} \left| R_{N,0}\left(f, \frac{2N+1}{2N} - \frac{x}{2N}\right)\right|^2 dx,$$

and

$$I_3 = \frac{1}{2N} \int_{0}^{1} \left| R_{N,0}\left(f, \frac{x}{2N} - \frac{2N+1}{2N}\right)\right|^2 dx.$$

First we estimate I_1. We have

$$R_{N,0}(f,x) = \sum_{n=-N}^{N} \left(f_n^* - \check{f}_n^*\right) e^{i\pi n \frac{2N}{2N+1} x} + \sum_{|n|>N} f_n^* e^{i\pi n \frac{2N}{2N+1} x}. \tag{23}$$

Therefore

$$I_1 = \frac{2N+1}{N} \sum_{n=-N}^{N} \left| f_n^* - \check{f}_n^* \right|^2 + \frac{2N+1}{N} \sum_{|n|>N} |f_n^*|^2.$$

In view of Lemmas 1 and 2 we obtain

$$I_1 = \frac{1}{(2N+1)N^{2q+1}} \sum_{n=-N}^{N} \left| \nu_{q,0}\left(\frac{2n}{2N+1}\right)\right|^2 + \frac{1}{(2N+1)N^{2q+1}} \sum_{|n|>N} \left| \mu_{q,0}\left(\frac{2n}{2N+1}\right)\right|^2 + o(N^{-2q-1}), \ \ N \to \infty.$$

Tending N to infinity and replacing the sums by the corresponding integrals we get

$$\lim_{N\to\infty} N^{2q+1} I_1 = \frac{1}{2}\int_{-1}^{1} |\nu_{q,0}(x)|^2 dx + \frac{1}{2}\int_{|x|>1} |\mu_{q,0}(x)|^2 dx.$$

Now we estimate I_2. From (23) we get

$$R_{N,0}\left(f, \frac{2N+1}{2N} - \frac{x}{2N}\right) = \sum_{n=-N}^{N} (-1)^n \left(f_n^* - \check{f}_n^*\right) e^{-\frac{i\pi n x}{2N+1}} + \sum_{|n|>N} (-1)^n f_n^* e^{-\frac{i\pi n x}{2N+1}}.$$

According Lemmas 1 and 2 we derive

$$R_{N,0}\left(f, \frac{2N+1}{2N} - \frac{x}{2N}\right) = \frac{1}{(2N+1)N^q} \sum_{n=-N}^{N} \nu_{q,0}\left(\frac{2n}{2N+1}\right) e^{-i\pi x \frac{n}{2N+1}} - \frac{1}{(2N+1)N^q} \sum_{|n|>N} \mu_{q,0}\left(\frac{2n}{2N+1}\right) e^{-i\pi x \frac{n}{2N+1}} + o(N^{-q}), \ N \to \infty.$$

Tending N to infinity and replacing the sums by the corresponding integrals, we derive

$$\lim_{N\to\infty} N^q R_{N,0}\left(f, \frac{2N+1}{2N} - \frac{x}{2N}\right) = \frac{1}{2}\int_{-1}^{1} \nu_{q,0}(h) e^{-i\pi x h/2} dh - \frac{1}{2}\int_{|h|>1} \mu_{q,0}(h) e^{-i\pi x h/2} dh.$$

Hence

$$\lim_{N\to\infty} N^{2q+1} I_2 = \frac{1}{8}\int_0^1 \left| \int_{-1}^{1} \nu_{q,0}(h) e^{-i\pi x h/2} dh - \int_{|h|>1} \mu_{q,0}(h) e^{-i\pi x h/2} dh \right|^2 dx.$$

Similarly

$$\lim_{N\to\infty} N^{2q+1} I_3 = \frac{1}{8}\int_0^1 \left| \int_{-1}^{1} \nu_{q,0}(h) e^{i\pi x h/2} dh - \int_{|h|>1} \mu_{q,0}(h) e^{i\pi x h/2} dh \right|^2 dx,$$

which completes the proof. □

The next theorem is analog of the previous one for $q = 0$.

Theorem 3 *Let $f' \in L_2[-1,1]$. Then the following estimate holds*

$$\lim_{N\to\infty} N^{\frac{1}{2}} ||R_{N,0}(f,x)||_{L_2(-1,1)} = c_{0,0}(f), \tag{24}$$

where

$$c_{0,0,}(f) = \left(-\frac{1}{8}\int_{-1}^{1} \left| \int_{-1}^{1} \nu_{0,0}(h) e^{i\pi x h/2} dh - \int_{|h|>1} \mu_{0,0}(h) e^{i\pi x h/2} dh \right|^2 dx + \frac{1}{2}\int_{-1}^{1} |\nu_{0,0}(x)|^2 dx + \frac{1}{2}\int_{|x|>1} |\mu_{0,0}(x)|^2 dx \right)^{\frac{1}{2}}, \tag{25}$$

and functions $\mu_{0,0}(x)$ and $\nu_{0,0}(x)$ are defined in Lemmas 1 and 2.

Proof. The proof is similar to the one of Theorem 2. □

Now we investigate the behavior of the quasi-periodic interpolation at the endpoints of the interval by means of the limit function.

Theorem 4 *Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \; k = 0, ..., q-1.$$

Then the following estimate holds

$$\lim_{N\to\infty} N^q R_{N,0}\left(f, 1 - \frac{h}{N}\right) = \ell_{q,0}(h), \tag{26}$$

where

$$\ell_{q,0}(h) = \frac{1}{2}\int_{-1}^{1} \nu_{q,0}(x) e^{i\pi(-1-2h)x/2} dx - \frac{1}{2}\int_{|x|>1} \mu_{q,0}(x) e^{i\pi(-1-2h)x/2} dx$$

and functions $\mu_{q,0}$ *and* $\nu_{q,0}$ *are defined in Lemmas 1 and 2.*

Proof. From definition of $R_{N,0}(f,x)$ we have

$$R_{N,0}(f, 1-h/N) = \sum_{|n|>N} (-1)^n f_n^* e^{\frac{i\pi n}{2N+1}(-1-2h)} + \sum_{n=-N}^{N} (-1)^n (f_n^* - \check{f}_n^*) e^{\frac{i\pi n}{2N+1}(-1-2h)}. \tag{27}$$

Taking into account Lemmas 1 and 2

$$N^q R_{N,0}(f, 1-h/N) = \frac{1}{(2N+1)} \sum_{n=-N}^{N} \nu_{q,0}\left(\frac{2n}{2N+1}\right) e^{\frac{i\pi n}{2N+1}(-1-2h)} - \frac{1}{(2N+1)} \sum_{|n|>N} \mu_{q,0}\left(\frac{2n}{2N+1}\right) e^{\frac{i\pi n}{2N+1}(-1-2h)} + o(1), \ N \to \infty. \tag{28}$$

Tending N to infinity and replacing the sums by corresponding integrals we get the statement of the theorem. □

For $q = 0$ we have the following result.

Theorem 5 *Let* $f' \in L_2[-1,1]$. *Then the following estimate holds*

$$\lim_{N\to\infty} R_{N,0}\left(f, 1 - \frac{h}{N}\right) = \ell_{0,0}(h),$$

where

$$\ell_{0,0}(h) = \frac{1}{2}\int_{-1}^{1} \nu_{0,0}(x) e^{i\pi(-1-2h)x/2} dx - \frac{1}{2}\int_{|x|>1} \mu_{0,0}(x) e^{i\pi(-1-2h)x/2} dx$$

and functions $\mu_{0,0}$ *and* $\nu_{0,0}$ *are defined in Lemmas 1 and 2.*

Proof. The proof is similar to the previous one. □

2 Analysis of the case $m \geq 1$

In this section we investigate the quasi-periodic interpolation $I_{N,m}(f,x)$ in general case $m > 0$, which is exact for the system (2) and interpolates a function on grid (3). For completeness we show derivation of the interpolation and repeat the procedure described in [7].

Consider the following formula

$$I_{N,m}(f,x) = \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) a_k(x), \quad x \in [-1,1] \tag{29}$$

with unknowns $a_k(x)$.

As (29) is exact for the system (2) we get the following system of equations for determination of the unknowns

$$e^{\frac{2i\pi\ell Nx}{2N+m+1}} = \sum_{k=-N}^{N} e^{\frac{2i\pi\ell k}{2N+m+1}} a_k(x), \quad |\ell| \leq N. \tag{30}$$

For formally solution of (30) we add some new unknowns and equations getting the new system

$$e^{\frac{2i\pi\ell Nx}{2N+m+1}} = \sum_{k=-N}^{N+m} e^{\frac{2i\pi\ell k}{2N+m+1}} a_k^*(x) + \varepsilon_\ell(x), \quad \ell = -N, ..., N+m, \tag{31}$$

where

$$\begin{aligned} &a_k^*(x) = a_k(x), \ |k| \leq N, \\ &a_k^*(x) = 0, \ k = N+1, ..., N+m, \\ &\varepsilon_\ell(x) = 0, \ \ |\ell| \leq N. \end{aligned} \tag{32}$$

We multiply the both sides of equation (31) by $e^{-\frac{2i\pi\ell s}{2N+m+1}}$ and sum over ℓ

$$\sum_{\ell=-N}^{N+m} e^{\frac{2i\pi\ell(Nx-s)}{2N+m+1}} = \sum_{\ell=-N}^{N+m}\sum_{k=-N}^{N+m} e^{\frac{2i\pi\ell(k-s)}{2N+m+1}} a_k^*(x) + \sum_{\ell=N+1}^{N+m} e^{-\frac{2i\pi\ell s}{2N+m+1}} \varepsilon_\ell(x).$$

Then

$$a_s^*(x) = \frac{1}{2N+m+1}\left(\sum_{\ell=-N}^{N+m} e^{\frac{2i\pi\ell(Nx-s)}{2N+m+1}} - \sum_{\ell=N+1}^{N+m} e^{-\frac{2i\pi\ell s}{2N+m+1}} \varepsilon_\ell(x)\right). \tag{33}$$

Applying conditions (32), we derive the following system with the Vandermonde matrix for determination of $\varepsilon_\ell(x)$

$$\sum_{\ell=N+1}^{N+m} e^{-\frac{2i\pi\ell s}{2N+m+1}} \left(\varepsilon_\ell(x) - e^{\frac{2i\pi\ell Nx}{2N+m+1}}\right) = \sum_{t=-N}^{N} e^{\frac{2i\pi t(Nx-s)}{2N+m+1}}, \quad s = N+1, ..., N+m. \tag{34}$$

After some transformations, we obtain

$$\sum_{\ell=1}^{m} v_{s+1,\ell}\tilde{\varepsilon}_\ell(x) = \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m)}{2N+m+1}}, \quad s = 0, ..., m-1,$$

where

$$\tilde{\varepsilon}_\ell(x) = e^{-\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \left(\varepsilon_{\ell+N}(x) - e^{\frac{2i\pi(\ell+N)Nx}{2N+m+1}} \right),$$

and

$$v_{s,\ell} = \alpha_\ell^{s-1}, \ \ \alpha_\ell = e^{\frac{2i\pi(\ell+N)}{2N+m+1}}.$$

Following [8](see also [9], [10]) where the explicit form of the inverse of the Vandermonde matrix was constructed, we derive

$$v_{s,\ell}^{-1} = -\frac{1}{\alpha_s^\ell \prod_{i=1,i\neq s}^m (\alpha_s - \alpha_i)} \sum_{j=0}^{\ell-1} \gamma_j \alpha_s^j, \ \ \ell, s = 1, ..., m,$$

where γ_j are the coefficients of the polynomial

$$\prod_{i=1}^m (x - \alpha_i) = \sum_{i=0}^m \gamma_i x^i.$$

Hence the solution of (34) can be written explicitly

$$\varepsilon_\ell(x) = e^{\frac{2i\pi\ell Nx}{2N+m+1}} + e^{\frac{2i\pi\ell(N+m)}{2N+m+1}} \sum_{s=0}^{m-1} v_{\ell-N,s+1}^{-1} \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m)}{2N+m+1}}, \ \ \ell = N+1, ..., N+m.$$

Substituting $\varepsilon_\ell(x)$ into (33), we get

$$\begin{aligned} a_k(x) = \frac{1}{2N+m+1} & \left(\sum_{\ell=-N}^{N} e^{\frac{2i\pi\ell Nx}{2N+m+1}} e^{-\frac{2i\pi\ell k}{2N+m+1}} - \sum_{\ell=N+1}^{N+m} e^{\frac{2i\pi\ell(N+m)}{2N+m+1}} e^{-\frac{2i\pi\ell k}{2N+m+1}} \right. \\ & \left. \times \sum_{s=0}^{m-1} v_{\ell-N,s+1}^{-1} \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m)}{2N+m+1}} \right), \ \ k = -N, \ldots, N. \end{aligned} \tag{35}$$

Substituting this into (29), we derive

$$I_{N,m}(f, x) = \sum_{n=-N}^{N} F_{n,m} e^{\frac{2i\pi nNx}{2N+m+1}}, \tag{36}$$

where

$$F_{n,m} = \check{f}_{n,m} - \sum_{\ell=1}^{m} \theta_{n,\ell} \check{f}_{\ell+N,m}, \tag{37}$$

$$\check{f}_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-\frac{2i\pi nk}{2N+m+1}} \tag{38}$$

and

$$\theta_{n,\ell} = e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} e^{\frac{2i\pi n(s-N-m)}{2N+m+1}}, \tag{39}$$

with the error

$$R_{N,m}(f, x) = f(x) - I_{N,m}(f, x). \tag{40}$$

Note that in (37) the sum vanishes in case of $m = 0$, and $I_{N,m}(f,x)$ coincides with $I_{N,0}(f,x)$, defined by (6).

Let us check that $I_{N,m}(f,x)$ interpolates f.

Theorem 6 *Let $f \in C[-1,1]$. Then $I_{N,m}(f,x)$ is interpolation of f on grid* (3).

Proof. We will proof straightforwardly. Let $x = k/N$

$$\begin{aligned}
I_{N,m}\left(f, \frac{k}{N}\right) &= \sum_{n=-N}^{N} \left(\check{f}_{n,m} - \sum_{\ell=1}^{m} \theta_{n,\ell} \check{f}_{\ell+N,m} \right) e^{\frac{2i\pi nk}{2N+m+1}} \\
&= \sum_{n=-N}^{N} \left(\check{f}_{n,m} - \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} e^{\frac{2i\pi n(s-N-m)}{2N+m+1}} \check{f}_{\ell+N,m} \right) e^{\frac{2i\pi nk}{2N+m+1}} \\
&= \sum_{n=-N}^{N} \check{f}_{n,m} e^{\frac{2i\pi nk}{2N+m+1}} - \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \check{f}_{\ell+N,m} \sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} \sum_{n=-N}^{N} e^{\frac{2i\pi n(s-N-m+k)}{2N+m+1}}.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
\sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} \sum_{n=-N}^{N} e^{\frac{2i\pi n(s-N-m+k)}{2N+m+1}} &= -\sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} \sum_{t=1}^{m} e^{\frac{2i\pi(t+N)(s-N-m+k)}{2N+m+1}} \\
&= -\sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} \sum_{t=1}^{m} e^{\frac{i\pi(2t+2N)(k+s+N+1)}{2N+m+1}} \\
&= -\sum_{t=1}^{m} e^{\frac{2i\pi(t+N)(k+N+1)}{2N+m+1}} \sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} e^{\frac{2i\pi(t+N)s}{2N+m+1}} \\
&= -e^{\frac{2i\pi(l+N)(k+N+1)}{2N+m+1}}
\end{aligned}$$

we obtain

$$\begin{aligned}
I_{N,m}\left(f, \frac{k}{N}\right) &= \sum_{n=-N}^{N} \check{f}_{n,m} e^{\frac{2i\pi nk}{2N+m+1}} + \sum_{\ell=1}^{m} \check{f}_{\ell+N,m} e^{\frac{2i\pi(\ell+N)k}{2N+m+1}} \\
&= \frac{1}{2N+m+1} \sum_{t=-N}^{N} f\left(\frac{t}{N}\right) \left(\sum_{n=-N}^{N} e^{\frac{2i\pi n(k-t)}{2N+m+1}} + \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(k-t)}{2N+m+1}} \right) \\
&= \frac{1}{2N+m+1} \sum_{t=-N}^{N} f\left(\frac{t}{N}\right) \sum_{n=-N}^{N+m} e^{\frac{2i\pi n(k-t)}{2N+m+1}} \\
&= \sum_{t=-N}^{N} f\left(\frac{t}{N}\right) \delta_{k,t} \\
&= f\left(\frac{k}{N}\right).
\end{aligned}$$

□

First we investigate L_2-convergence. As in case $m=0$ we introduce the function f^*. Let $f \in C^{q+p}[-1,1]$. We put

$$f^*(x) = \begin{cases} f_\ell(x), & x \in \left[-1, -\frac{2N}{2N+m+1}\right), \\ f\left(\frac{2N+m+1}{2N}x\right), & x \in \left[-\frac{2N}{2N+m+1}, \frac{2N}{2N+m+1}\right], \\ f_r(x), & x \in \left(\frac{2N}{2N+m+1}, 1\right], \end{cases} \tag{41}$$

where

$$f_\ell(x) = \sum_{j=0}^{q+p} \frac{f^{(j)}(-1)}{j!} \left(\frac{2N+m+1}{2N}x + 1\right)^j,$$

and

$$f_r(x) = \sum_{j=0}^{q+p} \frac{f^{(j)}(1)}{j!} \left(\frac{2N+m+1}{2N}x - 1\right)^j.$$

Obviously $f^* \in C^{q+p}[-1,1]$.

We need some preliminary lemmas. Lemmas 4 and 5 were introduced in [7] without proofs. Here we present them.

Lemma 4 *Let $f^{(q+p)} \in AC[-1,1]$ for some $q,p \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, \dots, q-1. \tag{42}$$

Then the following estimate holds as $N \to \infty$ and $|n| > N$

$$f_n^* = \frac{(-1)^{n+1}}{2N+m+1} \sum_{j=q}^{q+p} \frac{1}{N^j} \mu_{j,m}\left(\frac{2n}{2N+m+1}\right) + o(n^{-q-p-1}), \tag{43}$$

where

$$\mu_{j,m}(x) = \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(j-k)!(i\pi x)^{k+1}}. \tag{44}$$

Proof. We proceed as in the proof of Lemma 1 and write

$$f_n^* = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q+p} \frac{f_r^{(k)}(1) - f_\ell^{(k)}(-1)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^{q+p+1}} \int_{-\frac{2N}{2N+m+1}}^{\frac{2N}{2N+m+1}} f^{*(q+p+1)}(x) e^{-i\pi n x} dx$$
$$+ \frac{1}{2(i\pi n)^{q+p+1}} \int_{-1}^{-\frac{2N}{2N+m+1}} f_\ell^{(q+p+1)}(x) e^{-i\pi n x} dx$$
$$+ \frac{1}{2(i\pi n)^{q+p+1}} \int_{\frac{2N}{2N+m+1}}^{1} f_r^{(q+p+1)}(x) e^{-i\pi n x} dx.$$

Then

$$f_n^* = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q+p} \frac{f_r^{(k)}(1) - f_\ell^{(k)}(-1)}{(i\pi n)^{k+1}}$$
$$+ \left(\frac{2N+m+1}{2N}\right)^{q+p} \frac{1}{2(i\pi n)^{q+p+1}} \int_{-1}^{1} f^{(q+p+1)}(x) e^{-i\pi n \frac{2N}{2N+m+1} x} dx, \tag{45}$$

where

$$f_\ell^{(k)}(-1)=\left(\frac{2N+m+1}{2N}\right)^k\sum_{j=k}^{q+p}(-1)^{j-k}f^j(-1)\frac{(m+1)^{j-k}}{(j-k)!(2N)^{j-k}}$$

and

$$f_r^{(k)}(1)=\left(\frac{2N+m+1}{2N}\right)^k\sum_{j=k}^{q+p}f^j(1)\frac{(m+1)^{j-k}}{(j-k)!(2N)^{j-k}}.$$

Substituting this into (45) we get

$$f_n^*=\frac{(-1)^{n+1}}{2}\sum_{k=0}^{q+p}\frac{1}{(i\pi n)^{k+1}}\left(\frac{2N+m+1}{2N}\right)^k\sum_{j=k}^{q+p}\frac{A_{kj}(f)(m+1)^{j-k}}{(j-k)!(2N)^{j-k}}+o(n^{-q-p-1}).$$

Now in view of conditions (42) we derive

$$f_n^*=\frac{(-1)^{n+1}}{2}\sum_{j=q}^{q+p}\sum_{k=0}^{j}\left(\frac{2N+m+1}{2N}\right)^k\frac{A_{kj}(f)(m+1)^{j-k}}{(j-k)!(2N)^{j-k}(i\pi n)^{k+1}}+o(n^{-q-p-1}) \tag{46}$$

which completes the proof. □

Let

$$\Phi_{k,m}(e^{i\pi x})=e^{\frac{i\pi}{2}(m-1)x}\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}}. \tag{47}$$

Lemma 5 *Let $f^{(q+m)}\in AC[-1,1]$ for some $q\ge 0$, $m\ge 1$ and*

$$f^{(k)}(-1)=f^{(k)}(1)=0,\ k=0,...,q-1.$$

Then the following estimate holds as $|n|\le N$ and $N\to\infty$

$$F_{n,m}-f_n^*=\frac{(-1)^{n+1}}{(2N+m+1)N^q}\nu_{q,m}\left(\frac{2n}{2N+m+1}\right)+o\left(N^{-q-1}\right), \tag{48}$$

where

$$\begin{aligned}\nu_{q,m}(x)&=\sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(i\pi)^{k+1}(q-k)!}\\&\times\left(\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}}-e^{-i\pi\frac{m-1}{2}x}\sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi_{k,m}^{(\tau)}(-1)\left(e^{i\pi x}+1\right)^\tau\right).\end{aligned} \tag{49}$$

Proof. First, let us show that

$$F_{n,m}=\sum_{r=-\infty}^{\infty}f^*_{n+r(2N+m+1)}-\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty}f^*_{N+\ell+r(2N+m+1)},\ -N\le n\le N+m. \tag{50}$$

We have

$$F_{n,m}=\frac{1}{2N+m+1}\sum_{k=-N}^{N}f\left(\frac{k}{N}\right)\left(e^{-\frac{2i\pi nk}{2N+m+1}}-\sum_{\ell=1}^{m}\theta_{n,\ell}e^{-\frac{2i\pi(N+\ell)k}{2N+m+1}}\right).$$

Then

$$f\left(\frac{k}{N}\right) = f^*\left(\frac{2k}{2N+m+1}\right) = \sum_{t=-\infty}^{\infty} f_t^* e^{\frac{2i\pi kt}{2N+m+1}}.$$

Hence

$$F_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} \sum_{t=-\infty}^{\infty} f_t^* \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right).$$

First, we take into account that

$$\frac{1}{2N+m+1} \sum_{k=-N}^{N+m} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} = \delta_{t,N+\ell}, \ \ -N \le t \le N+m$$

and write

$$\begin{aligned} F_{n,m} &= \frac{1}{2N+m+1} \sum_{k=-N}^{N+m} \sum_{t=-N}^{N+m} \sum_{r=-\infty}^{\infty} f^*_{t+r(2N+m+1)} \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right) \\ &\qquad - \frac{1}{2N+m+1} \sum_{k=N+1}^{N+m} \sum_{t=-\infty}^{\infty} f_t^* \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right) \\ &= \sum_{r=-\infty}^{\infty} f^*_{n+r(2N+m+1)} - \sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} \\ &\qquad - \frac{1}{2N+m+1} \sum_{k=N+1}^{N+m} \sum_{t=-\infty}^{\infty} f_t^* \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right). \end{aligned} \tag{51}$$

Second, we show that the last term in the right-hand side of (51) vanishes

$$\begin{aligned} \sum_{k=N+1}^{N+m} \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} &= \sum_{\ell=1}^{m} e^{\frac{2\imath\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=0}^{m-1} v^{-1}_{\ell,s+1} e^{\frac{2\imath\pi n(s-N-m)}{2N+m+1}} \sum_{p=1}^{m} e^{\frac{2i\pi(t-N-\ell)(p+N)}{2N+m+1}} \\ &= \sum_{p=1}^{m} e^{\frac{2i\pi t(p+N)}{2N+m+1}} \sum_{s=0}^{m-1} e^{\frac{2\imath\pi n(s-N-m)}{2N+m+1}} \sum_{\ell=1}^{m} v^{-1}_{\ell,s+1} e^{\frac{2\imath\pi(\ell+N)(m-p)}{2N+m+1}} \\ &= \sum_{p=1}^{m} e^{\frac{2i\pi t(p+N)}{2N+m+1}} e^{-\frac{2\imath\pi n(p+N)}{2N+m+1}} \\ &= \sum_{p=1}^{m} e^{\frac{2i\pi(t-n)(p+N)}{2N+m+1}} \\ &= \sum_{k=N+1}^{N+m} e^{\frac{2i\pi(t-n)k}{2N+m+1}} \end{aligned}$$

which proves equation (50).

Equation (50) shows that

$$F_{n,m} - f_n^* = \sum_{r\neq 0} f^*_{n+r(2N+m+1)} - \sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)}. \tag{52}$$

According to Lemma 4

$$\begin{aligned}\sum_{r\neq 0} f^*_{n+r(2N+m+1)} &= \frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+m}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}\\ &\times\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{\left(2r+\frac{2n}{2N+m+1}\right)^{k+1}}\\ &+o\left(N^{-q-m-1}\right),\ N\to\infty,\ |n|\le N.\end{aligned}\tag{53}$$

Then, similarly

$$\begin{aligned}\sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} &= \frac{(-1)^{N+\ell+1}}{2N+m+1}\sum_{j=q}^{q+m}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}\\ &\times\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{\left(2r+\frac{2(N+\ell)}{2N+m+1}\right)^{k+1}}\\ &+o\left(N^{-q-m-1}\right),\ N\to\infty.\end{aligned}\tag{54}$$

Taking into account that $\theta_{n,\ell}=O\left(N^{m-1}\right)$ we obtain

$$\begin{aligned}\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} &= \sum_{j=q}^{q+m}\frac{1}{(2N+m+1)N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}\\ &\times\sum_{\ell=1}^{m}(-1)^{N+\ell+1}\theta_{n,\ell}\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{\left(2r+\frac{2(N+\ell)}{2N+m+1}\right)^{k+1}}\\ &+o\left(N^{-q-2}\right).\end{aligned}\tag{55}$$

According to (39)

$$\begin{aligned}\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} &= \frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+m}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}\\ &\times e^{-\frac{i\pi(m-1)n}{2N+m+1}}\sum_{\ell=1}^{m}\Phi_{k,m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}\right)\sum_{s=0}^{m-1}v^{-1}_{\ell,s+1}e^{\frac{2\imath\pi ns}{2N+m+1}}\\ &+o\left(N^{-q-2}\right),\ |n|\le N,\ N\to\infty.\end{aligned}\tag{56}$$

According to the Taylor expansion

$$\begin{aligned}\Phi_{k,m}\left(e^{\frac{2\imath\pi(N+\ell)}{2N+m+1}}\right) &= \sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\left(e^{\frac{2\imath\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\\ &+O(N^{-m}),\ |n|\le N,\ N\to\infty\end{aligned}$$

we write

$$\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty}f^{*}_{N+\ell+r(2N+m+1)}=\frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+m}\frac{1}{N^{j}}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}e^{-\frac{i\pi(m-1)n}{2N+m+1}}$$
$$\times\sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=0}^{m-1}v^{-1}_{\ell,s+1}e^{\frac{2\imath\pi ns}{2N+m+1}}$$
$$+\,o\left(N^{-q-1}\right),\ \ |n|\le N,\ \ N\to\infty. \tag{57}$$

Finally, taking into account the relations

$$\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=0}^{m-1}v^{-1}_{\ell,s+1}e^{\frac{2\imath\pi ns}{2N+m+1}}=\left(e^{\frac{2i\pi n}{2N+m+1}}+1\right)^{\tau},\ \ \tau=0,\dots,m-1$$

we get

$$F_{n,m}-f^{*}_{n}=\frac{(-1)^{n+1}}{(2N+m+1)N^{q}}\sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(i\pi)^{k+1}(q-k)!}\sum_{r\neq0}\frac{(-1)^{r(m+1)}}{\left(2r+\frac{2n}{2N+m+1}\right)^{k+1}}$$
$$-\frac{(-1)^{n+1}}{(2N+m+1)N^{q}}\sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(i\pi)^{k+1}(q-k)!}e^{-\frac{i\pi(m-1)n}{2N+m+1}}$$
$$\times\sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\left(e^{\frac{2i\pi n}{2N+m+1}}+1\right)^{\tau}+o\left(N^{-q-1}\right) \tag{58}$$

which completes the proof. □

Theorem 7 *[7] Let $f^{(q+m)}\in AC[-1,1]$ for some $q\ge0$ and $m\ge1$ and*

$$f^{(k)}(-1)=f^{(k)}(1)=0,\ \ k=0,...,q-1.$$

Then the following estimate holds

$$\lim_{N\to\infty}N^{q+\frac{1}{2}}||R_{N,m}(f,x)||_{L_2(-1,1)}=c_{m,q}(f), \tag{59}$$

where

$$c_{m,q}(f)=\left(-\frac{m+1}{8}\int_{-1}^{1}\left|\int_{-1}^{1}\nu_{q,m}(h)e^{i\pi(m+1)xh/2}dh-\int_{|h|>1}\mu_{q,m}(h)e^{i\pi(m+1)xh/2}dh\right|^{2}dx\right.$$
$$\left.+\frac{1}{2}\int_{-1}^{1}\left|\nu_{q,m}\left(x\right)\right|^{2}dx+\frac{1}{2}\int_{|x|>1}\left|\mu_{q,m}\left(x\right)\right|^{2}dx\right)^{\frac{1}{2}} \tag{60}$$

and functions $\mu_{q,m}$ and $\nu_{q,m}$ are defined in Lemmas 4 and 5.

Proof. We divide $R_{N,m}(f,x)$ into three parts

$$\begin{aligned}
||R_{N,m}(f,x)||^2_{L_2(-1,1)} &= \int_{-1}^{1} |R_{N,m}(f,x)|^2\, dx \\
&= \frac{2N+m+1}{2N} \int_{-\frac{2N}{2N+m+1}}^{\frac{2N}{2N+m+1}} \left| R_{N,m}\left(f, \frac{2N+m+1}{2N}x\right)\right|^2 dx \\
&= \frac{2N+m+1}{2N} \int_{-1}^{1} \left| R_{N,m}\left(f, \frac{2N+m+1}{2N}x\right)\right|^2 dx \\
&- \frac{2N+m+1}{2N} \int_{\frac{2N}{2N+m+1}}^{1} \left| R_{N,m}\left(f, \frac{2N+m+1}{2N}x\right)\right|^2 dx \\
&- \frac{2N+m+1}{2N} \int_{-1}^{-\frac{2N}{2N+m+1}} \left| R_{N,m}\left(f, \frac{2N+m+1}{2N}x\right)\right|^2 dx \\
&= I_1 - I_2 - I_3,
\end{aligned}$$

where

$$I_1 = \frac{2N+m+1}{2N} \int_{-1}^{1} \left| R_{N,m}\left(f, \frac{2N+m+1}{2N}x\right)\right|^2 dx,$$

$$I_2 = \frac{m+1}{2N} \int_{0}^{1} \left| R_{N,m}\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x\right)\right|^2 dx,$$

and

$$I_3 = \frac{m+1}{2N} \int_{0}^{1} \left| R_{N,m}\left(f, \frac{m+1}{2N}x - \frac{2N+m+1}{2N}\right)\right|^2 dx.$$

First we estimate I_1. We have

$$R_{N,m}(f,x) = \sum_{n=-N}^{N} \left(f_n^* - F_{n,m}\right) e^{i\pi n \frac{2N}{2N+m+1}x} + \sum_{|n|>N} f_n^* e^{i\pi n \frac{2N}{2N+m+1}x}. \tag{61}$$

Therefore

$$I_1 = \frac{2N+m+1}{N} \sum_{n=-N}^{N} |f_n^* - F_{n,m}|^2 + \frac{2N+m+1}{N} \sum_{|n|>N} |f_n^*|^2 .$$

In view of Lemmas 4 and 5 we obtain

$$\begin{aligned}
I_1 =& \frac{1}{(2N+m+1)N^{2q+1}} \sum_{n=-N}^{N} \left| \nu_{q,m}\left(\frac{2n}{2N+m+1}\right)\right|^2 \\
&+ \frac{1}{(2N+m+1)N^{2q+1}} \sum_{|n|>N} \left| \mu_{q,m}\left(\frac{2n}{2N+m+1}\right)\right|^2 \\
&+ o\left(N^{-2q-1}\right).
\end{aligned}$$

Tending N to infinity and replacing the sums by the corresponding integrals we get

$$\lim_{N\to\infty} N^{2q+1} I_1 = \frac{1}{2}\int_{-1}^{1} |\nu_{q,m}(x)|^2\, dx + \frac{1}{2}\int_{|x|>1} |\mu_{q,m}(x)|^2\, dx.$$

Now we estimate I_2. From (61) we get

$$R_{N,m}\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x\right) = \sum_{n=-N}^{N} (-1)^n \left(f_n^* - F_{n,m}\right) e^{-i\pi x \frac{(m+1)n}{2N+m+1}} + \sum_{|n|>N} (-1)^n f_n^* e^{-i\pi x \frac{(m+1)n}{2N+m+1}}.$$

According to Lemmas 4 and 5 we derive

$$R_{N,m}\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x\right) = \frac{1}{(2N+m+1)N^q} \sum_{n=-N}^{N} \nu_{q,m}\left(\frac{2n}{2N+m+1}\right) e^{-i\pi x \frac{(m+1)n}{2N+m+1}} - \frac{1}{(2N+m+1)N^q} \sum_{|n|>N} \mu_{q,m}\left(\frac{2n}{2N+m+1}\right) e^{-i\pi x \frac{(m+1)n}{2N+m+1}} + o(N^{-q}),\ N \to \infty.$$

Tending N to infinity and replacing the sums by the corresponding integrals, we derive

$$\lim_{N\to\infty} N^q R_N\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x\right) = \frac{1}{2}\int_{-1}^{1} \nu_{q,m}(h) e^{-i\pi(m+1)xh/2} dh - \frac{1}{2}\int_{|h|>1} \mu_{q,m}(h) e^{-i\pi(m+1)xh/2} dh.$$

Hence

$$\lim_{N\to\infty} N^{2q+1} I_2 = \frac{m+1}{8} \int_0^1 \left| \int_{-1}^{1} \nu_{q,m}(h) e^{-i\pi(m+1)xh/2} dh - \int_{|h|>1} \mu_{q,m}(h) e^{-i\pi(m+1)xh/2} dh \right|^2 dx.$$

Similarly

$$\lim_{N\to\infty} N^{2q+1} I_3 = \frac{m+1}{8} \int_0^1 \left| \int_{-1}^{1} \nu_{q,m}(h) e^{i\pi(m+1)xh/2} dh - \int_{|h|>1} \mu_{q,m}(h) e^{i\pi(m+1)xh/2} dh \right|^2 dx,$$

which completes the proof. □

Now we investigate the behavior of the quasi-periodic interpolation at the endpoints of the interval in terms of the limit function.

Theorem 8 *Let $f^{(q+m)} \in AC[-1,1]$ for some $q \geq 0$, $m \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0,\ k = 0, ..., q-1.$$

Then the following estimate holds

$$\lim_{N\to\infty} N^q R_{N,m}\left(f, 1 - \frac{h}{N}\right) = \ell_{q,m}(f,h), \tag{62}$$

where

$$\ell_{q,m}(f,h) = \frac{1}{2}\int_{-1}^{1} \nu_{q,m}(x) e^{i\pi(-m-1-2h)x/2} dx \\ - \frac{1}{2}\int_{|x|>1} \mu_{q,m}(x) e^{i\pi(-m-1-2h)x/2} dx \tag{63}$$

and functions $\mu_{q,m}$ *and* $\nu_{q,m}$ *are defined in Lemmas 4 and 5.*

Proof. From definition of $R_{N,m}(f,x)$ we have

$$R_{N,m}\left(f, 1-\frac{h}{N}\right) = \sum_{|n|>N} (-1)^n f_n^* e^{-\frac{i\pi n}{2N+m+1}(m+1+2h)} \\ + \sum_{n=-N}^{N} (-1)^n (f_n^* - F_{n,m}) e^{-\frac{i\pi n}{2N+m+1}(m+1+2h)}.$$

Taking into account Lemmas 4 and 5, we get

$$N^q R_{N,m}\left(f, 1-\frac{h}{N}\right) = \frac{1}{(2N+m+1)} \sum_{n=-N}^{N} \nu_{q,m}\left(\frac{2n}{2N+m+1}\right) e^{\frac{i\pi n}{2N+m+1}(-m-1-2h)} \\ - \frac{1}{(2N+m+1)} \sum_{|n|>N} \mu_{q,m}\left(\frac{2n}{2N+m+1}\right) e^{\frac{i\pi n}{2N+m+1}(-m-1-2h)} \\ + o(1).$$

Tending N to infinity and replacing the sums by corresponding integrals we get the statement of the theorem. □

3 Numerical results

In this section we present some numerical results for the following specific functions

$$f(x) = (x^2-1)^q \sin(x-1),\ q \geq 0.$$

First we show some results concerning the behavior of the quasi-periodic interpolation at the endpoints of the interval. Table 1 shows the values of $\max_{h\geq 0} |\ell_{m,q}(f,h)|$ (see Theorem 8) and the points h_{max} where the maximums are achieved.

Figure 1 shows the graphs of the absolute errors $|R_{512,m}(f,x)|$ for different values of q and m at the point $x=1$. Similar pictures we have at the point $x=-1$. It is interesting to compare the results of Figure 1 and Table 1. For that, we need to calculate the values

$$N^q \max_{x\in[-1,1]} |R_{512,m}(f,x)|$$

	$q=0$	$q=1$	$q=2$	$q=3$
$m=0$	0.12835	0.08443	0.04146	0.11949
h_{max}	0.4178	0.4432	0.4039	0.4749
$m=1$	0.02991	0.04488	0.01873	0.10713
h_{max}	0.3746	0.3932	0.4092	0.3923
$m=2$	0.00947	0.0228	0.03081	0.03828
h_{max}	0.3512	0.353	0.3712	0.3553
$m=3$	0.00341	0.01141	0.0263	0.01019
h_{max}	0.3312	0.3398	0.3412	0.3912
$m=4$	0.00132	0.00568	0.01857	0.03018
h_{max}	0.3212	0.3212	0.3212	0.3112
$m=5$	0.00054	0.00282	0.01198	0.03314
h_{max}	0.3112	0.3012	0.3012	0.3106

Table 1: Numerical values of $\max_{h\geq 0}|\ell_{m,q}(f,h)|$ and the points h_{max} where the maximums are achieved.

from Figure 1 and then compare them with the corresponding values in Table 1. For example, when $q=3$ and $m=5$ we get

$$\max_{x\in[-1,1]}|R_{512,m}(f,x)| = 2.5\times 10^{-10}.$$

As $q=3$ and $N=512$ we calculate that

$$N^q \max_{x\in[-1,1]}|R_{512,m}(f,x)| = 0.033554.$$

The asymptotic value corresponding to $q=3$ and $m=5$ in Table 1 is 0.03314 which is close to the calculated one. Similarly can be verified that for other values of m and q the asymptotic value is rather close to the values derived from the figure.

Now let us show some results concerning the L_2-convergence. Numerical values of $c_{m,q}(f)$ are presented in Table 2 (see the last column). We see that as smaller is the value of q as bigger are the differences in accuracies while increasing m. Note that $c_{m,q}(f)$ is an asymptotic estimate so it is interesting to investigate the accuracies also for moderate values of N. We denote

$$c_{N,m,q}(f) = N^{q+1/2}||R_{N,m}(f,x)||_{L_2(-1,1)}$$

and in Table 2 present also the values of $c_{N,m,q}(f)$ for different N. We see that in all cases the values $c_{128,m,q}(f)$ coincide with $c_{m,q}(f)$ rather precisely.

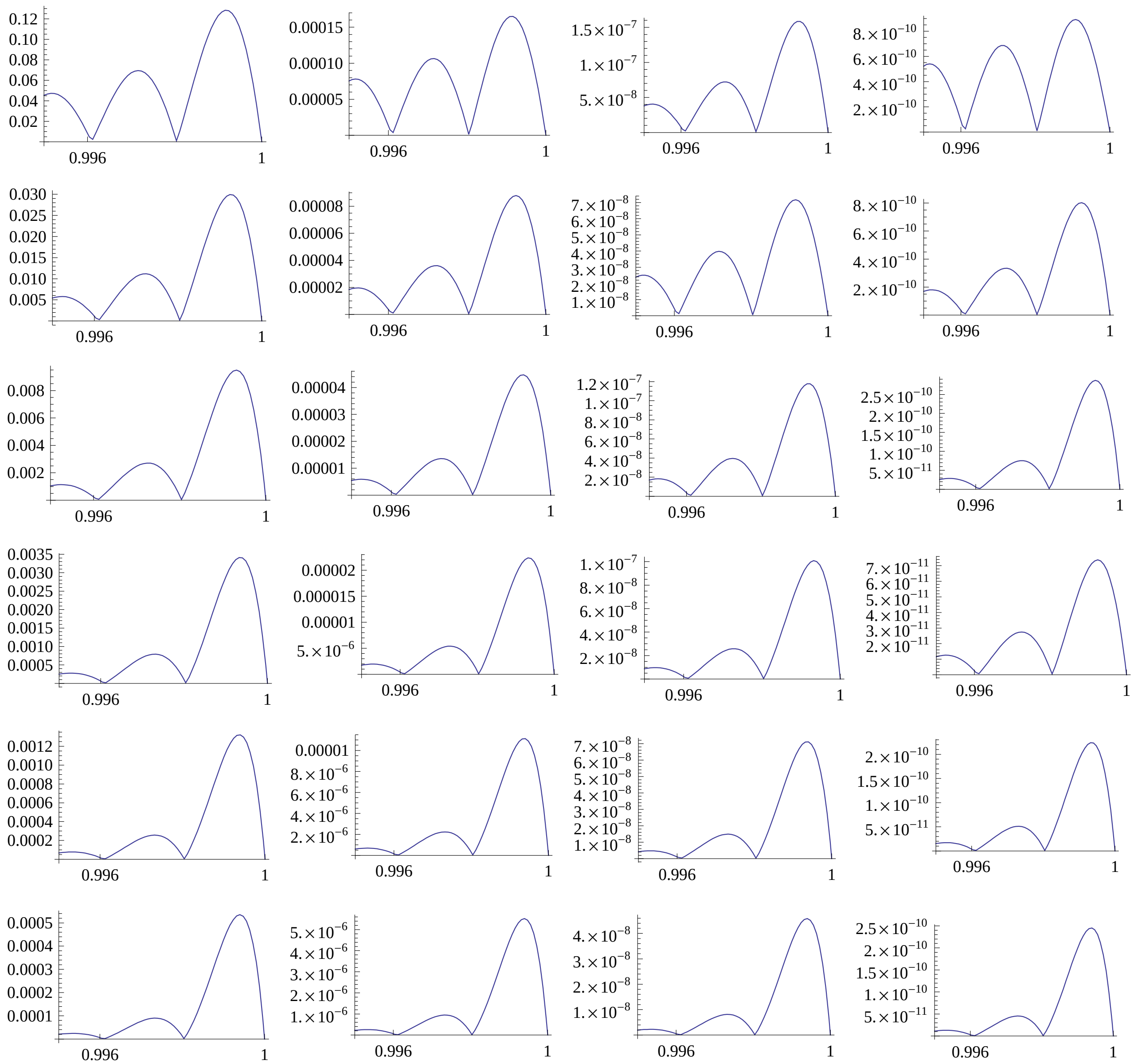

Figure 1: The graphs of the absolute errors $|R_{512,m}(f,x)|$ at the point $x=1$ for $q=0,1,2,3$ (from left to right) and $m=0,1,2,3,4,5$ (from top to bottom).

$q=0$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{m,q}(f)$
$m=0$	0.174552	0.1729474	0.172117	0.171695	0.171268
$m=1$	0.034082	0.03315	0.03275	0.032568	0.032399
$m=2$	0.01044	0.010017	0.009836	0.009754	0.009678
$m=3$	0.003763	0.003545	0.003455	0.003415	0.003379
$m=4$	0.00148	0.001364	0.001318	0.001298	0.00128
$m=5$	0.000617	0.000554	0.000529	0.000519	0.00051
$m=6$	0.000268	0.000233	0.00022	0.000215	0.000211
$m=7$	0.00012	0.000101	0.000094	0.000091	0.000089
$q=1$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{m,q}(f)$
$m=0$	0.17819	0.177975	0.177851	0.177784	0.177713
$m=1$	0.069808	0.06798	0.067213	0.06687	0.066555
$m=2$	0.032281	0.030764	0.03012	0.029828	0.029557
$m=3$	0.015675	0.014553	0.014087	0.013879	0.013689
$m=4$	0.007805	0.007031	0.00672	0.006584	0.006462
$m=5$	0.003955	0.003443	0.003243	0.003159	0.003085
$m=6$	0.002032	0.001702	0.001578	0.001527	0.001483
$m=7$	0.001057	0.000848	0.000772	0.000742	0.000717
$q=2$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{m,q}(f)$
$m=0$	0.1042	0.105212	0.105878	0.106252	0.106652
$m=1$	0.101363	0.101854	0.102198	0.1024	0.102626
$m=2$	0.076590	0.07468	0.073913	0.073583	0.073291
$m=3$	0.052277	0.049209	0.047945	0.047388	0.046884
$m=4$	0.033884	0.03067	0.029373	0.028808	0.028304
$m=5$	0.021311	0.018477	0.017368	0.016895	0.016481
$m=6$	0.013158	0.010885	0.010026	0.009668	0.009362
$m=7$	0.008032	0.006314	0.005689	0.005435	0.00522
$q=3$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{m,q}(f)$
$m=0$	0.237194	0.23589	0.23477	0.234085	0.233313
$m=1$	0.117628	0.110218	0.107563	0.1065	0.105611
$m=2$	0.119307	0.120875	0.122202	0.12299	0.123861
$m=3$	0.121601	0.120925	0.120863	0.120929	0.121074
$m=4$	0.108691	0.103324	0.101138	0.100217	0.099435
$m=5$	0.088543	0.079819	0.076301	0.074803	0.073504
$m=6$	0.067792	0.0576963	0.053769	0.052124	0.050715
$m=7$	0.049741	0.0398189	0.03612	0.034606	0.033362

Table 2: Numerical values of $c_{N,m,q}(f)$ and $c_{m,q}(f)$.

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