Armenian Journal of Mathematics
Volume 8, Number 2, 2016, 104–119

Perturbations of Operator Banach frames in Banach spaces

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Abstract. Casazza and Christensen [3] studied perturbation of operators in the context of frames. Also, Chistensen and Heil [6] studied perturbation of frames and atomic decompositions. In the present paper, we study perturbation of operator Banach frames (OBFs) for Banach spaces and obtained perturbation results for operator Banach frames and operator Bessel sequences. Also, we give a condition under which the sum of finite number of sequences of operators is an OBF by comparing each of the sequences with another system of OBFs. Finally, we define similar OBFs and prove that if a sequence of operators is similar to an OBF, then it has to be an OBF.

Key Words: Frames, Banach frames, Operator frames, Operator Banach frames
Mathematics Subject Classification 2000: 42C15, 42A38

Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [9] to study some deep problems in non-harmonic Fourier series. For a nice introduction to frames, one may refer to [7]. Frames were extended to Banach spaces by Feichtinger and Grochenig [11] and introduced the notion of atomic decompositions for Banach spaces. Later, Grochenig [12] introduced a more general concept called Banach frame for Banach Spaces. He gave the following definition of a Banach frame.

Let E be a Banach space and E_d be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d if

1. $\{f_n(x)\} \in E_d, \quad x \in E.$

2. there exist constants A and B with $0 < A \leq B < \infty$ such that
$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E.$$

3. S is a bounded linear operator such that
$$S(\{f_n(x)\}) = x, \quad x \in E.$$

Banach frames were further studied in [15, 16, 17, 18, 19].
The concept of fusion Banach frame for Banach spaces was introduced and studied by Jain and Kaushik [14] as a generalisation of Banach frame. They gave the following definition of a fusion Banach frame.

Definition 1 *Let E be a Banach space, $\{E_i\}$ be a sequence of closed subspaces of E and let $\{T_i\}$ be non trivial linear projections of E onto E_i. Let $\mathcal{A}$ be an associated Banach space and $S : \mathcal{A} \to E$ be an operator. Then $(\{E_i,\ T_i\},\ S)$ is called a fusion Banach frame for E with respect to $\mathcal{A}$ if*

1. $\{T_i f\} \in \mathcal{A},\ f \in E.$

2. there exist constants A and B with $0 < A \leq B < \infty$ such that
$$A\|f\|_E \leq \|\{T_i f\}\|_{\mathcal{A}} \leq B\|f\|_E, \quad f \in E.$$

3. S is a bounded linear operator such that
$$S(\{T_i f\}) = f, \quad f \in E.$$

The notions of (p, Y)-Bessel operator sequences, operator frames and (p, Y)-Reisz bases for Banach spaces were introduced and studied by Cao et al. [2] as a generalization of usual concepts in Hilbert spaces and of the g-frames introduced by W. Sun [20].

Later, Chun-Yan-Li [8] introduced and studied operator Bessel sequences, operator frames, Banach operator frames and observed that frames, g-frames for Hilbert spaces, E_d-frames and (p, Y)-operator frames for Banach spaces can be regarded as special cases of operator frames.

The concept of operator Banach frame for a Banach space is introduced and studied in [4] as an amalgamation of the notions of operator frame and Banach operator frame. It is observed that Banach frames, p-frames, E_d-frames, fusion Banach frames for Banach spaces and frames, g-frames for Hilbert spaces can be regarded as special cases of Operator Banach frames.

In this paper, we study perturbations of operator Banach frames (OBFs) for Banach spaces and obtain perturbation results for operator Banach frames and operator Bessel sequences. Also, we give a condition under which the sum of finite number of sequences of operators is an OBF by comparing each of the sequences with another system of OBFs. Finally, we define similar OBFs and prove that if a sequence of operators is similar to an OBF, then it has to be an OBF.

1 Preliminaries

Throughout this paper, E and $E_i,\ i \in \mathbb{N}$ will denote Banach spaces and E^* denotes the dual space of the Banach space E, for each $i \in \mathbb{N}$, $T_i \in B(E, E_i)$ denotes a bounded linear operator from E into E_i and ran(P) denotes the range of bounded linear operator P.

We begin this section with the following definition of an operator Banach frame given in [4].

Definition 2 *Let E be a Banach space, $\{E_i\}$ be a sequence of Banach spaces and $T_i \in B(E, E_i), i \in \mathbb{N}$. Let $\mathcal{A}$ be an associated Banach space and $S : \mathcal{A} \to E$ be an operator. Then $(\{T_i\},\ S)$ is called an Operator Banach frame (OBF) for E with respect to $\mathcal{A}$ if*

1. $\{T_i f\} \in \mathcal{A}, f \in E$.

2. *there exist constants A and B with $0 < A \leq B < \infty$ such that*

$$A\|f\|_E \leq \|\{T_i f\}\|_{\mathcal{A}} \leq B\|f\|_E, \quad f \in E. \tag{1}$$

3. *S is a bounded linear operator such that*

$$S(\{T_i f\}) = f, \quad f \in E.$$

The positive constants A and B respectively, are called lower and upper frame bounds for the OBF $(\{T_i\},\ S)$. The inequality (1) is called the frame inequality for the OBF. The operator $S : \mathcal{A} \to\ E$ is called the reconstruction operator. If condition (1) in Definition 2 and the upper inequality in (1) are satisfied, then we call $\{T_i\}$ to be an operator Bessel sequence for E with respect to $\mathcal{A}$. Let us denote by $Bess(E)$ the set of all operator Bessel sequences for E with respect to $\mathcal{A}$. For a sequence $\mathcal{T} = \{T_i\} \in Bess(E)$ define $\mathcal{R}_{\mathcal{T}} : E \to \mathcal{A}$ by $\mathcal{R}_{\mathcal{T}} f = \{T_i f\}$, for all $f \in E$. Then, $\mathcal{R}_{\mathcal{T}} \in B(E, \mathcal{A})$. We call $\mathcal{R}_{\mathcal{T}}$ the analysis operator for the operator Bessel sequence $\{T_i\}$.
If condition $(1), (3)$ and only upper inequality in (1) are satisfied, then $(\{T_i\}, S)$ is called Banach operator frames (which was introduced by Chun-Yan-Li [8]).

Observation 1

1. *One may observe that an operator Banach frame is a Banach operator frame. The converse need not be true.*

2. *In case $E_i = \mathbb{K}$, for each $i \in \mathbb{N}$, the notion of operator Banach frame coincide with the notion of Banach frame.*

3. *If E_i is closed subspace of E and $\{T_i\}$ is a sequence of non trivial projection of E onto E_i in Definition 2, then $(\{E_i,\ T_i\},\ S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$.*

Definition 3 *An OBF $(\{T_i\},\ S)$ for E with respect to $\mathcal{A}$ with frame bounds A and B is called*

1. *tight if it is possible to choose $A = B$ satisfying* (1).

2. *normalized tight, if it is possible to choose $A = B = 1$ satisfying* (1).

3. *exact, if for each i_0 there exists no reconstruction operator S_0 such that $(\{T_i\}_{i\neq i_0},\ S_0)$ is an OBF for E.*

Definition 4 *Let E be a Banach space, $\{E_i\}$ be a sequence of Banach spaces and $T_i \in B(E,\ E_i)$, $i \in \mathbb{N}$. Then $\{T_i\}$ is called total on E if $\{f \in E : T_i f = 0, \forall i \in \mathbb{N}\} = \{0\}$.*

The following lemma proved in [4] will be used in subsequent results.

Lemma 1 *Let E be a Banach space, $\{E_i\}$ be a sequence of Banach spaces and $T_i \in B(E,\ E_i)$, $i \in \mathbb{N}$. If $\{T_i\}$ is total over E, then $\mathcal{A} = \{\{T_i f\} : f \in E\}$ is a Banach space with norm given by $\|\{T_i f\}\|_{\mathcal{A}} = \|f\|_E$, $f \in E$.*

For $1 \le p < \infty$, define

$$\bigoplus\nolimits_p E_i = \Big\{\{f_i\} : f_i \in E_i \ \ (i \in \mathbb{N}),\ \ \|\{f_i\}\|_p = \Big(\textstyle\sum_{i=1}^{\infty} \|f_i\|^p\Big)^{\frac{1}{p}} < \infty\Big\}$$
$$\bigoplus\nolimits_\infty E_i = \Big\{\{f_i\} : f_i \in E_i \ \ (i \in \mathbb{N}),\ \ \|\{f_i\}\|_\infty = \sup \|f_i\| < \infty\Big\}$$

Example 1 *Let $E = \ell^\infty(\mathbb{N})$, $E_i = \ell^1(\mathbb{N})$ and $\mathcal{A} = \bigoplus_\infty E_i$. Define $T_i : E \to E_i$ by $T_i f = \{0, 0, ..., \underbrace{\xi_i}_{i^{th} place}, 0, ...\}$, $f = \{\xi_i\} \in E$. Then $T_i \in B(E,\ E_i)$ such that $\{T_i\}$ is an operator Bessel sequence for E with respect to $\mathcal{A}$. Also $\{T_i\}$ is total over E. Hence, by Lemma 1, there is an associated Banach space $\mathcal{A}_0 \subset \mathcal{A}$ with norm given by $\|\{T_i f\}\|_{\mathcal{A}} = \|f\|_E$, $f \in E$. Define $S : \mathcal{A}_0 \to E$ by $S(\{T_i f\}) = f$, $f \in E$. Then S is a bounded linear operator such that $(\{T_i\},\ S)$ is an OBF for E with respect to $\mathcal{A}_0$.*

Definition 5 *A sequence $\{a_n\} \subset \mathbb{R}$ is said to be positively confined if $0 < \inf_{1\le n<\infty} a_n \le \sup_{1\le n<\infty} a_n < \infty$. For $x = \{x_n\}, y = \{y_n\}$ in E and $\alpha \in \mathbb{K}$, we define $x \pm y = \{x_n \pm y_n\}$, $x.y = \{x_n y_n\}$ and $\alpha x = \{\alpha x_n\}$.*

2 Perturbations of Operator Banach frames

Perturbation theory is an important tool in various areas of applied mathematics. The fundamental results of Paley and Weiner states that a system that is sufficiently close to a Reisz basis for Hilbert spaces in some sense, is also a Reisz basis. Since then a number of results similar to that of Paley and Weiner in various contexts have been appeared in [1, 3, 5, 6, 10, 13, 16]. In this section, we discuss the perturbation of OBF and obtain various results in this direction.
We begin this section with the following result.

Theorem 2 *Let $(\{T_i\},\ S)$ be an OBF for E with respect to $\mathcal{A}$. Let $R_i \in B(E,\ E_i),\ i \in \mathbb{N}$ such that $\{R_i f\} \in \mathcal{A},\ f \in E$. Let $\mathcal{U} : \mathcal{A} \to \mathcal{A}$ be a bounded linear operator such that $\mathcal{U}\{R_i f\} = \{T_i f\}$, for all $f \in E$. Then there exists a bounded linear operator $P : \mathcal{A} \to\ E$ such that $(\{R_i\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ if and only if there exists a constant $K > 1$ such that*

$$\|\{T_i f - R_i f\}\|_{\mathcal{A}} \ \leq\ K \min\{\|T_i f\|_{\mathcal{A}},\ \|R_i f\|_{\mathcal{A}}\}, \quad f \in E.$$

Proof. Let $Q_0 : E \to \mathcal{A}$ be defined by $Q_0 f = \{T_i f\},\ f \in E$ and $Q : E \to \mathcal{A}$ be defined by $Qf = \{R_i f\},\ \ f \in E$. Let $A_T,\ B_T;\ A_R,\ B_R$ be the frame bounds for OBFs $(\{T_i\},\ S)$ and $(\{R_i\}, P)$. Then

$$A_T \|f\|_E \ \leq\ \|\{T_i f\}\|_{\mathcal{A}} \ \leq\ B_T \|f\|_E, \quad f \in E. \tag{2}$$

and

$$A_R \|f\|_E \ \leq\ \|\{R_i f\}\|_{\mathcal{A}} \ \leq\ B_R \|f\|_E, \quad f \in E. \tag{3}$$

Therefore, we get

$$\|Q_0 f - Qf\|_{\mathcal{A}} \ \leq\ \Big(1 + \frac{B_R}{A_T}\Big) \|\{T_i f\}\|_{\mathcal{A}}, \quad f \in E.$$

Also, using (3.2) and (3.3), we have

$$\|Q_0 f - Qf\|_{\mathcal{A}} \ \leq\ \Big(1 + \frac{B_T}{A_R}\Big) \|\{R_i f\}\|_{\mathcal{A}}, \quad f \in E.$$

Choose $K = \max\Big\{\Big(1 + \frac{B_R}{A_T}\Big),\ \Big(1 + \frac{B_T}{A_R}\Big)\Big\}$. Then

$$\|\{T_i f - R_i f\}\|_{\mathcal{A}} \ \leq\ K \min\{\|T_i f\|_{\mathcal{A}},\ \|R_i f\|_{\mathcal{A}}\}, \quad f \in E.$$

Conversely, suppose that there exists $K > 1$ such that

$$\|\{T_i f - R_i f\}\|_{\mathcal{A}} \ \leq\ K \min\{\|T_i f\|_{\mathcal{A}},\ \|R_i f\|_{\mathcal{A}}\}, \quad f \in E.$$

Then, for each $f \in E$, we have

$$\begin{aligned} A_T\|f\|_E \ \le \ \|Q_0 f\|_{\mathcal{A}} &\le \|Q_0 f - Qf\|_{\mathcal{A}} + \|Qf\|_{\mathcal{A}} \\ &\le (1+K)\|Qf\|_{\mathcal{A}} \\ &\le (1+K)(\|Q_0 f - Qf\|_{\mathcal{A}} + \|Q_0 f\|_{\mathcal{A}}) \\ &\le (1+K)^2\|Q_0 f\|_{\mathcal{A}} \\ &\le (1+K)^2 B_T\|f\|_E. \end{aligned}$$

Write $P = S\mathcal{U}$. Then $P : \mathcal{A} \to\ E$ is a bounded linear operator such that $P(\{R_i f\}) = f, \ \ f \in E$. Thus $(\{R_i\}, P)$ is an OBF for E with respect to $\mathcal{A}$. □

Next, we give a condition under which the perturbation of a given OBF by uniformly scaled version of a given operator Bessel sequence (by an appropriately chosen scalar number) is still an OBF.

Theorem 3 *Let $(\{T_i\},\ S)$ be an OBF for E with respect to $\mathcal{A}$. Let $R_i \in B(E,\ E_i),\ i \in \mathbb{N}$ such that $\{R_i f\} \in \mathcal{A}$ and for some constant $K > 0$,*

$$\|\{R_i f\}\|_{\mathcal{A}} \ \le \ K\|f\|_E, \quad f \in E.$$

Then, for any non-zero constant λ with $|\lambda| < \dfrac{\|S\|^{-1}}{K}$, there exists a reconstruction operator $P : \mathcal{A} \to\ E$ such that $(\{T_i + \lambda R_i\}, P)$ is an OBF for E with respect to $\mathcal{A}$ with frame bounds $\|S\|^{-1} - |\lambda|K$ and $\|Q_0\| + |\lambda|K$, where $Q_0 : E \to \mathcal{A}$ is defined by $Q_0 f = \{T_i f\}, \ \ f \in E$.

Proof. Let $Q : E \to \mathcal{A}$ be defined by $Qf = \{R_i f\},\ f \in E$. Clearly $\{(T_i + \lambda R_i)f\} \in \mathcal{A}$, for all $f \in E$. Also, we have

$$\begin{aligned} \|Q_0 f + \lambda Qf\|_{\mathcal{A}} \ &\le \|Q_0 f\|_{\mathcal{A}} + |\lambda|K\|f\|_E \\ &\le (\|Q_0\| + |\lambda|K)\|f\|_E, \quad f \in E \end{aligned}$$

and

$$\begin{aligned} (\|S\|^{-1} - |\lambda|K)\|f\|_E &\le \|Q_0 f\|_{\mathcal{A}} - |\lambda|\|Qf\|_{\mathcal{A}} \\ &\le \|Q_0 f + \lambda Qf\|_{\mathcal{A}}, \quad f \in E. \end{aligned}$$

Then

$$(\|S\|^{-1} - |\lambda|K)\|f\|_E \ \le \|Q_0 f + \lambda Qf\|_{\mathcal{A}} \ \le (\|Q_0\| + |\lambda|K)\|f\|_E.$$

Define $L : E \to\ \mathcal{A}$ by $Lf = \{(T_i + \lambda R_i)f\}, \ \ f \in E$. Then, L is a bounded linear operator such that

$$\|Q_0 f - Lf\|_{\mathcal{A}} \le |\lambda|K\|f\|_E, \quad f \in E.$$

This gives $\|Q_0 - L\| \leq |\lambda|K$. Now, since $SQ_0 = I$, I denote the identity operator on E, we have $\|I - SL\| \leq \|S\|\|Q_0 - L\| < 1$. Thus, SL is invertible. Write $P = (SL)^{-1}S$. Then $P : \mathcal{A} \to E$ is a bounded linear operator such that $P(\{(T_i + \lambda R_i)f\}) = f, \quad f \in E$. Hence $(\{T_i + \lambda R_i\}, P)$ is an OBF for E with respect to $\mathcal{A}$ with desired frame bounds. $\square$

The following result gives a sufficient condition for the perturbation of OBF by a sequence of type $\{\alpha_i R_i\}$ (where $R_i \in B(E, E_i)$ and $\{\alpha_i\}$ is a positively confined sequence in $\mathbb{R}$) to be an OBF.

Theorem 4 *Let $(\{T_i\},\ S)$ be an OBF for E with respect to $\mathcal{A} \subset \bigoplus_p E_i (1 \leq p \leq \infty)$. Let $R_i \in B(E,\ E_i),\ i \in \mathbb{N}$ and $\{\gamma_i\} \subset \mathbb{R}$ be any positively confined sequence such that $\{(\gamma_i R_i)f\} \in \mathcal{A}, \quad f \in E$. If $Q : E \to \mathcal{A}$ defined by $Qf = \{R_i f\}, \quad f \in E$ such that $\|Q\| < \dfrac{\|S\|^{-1}}{\sup_{1 \leq i < \infty} \gamma_i}$, then there exists a reconstruction operator $P : \mathcal{A} \to E$ such that $(\{T_i + \gamma_i R_i\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ with frame bounds $\Big(\|S\|^{-1} - \|Q\|\Big(\sup_{1 \leq i < \infty} \gamma_i\Big)\Big)$ and $\Big(\|Q_0\| - \|Q\|\Big(\sup_{1 \leq i < \infty} \gamma_i\Big)\Big)$, where $Q_0 : E \to \mathcal{A}$ is defined by $Q_0 f = \{T_i f\}, \quad f \in E$.*

Proof. Clearly, $\{T_i + \gamma_i R_i f\} \in \mathcal{A}$, for all $f \in E$. Now, for each $f \in E$, we have

$$\begin{aligned}
\|\{(T_i + \gamma_i R_i)f\}\|_{\mathcal{A}} &\leq \|\{T_i f\}\|_{\mathcal{A}} + \|\{\gamma_i R_i f\}\|_{\mathcal{A}} \\
&\leq \|\{T_i f\}\|_{\mathcal{A}} + \Big(\sup_{1 \leq i < \infty} \gamma_i\Big)\|R_i f\|_{\mathcal{A}} \\
&\leq \Big(\|Q_0\| + \|Q\|\Big(\sup_{1 \leq i < \infty} \gamma_i\Big)\Big)\|f\|_E.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|\{T_i + \gamma_i R_i\}f\|_{\mathcal{A}} &\geq \|\{T_i f\}\|_{\mathcal{A}} - \|\{\gamma R_i f\}\|_{\mathcal{A}} \\
&\geq \|\{T_i f\}\|_{\mathcal{A}} - \Big(\sup_{1 \leq i < \infty} \gamma_i\Big)\|R_i f\|_{\mathcal{A}} \\
&\geq \Big(\|S\|^{-1} - \|Q\|\Big(\sup_{1 \leq i < \infty} \gamma_i\Big)\Big)\|f\|_E, \quad f \in E.
\end{aligned}$$

Define $L : E \to \mathcal{A}$ by $Lf = \{(T_i + \gamma R_i)f\}, f \in E$. Then, L is a bounded linear operator such that

$$\begin{aligned}
\|Q_0 f - Lf\|_{\mathcal{A}} &= \|\{T_i f\} - \{(T_i + \gamma_i R_i)f\}\| \\
&\leq \Big(\sup_{1 \leq i < \infty} \gamma_i\Big)\|Q\|\|f\|, \quad f \in E.
\end{aligned}$$

This gives $\|Q_0 - L\| \leq \Big(\sup_{1\leq i<\infty} \gamma_i\Big)\|Q\|$. Since $S\mathcal{Q}_0 = I$, I denoting the identity mapping on E,

$$\|I - SL\| < \|S\|\|Q\|\left(\sup_{1\leq i<\infty} \gamma_i\right) < 1.$$

Therefore, SL is invertible. Write $P = (SL)^{-1}S$. Then $P : \mathcal{A} \to E$ is a bounded linear operator such that $P(\{(T_i + \gamma_i R_i)f\}) = f,\ f \in E$. Hence $(\{T_i + \gamma_i R_i\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ with frame bounds $\Big(\|S\|^{-1} - \|Q\|\Big(\sup_{1\leq i<\infty} \gamma_i\Big)\Big)$ and $\Big(\|Q_0\| - \|Q\|\Big(\sup_{1\leq i<\infty} \gamma_i\Big)\Big)$. □

Next, we give a perturbation result for operator Bessel sequences.

Theorem 5 *Let $\{T_i\}$ be an operator Bessel sequence for E with respect to $\mathcal{A}$. Let $R_i \in B(E,\ E_i),\ i \in \mathbb{N}$ such that $\{R_i f\} \in \mathcal{A}, f \in E$. Then $\{R_i\}$ is an operator Bessel sequence for E with respect to $\mathcal{A}$, if there exist non-negative constants $\alpha, \beta\ \ (\beta < 1)$ and γ such that*

$$\|\{(T_i - R_i)f\}\|_{\mathcal{A}} \leq \alpha\|\{T_i f\}\|_{\mathcal{A}} + \beta\|\{R_i f\}\|_{\mathcal{A}} + \gamma\|f\|_E, \quad f \in E.$$

Proof. Let K be the Bessel bound for the operator Bessel sequence $\{T_i\}$. Then $\|\{T_i f\}\| \leq K\|f\|, f \in E$. Thus, we have

$$\begin{aligned}\|\{R_i f\}\|_{\mathcal{A}} &= \|\{T_i f\} - \{(T_i - R_i)f\}\|_{\mathcal{A}} \\ &\leq (1+\alpha)\|\{T_f\}\|_{\mathcal{A}} + \beta\|\{R_i f\}\|_{\mathcal{A}} + \gamma\|f\|_E, \quad f \in E.\end{aligned}$$

This gives

$$\|\{R_i f\}\|_{\mathcal{A}} \leq \frac{(1+\alpha)K + \gamma}{1-\beta}\|f\|_E.$$

Hence, $\{R_i\}$ is a operator Bessel sequence for E with respect to $\mathcal{A}$. □

In the following result, we show that OBFs are stable under perturbation of frame elements by positively confined sequence of scalars.

Theorem 6 *Let $(\{T_i\},\ S)$ be an OBF for E with respect to $\mathcal{A}$, where $\mathcal{A} \subset \bigoplus_p E_i (1 \leq p \leq \infty)$. Let $R_i \in B(E,\ E_i),\ i \in \mathbb{N}$ such that $\{R_i f\} \in \mathcal{A},\ \ f \in E$. Let $U : \mathcal{A} \to \mathcal{A}$ be a bounded linear operator such that $U(\{R_i f\}) = \{T_i f\},\ \ f \in E$. Let $\{\alpha_i\}$ and $\{\beta_i\}$ be two positively confined sequences. If there exists constants $\lambda,\ \mu\ \ (0 \leq \mu < 1)$ and γ such that $\gamma < (1-\lambda)\|S\|^{-1}(\inf_{i\in\mathbb{N}} \alpha_i)$ and $\|\{(\alpha_i T_i f - \beta_i R_i)f\}\|_{\mathcal{A}} \leq \lambda\|\{\alpha_i T_i f\}\|_{\mathcal{A}} + \mu\|\{\beta_i R_i f\}\|_{\mathcal{A}} + \gamma\|f\|_E,\ \ f \in E$, then there exists a bounded linear operator $P : \mathcal{A} \to E$ such that $(\{R_i\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ with frame bounds $\dfrac{(1-\lambda)\|S\|^{-1}(\inf_{i\in\mathbb{N}} \alpha_i) - \gamma}{(1+\mu)\sup_{i\in\mathbb{N}} \beta_i}$ and $\dfrac{(1+\lambda)\|Q_0\|(\sup_{i\in\mathbb{N}} \alpha_i) + \gamma}{(1-\mu)\inf_{i\in\mathbb{N}} \beta_i}$, where $Q_0 : E \to \mathcal{A}$ is a bounded linear operator defined by $Q_0 f = \{T_i f\},\ \ f \in E$.*

Proof. Since the operator $SQ_0 : E \to E$ is an identity operator, we have $\|f\|_E = \|SQ_0(f)\|_E \leq \|S\|\|\{T_i f\}\|_{\mathcal{A}}, \quad f \in E$. Note that

$$\begin{aligned}\|\{(\beta_i R_i)f\}\|_{\mathcal{A}} &\leq \|\{(\alpha_i T_i)f\}\|_{\mathcal{A}} + \|\{(\alpha_i T_i f - \beta_i R_i)f\}\|_{\mathcal{A}} \\ &\leq \|\{(\alpha_i T_i)f\}\|_{\mathcal{A}} + \lambda\|\{\alpha_i T_i f\}\|_{\mathcal{A}} + \mu\|\{\beta_i R_i f\}\|_{\mathcal{A}} \\ &\quad + \gamma\|f\|_E, \quad f \in E.\end{aligned}$$

This gives

$$(1-\mu)\|\{(\beta_i R_i)f\}\|_{\mathcal{A}} \leq \Big((1+\lambda)\|Q_0\|(\sup_{i\in\mathbb{N}} \alpha_i) + \gamma\Big)\|f\|_E, \quad f \in E.$$

Since $\mathcal{A} \subset \bigoplus_p E_i$, we get

$$\begin{aligned}(1-\mu)\Big(\inf_{i\in\mathbb{N}} \beta_i\Big)\|\{R_i f\}\|_{\mathcal{A}} &\leq (1-\mu)\|\{(\beta_i R_i)f\}\|_{\mathcal{A}} \\ &\leq \Big((1+\lambda)\|Q_0\|(\sup_{i\in\mathbb{N}} \alpha_i) + \gamma\Big)\|f\|_E, \quad f \in E.\end{aligned}$$

Also, we have

$$\begin{aligned}(1+\mu)\Big(\sup_{i\in\mathbb{N}} \beta_i\Big)\|\{R_i f\}\|_{\mathcal{A}} &\geq (1+\mu)\|\{(\beta_i R_i)f\}\|_{\mathcal{A}} \\ &\geq \Big((1-\lambda)\|S\|^{-1}(\inf_{i\in\mathbb{N}} \alpha_i) - \gamma\Big)\|f\|_E.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{(1-\lambda)\|S\|^{-1}(\inf_{i\in\mathbb{N}} \alpha_i) - \gamma}{(1+\mu)\sup_{i\in\mathbb{N}} \beta_i}\|f\|_E &\leq \|\{R_i f\}\|_{\mathcal{A}} \\ &\leq \frac{(1+\lambda)\|Q_0\|(\sup_{i\in\mathbb{N}} \alpha_i) + \gamma}{(1-\mu)\inf_{i\in\mathbb{N}} \beta_i}\|f\|_E.\end{aligned}$$

Write $P = SU$. Then $P : \mathcal{A} \to E$ is a bounded linear operator such that $P(\{R_i f\}) = f, \quad f \in E$. Hence $(\{R_i\}, P)$ is an OBF for E with respect to $\mathcal{A}$ and with desired frame bounds. □

Remark 1 *Positive confinedness of sequences $\{\alpha_i\}$ and $\{\beta_i\}$ in $\mathbb{R}$ is necessary. Indeed, if $\{\alpha_i\}$ is not positively confined, then either $\inf_{1\leq i<\infty} \alpha_i \leq 0$ or $\sup_{1\leq <\infty} \alpha_i$ is infinite. So we get either negative lower frame bounds or an infinite upper frame bounds for the OBF $(\{T_i\}, S)$. In this case, the frame inequality is lost. Similar argument is valid for the case when the sequence $\{\beta_i\}$ is not positively confined.*

In the following result, we give sufficient conditions for the stability of an OBF.

Theorem 7 *Let $(\{T_i\},\ S)$ be an OBF for E with respect to $\mathcal{A}$. Let $R_i \in B(E, E_i)$, $i \in \mathbb{N}$ be such that $\{R_i f\} \in \mathcal{A}$, $\ f \in E$. If there exists constants $\alpha, \beta (\geq\ 0)$ and μ such that $\max\left\{\dfrac{\|S\|[(\alpha + \beta\|Q_0\|) + \mu]}{1-\beta}, \beta\right\} < 1$, where $Q_0 : E \to \mathcal{A}$ is a bounded linear operator defined by $Q_0 f = \{T_i f\}, f \in E$ and $\|\{(T_i - R_i)f\}\|_{\mathcal{A}} \leq \alpha\|\{T_i f\}\|_{\mathcal{A}} + \beta\|\{R_i f\}\|_{\mathcal{A}} + \mu\|f\|$, $f \in E$, then there exists a bounded linear operator $P : \mathcal{A} \to\ E$ such that $(\{R_i\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ with frame bounds $\|S\|^{-1}\Big(1 - \dfrac{[(\alpha+\beta)\|Q_0\| + \mu]\|S\|}{1-\beta}\Big)$ and $\dfrac{[(\alpha+1)\|Q_0\| + \mu]}{1-\beta}$.*

Proof. Let $Q : E \to \mathcal{A}$ be define by $Qf = \{R_i f\}$, $\ f \in E$. As $\|S\|^{-1}$ and $\|Q_0\|$ are frame bounds for the OBF $(\{T_i\}, S)$, we have $\|Q_0 f - Qf\|_{\mathcal{A}} \leq \alpha\|Q_0 f\|_{\mathcal{A}} + \beta\|Qf\|_{\mathcal{A}} + \|f\|_E$, $\ f \in E$. So

$$\|Qf\|_{\mathcal{A}} \leq \frac{[(1+\alpha)\|Q_0\| + \mu]}{1-\beta}\|f\|_E, \quad f \in E.$$

Thus Q is a bounded linear operator such that

$$\|Q_0 f - Qf\|_{\mathcal{A}} \leq \frac{[(\alpha+\beta)\|Q_0\| + \mu]}{1-\beta}\|f\|_E, \quad f \in E.$$

Therefore

$$\begin{aligned}\|I - SQ\| &\leq \|S\|\|Q_0 - Q\| \\ &\leq \frac{[(\alpha+\beta)\|Q_0\| + \mu]}{1-\beta} < 1\end{aligned}$$

and SQ is an invertible operator satisfying

$$\|SQ\|^{-1} \leq \frac{1}{1 - \dfrac{[(\alpha+\beta)\|Q_0\| + \mu]}{1-\beta}}.$$

Write $P = (SQ)^{-1}S$. Then $PQ = I$. Therefore

$$\|f\|_E = \|PQf\|_E \leq \|P\|\|Qf\|_{\mathcal{A}} \leq \frac{\|S\|}{1 - \dfrac{[(\alpha+\beta)\|Q_0\| + \mu]}{1-\beta}}\|Qf\|_{\mathcal{A}}$$

So, we have

$$\|S\|^{-1}\Big(1 - \frac{[(\alpha+\beta)\|Q_0\| + \mu]\|S\|}{1-\beta}\Big)\|f\|_E \leq \|Qf\|_{\mathcal{A}}, \quad f \in E.$$

Hence $(\{R_i\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ with desired frame bounds. □

3 Perturbation of Finite sum of OBFs

In this section, we give a condition under which the sum of finite number of sequences of operators is an OBF by comparing each of the sequences with another system of OBFs.
First, we give the following example to find the necessity of obtaining such a condition.

Example 2 *Let $(X, \|.\|)$ be a Banach space. Define*

$$E = \ell^\infty(X) = \Big\{ \{\xi_i\} : \xi_i \in X; \sup_{1 \le i < \infty} \|\xi_i\| < \infty \Big\}$$

Define norm $|||.|||$ on E by $|||\{\xi_i\}|||_E = \sup_{1\le i<\infty} \|\xi_i\|_X$, $\{\xi_i\} \in E$. Then E is a Banach space with this norm. Now, for each $i \in \mathbb{N}$, define $E_i = \Big\{ \{\eta_i\} : \eta_i = \{0,0,0, \underbrace{x}_{x \;\; at\ ith\ place}, 0,0,...\};\ x \in X \Big\}$. Define $T_{1,i} : E \to E_i$ by $T_{1,i}f = \{0,0,0,...\xi_i,0,0,....\}, f = \{\xi_i\} \in E$. Then $\{T_{1,i}\}$ is total over E. Therefore, by Lemma 1, there exists an associated Banach space $\mathcal{A} = \Big\{ \{T_{1,i}f\} : f \in E \Big\}$ with norm given by $\|\{T_{1,i}f\}\|_{\mathcal{A}} = \|f\|_E$, $f \in E$. Define $S_1 : \mathcal{A} \to E$ by $S_1(\{T_{1,i}f\}) = f$, $f \in E$. Then, S_1 is a bounded linear operator such that $(\{T_{1,i}\}, S_1)$ is an OBF for E. Define $\{T_{2,i}\} \subset B(E, E_i)$ by

$$\begin{cases} T_{2,1} = -T_{1,1} \\ T_{2,2} = T_{1,1} \\ T_{2,i} = T_{1,i-1} \quad , \quad (i = 3,4,....\ ). \end{cases}$$

Then there exists a reconstruction operator S_2 such that $(\{T_{2,i}\}, S_2)$ is an OBF for E with respect to $\mathcal{A}$. Define $\{R_{1,i}\}$ and $\{R_{2,i}\}$ in $B(E, E_i)$ by

$$\begin{cases} R_{1,1} = 0 \\ R_{1,i} = T_{1,i}, \quad (i = 2,3,4,....\ ) \\ R_{2,1} = 0 \\ R_{2,2} = 0 \\ R_{2,i} = T_{1,i-1} \quad , \quad (i = 3,4,....\ ) \end{cases}$$

Then, for suitable choice of α and β, $\|\{(T_{n,i} - R_{n,i})f\}\|_{\mathcal{A}} \le \alpha \|\{T_{n,i}f\}\|_{\mathcal{A}} + \beta\|f\|_E$, $f \in E$, $(n = 1,2)$ is satisfied. But there exists, in general, no reconstruction operator $P_0 : \mathcal{A}_0 \to E$ such that $(\{\sum_{n=1}^2 R_{n,i}\}, P_0)$ is an OBF for E with respect to $\mathcal{A}_0$. Indeed, if $(\{\sum_{n=1}^2 R_{n,i}\}, P_0)$ is an OBF for E with respect to $\mathcal{A}_0$ then there exists positive constants A and B such that

$$A\|f\|_E \le \left\| \left\{ \left(\sum_{n=1}^{2} R_{n,i} \right) f \right\} \right\| \le B\|f\|_E \tag{4}$$

Now, let $f = (x, 0, 0, ...)$ be a non zero element in E then

$$\Big(\sum_{n=1}^{2} R_{n,i}\Big)(f) = 0, \quad \text{for all} \quad i \in \mathbb{N}.$$

Hence by frame inequality (4), we have $f = 0$. This is a contradiction.

Thus, it is natural to ask for a condition under which $(\{\sum_{n=1}^{2} R_{n,i}\}, P_0)$ is an OBF for E. The following theorem gives such a condition in a more general set up.

Theorem 8 *Let $(\{T_{n,i}\},\ S_n)$, $n \in E_k = \{1, 2, ..., k\}$ be OBFs for E with respect to $\mathcal{A}$. Let $R_{n,i} \in B(E,\ E_i)$ with $\{R_{n,i}f\} \in \mathcal{A}$, $f \in E$, $n \in E_k$. Let $U : \mathcal{A} \to \mathcal{A}$ be a bounded linear operator such that $U(\{(\sum_{i=1}^{k} R_{n,i})f\}) = \{T_{p,i}f\}$ for some $p \in E_k$ and for each $n = 1, 2, ..., k$, let $Q_n : E \to \mathcal{A}$ be defined by $Q_n f = \{T_{n,i}f\}$, $f \in E$. If there exists constants $\alpha > 0$, $\beta > 0$ such that*

1. $\alpha \sum_{n\in E_k} \|Q_n\| + k\beta < \|S_{n_0}\|^{-1} - \sum_{n\in E_k,\ n\neq n_0} \|Q_n\|$, *for some* $n_0 \in E_k$.
2. $\|\{(T_{n,i} - R_{n,i})f\}\|_{\mathcal{A}} \leq \alpha \|\{T_{n,i}f\}\|_{\mathcal{A}} + \beta\|f\|$, $f \in E$, $n \in E_k$,

then there exists a bounded linear operator $P : \mathcal{A} \to E$ such that $(\{\sum_{n\in E_k} R_{n,i}\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ with frame bounds $\Big(\|S_{n_0}\|^{-1} - [\alpha \sum_{n\in E_k} \|Q_n\| + \sum_{n\in E_k,\ n\neq n_0} \|Q_n\| + k\beta]\Big)$ and $\Big((1 + \alpha) \sum_{n\in E_k} \|Q_n\| + k\beta\Big)$.

Proof. For each $n \in E_k$, $S_n Q_n$ is an identity operator on E. Therefore

$$\|f\|_E = \|S_n Q_n f\|_{\mathcal{A}} \leq \|S_n\| \|\{T_{n,i}f\}\|_{\mathcal{A}}, \quad f \in E.$$

Also, we have

$$\Big\| \sum_{n\in E_k} \{T_{n,i}f\} \Big\|_{\mathcal{A}} \leq \Big(\sum_{n\in E_k} \|Q_n\| \Big) \|f\|_E, \quad f \in E.$$

Now, for each $f \in E$, we have

$$\begin{aligned}
\Big\| \{(\sum_{n\in E_k} R_{n,i})f\} \Big\|_{\mathcal{A}} &= \Big\| \sum_{n\in E_k} \{(T_{n,i} - (T_{n,i} - R_{n,i}))f\} \Big\|_{\mathcal{A}} \\
&\geq \Big\| \{(\sum_{n\in E_k} T_{n,i})f\} \Big\|_{\mathcal{A}} - \Big\| \sum_{n\in E_k} \{(T_{n,i} - R_{n,i})f\} \Big\|_{\mathcal{A}} \\
&\geq \Big\| \{T_{n_0,i}\} + \sum_{n\neq n_0} \{T_{n,i}f\} \Big\|_{\mathcal{A}} - \sum_{n\in E_k} \Big\| \{(T_{n,i} - R_{n,i})f\} \Big\|_{\mathcal{A}} \\
&\geq \Big(\|S_{n_0}\|^{-1} - [\alpha \sum_{n\in E_k} \|Q_n\| + \sum_{n\in E_k, n\neq n_0} \|Q_n\| + k\beta] \Big) \|f\|_E
\end{aligned}$$

and

$$\Big\|\{(\sum_{n\in E_k} R_{n,i})f\}\Big\|_{\mathcal{A}} \leq \Big((1+\alpha)\sum_{n\in E_k}\|Q_n\| + k\beta\Big)\|f\|_E, \quad f\in E.$$

Therefore, we get

$$\Big(\|S_{n_0}\|^{-1} - [\alpha \sum_{n\in E_k}\|Q_n\| + \sum_{n\in E_k, n\neq n_0}\|Q_n\| + k\beta]\Big)\|f\|_E \leq \Big\|\{(\sum_{n\in E_k} R_{n,i})f\}\Big\|_{\mathcal{A}}$$
$$\leq \Big((1+\alpha)\sum_{n\in E_k}\|Q_n\| + k\beta\Big)\|f\|_E, \quad f\in E.$$

Write $P = S_pU$ where p is fixed index given in hypothesis. Then $P : \mathcal{A}\to\ E$ is a bounded linear operator such that $P(\{(\sum_{n\in E_k} R_{n,i})f\}) = f, \quad f\in E$. Hence $(\{\sum_{n\in E_k} R_{n,i}\},\ P)$ is an OBF for E with respect to $\mathcal{A}$ with desired frame bounds. □

Remark 2 *The condition* (1) *in Theorem 8 is not neccesary. Indeed, if* $(\{T_{1,i}\}, S_1)$ *is a normalized tight operator Banach frame for* E *with respect to* $\mathcal{A}$*. Let* $T_{2,i} = R_{1,i} = R_{2,i} = T_{1,i},\ i\in\mathbb{N}$*. Then* $\sum_{n=1}^{2} R_{n,i} = 2T_{1,i}$*. So, there exists a reconstruction operator* $P : \mathcal{A}\to\ E$ *such that* $(\{\sum_{n=1}^{2} R_{n,i}\},\ P)$ *is an OBF for* E *with respect to* $\mathcal{A}$*. Also, it can be easily verified that condition* (1) *of Theorem 8 does not hold.*

4 Similar Operator Banach frames

In this section, we define similar OBFs and give the following definition.

Definition 6 *Let* $(\{T_i\},\ S)$ *and* $(\{R_i\},\ P)$ *be OBFs for a Banach space* E*. We say that the OBF* $(\{R_i\},\ P)$ *is similar to the OBF* $(\{T_i\},\ S)$ *if there is an isomorphism* U *on* E *such that* $R_i = T_iU$*, for all* $i\in\mathbb{N}$*.*

In the following result, we prove that if a sequence of operators is similar to an OBF, then it is also an OBF.

Theorem 9 *Let* $(\{T_i\},\ S)$ *be an OBF for* E *with respect to* $\mathcal{A}$ *with frame bounds* A *and* B*. Let* U *be an isomorphism on* E*. Then there is a bounded linear operator* $P : \mathcal{A}\to\ E$ *such that* $(\{T_iU\},\ P)$ *is an OBF for* E *with respect to* $\mathcal{A}$*.*

Proof. Clearly, $\{T_iU(f)\}\in\mathcal{A}$. Also for each $f\in E$, we have

$$\|\{T_iUf\}\|_{\mathcal{A}} \leq B\|U\|\|f\|_E$$

Again

$$\begin{aligned} A\|f\| &= A\|U^{-1}Uf\| \\ &\leq \|U^{-1}\|\|\{T_iUf\}\|_{\mathcal{A}} \end{aligned}$$

So $A\|U^{-1}\|^{-1}\|f\|_E \leq \|\{T_iUf\}\|_{\mathcal{A}} \leq B\|U\|\|f\|_E$. Write $P = U^{-1}S$, then $(\{T_iU\},\ P)$ is an OBF for E with respect to $\mathcal{A}$. □

Finally, we give the following result related to similar OBFs.

Theorem 10 *Let $(\{T_i\}, S)$ and $(\{Q_i\}, P)$ be OBFs for E with respect to $\mathcal{A}$ with analysis operator $\mathcal{R}_T$ and $\mathcal{R}_Q$ respectively. Then the following statements are equivalent:*

1. *$(\{Q_i\}, P)$ is similar to $(\{T_i\}, S)$.*
2. *$\mathcal{R}_Q = \mathcal{R}_T U$, where U is an isomorphism on E.*

Proof. We omit the proof as it can worked out in few steps using the hypothesis. □

Acknowledgement

The authors pay their sincere thanks to the refree(s) for his/her critical remarks and many suggestions for the improvement of this paper.

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Please, cite to this paper as published in
Armen. J. Math., V. **8**, N. 2(2016), pp. 104–119