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Massless High-Spin Representations of Extended Poincaré Algebra

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Abstract. All physical phenomena in the four-dimensional spacetime are invariant under the Poincaré group. The Standard Model of fundamental interactions, Electroweak theory, and Quantum Chromodynamics are required to be invariant under Poincaré group. Any possible extension of the Poincaré group hints to the existence of a new physics beyond the Standard Model. In particular, the supersymmetric extension of the Poincaré group predicts the existence of new particles that are supersymmetric partners of the elementary particles of the Standard Model: leptons, quarks, W and Z bosons and gluons. In a recently suggested high-spin extension of the Poincaré group, new massless particles of increasing spins are predicted to exist. In that respect we are interested in investigating a massless representation of the Poincaré algebra that has high-spin states. The massless states are described by the helicity operator, which has only two polarisations equal to the components of spin along the direction of motion, as it takes place for photons and gravitons. This means that not all of the 2s+1 spin magnetic quantum states exist and the spin operator is not defined anymore. In order to eliminate the spin operator from a massless representation and ensure that only helicity operator is included into the representations, Schwinger suggested that new non-commuting coordinates should be defined. We investigate the uncertainty relations that follow from non-commutativity of these new coordinates. It is the average wavelength of a massless particle that sets the scale of the coordinate uncertainty.

Key Words: Poincaré Algebra, Massless Representations, Spin and Helicity Operators, Non Commutative Coordinates, Heisenberg Uncertainty Principle, Unbounded Helicity Spectrum
Mathematics Subject Classification 2020: 20-XX, 20A10, 20-08, 20Cxx, 20F05, 22E66

Introduction

The fundamental consequence of electromagnetic and mechanical experiments, and in accordance with the Maxwell equations, there is a physical equivalence of two coordinate systems that differ in the following ways: a translation of the spatial origin, a translation of the time origin, a rotation of the space axes, a constant relative velocity between the two systems [8, 22]. This fundamental property of the spacetime is formulated as the invariance of all physical systems with respect to the corresponding infinitesimal coordinate transformations defined in the following way:

$$\delta x^\mu = \delta\epsilon^\mu + \delta\omega^{\mu\nu} x_\mu, \qquad \delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu},$$

where $\delta\epsilon^\mu$ is a spacetime translation, $\delta\omega^{\mu\nu}$ is a four-dimensional rotation, and the space-time coordinates are:

$$x^\mu = (ct, \vec{r}), \qquad x^0 = -x_0 = ct, \qquad x^i = x_i = r_i.$$

The six independent parameters of this four-dimensional rotation are related to $\delta\vec{\omega}$ and $\delta\vec{v}$ by

$$\delta\omega_{ij} = \epsilon_{ijk}\delta\omega_k, \qquad \delta\omega_{0i} = \frac{1}{c}\delta v_i,$$

where $\vec{\omega}$ is the angular velocity and $\vec{v}$ is the velocity. The infinitesimal unitary transformation, $U = 1 + iG$, that is induced by an infinitesimal coordinate transformation is given by the expression [19]

$$U = 1 + i\frac{1}{\hbar}(P^\mu \delta\epsilon_\mu + \frac{1}{2} M^{\mu\nu}\delta\omega_{\mu\nu}),$$

where the space components of the Poincaré generators P^μ and $M^{\mu\nu}$ are

$$cP^0 = H + Mc^2, \qquad P_i, \qquad \frac{1}{c}M^{0i} = N_i, \qquad M_{ij} = \epsilon_{ijk}J_k.$$

The generators P_i, and J_k, are the linear and angular momentum operators, while H is the energy, or the Hamiltonian operator, and N_i is the boost operator, in total ten operators. The quantum unit of action $\hbar = 1.0545 \times 10^{-27}$ erg sec. The full set of commutators for the generators comprise the Poincaré algebra [8, 19, 22]:

$$\begin{aligned}
&[P_\mu,\ P_\nu] = 0,\\
&[P_\mu,\ M_{\kappa\lambda},\ ] = i\hbar(\eta_{\mu\lambda}\ P_\kappa - \eta_{\mu\kappa}\ P_\lambda),\\
&[M_{\mu\nu},\ M_{\kappa\lambda}] = i\hbar(\eta_{\mu\kappa}\ M_{\nu\lambda} - \eta_{\nu\kappa}\ M_{\mu\lambda} + \eta_{\nu\lambda}\ M_{\mu\kappa} - \eta_{\mu\lambda}\ M_{\nu\kappa}),
\end{aligned}$$

where

$$\eta_{00} = -1, \quad \eta_{0i} = 0, \quad \eta_{ij} = \delta_{ij}.$$

The commutators can also be presented in the following form:

$$
\begin{aligned}
&[P^{\nu}, \frac{1}{2} M^{\kappa\lambda} \delta\omega_{\kappa\lambda}] = i\hbar \delta\omega^{\mu\nu} P_{\mu}, \qquad [P^{\nu}, P^{\lambda}\delta\epsilon_{\lambda}] = 0,\\
&[M^{\mu\nu}, \frac{1}{2} M^{\kappa\lambda} \delta\omega_{\kappa\lambda}] = i\hbar (\delta\omega^{\lambda\mu} M_{\lambda}{}^{\nu} + \delta\omega^{\lambda\nu} M_{\lambda}{}^{\mu}),\\
&[M^{\mu\nu}, P^{\lambda}\delta\epsilon_{\lambda}] = i\hbar (\delta\epsilon^{\lambda} P^{\nu} - i\delta\epsilon^{\nu} P^{\mu}),
\end{aligned}
$$

indicating the response of vectors and tensors to infinitesimal Lorentz rotations, here $(1/2)M^{\kappa\lambda}\delta\omega_{\kappa\lambda}$ corresponds to the three-dimensional rotations and boosts. The response of these operators to the translational $P^{\lambda}\delta\epsilon_{\lambda}$ is given as well. In terms of a three-dimensional vectors the algebra will take the following form:

$$
[P_n,\ P_m] = 0, \qquad [J_n, J_m] = i\hbar \epsilon_{nmk} J_k, \tag{1}
$$

$$
[P_n,\ J_m] = i\hbar \epsilon_{nmk} P_k, \qquad [N_n,\ N_m] = -i\hbar \frac{1}{c^2} J_{nm}, \tag{2}
$$

$$
[N_n,\ J_m] = i\hbar \epsilon_{nmk} N_k, \qquad [P_n,\ N_m] = i\hbar \delta_{nm} \frac{P_0}{c}, \tag{3}
$$

$$
\begin{aligned}
&[H,\ P_n] = 0,\\
&[H,\ J_n] = 0,\\
&[H,\ N_n] = i\hbar P_n,
\end{aligned} \tag{4}
$$

When $P_0/c \ll 1$, the algebra reduces to the Galilean algebra.

1 High-spin extension of Poincaré algebra

The algebra we were interested in is the extension of the high-spin Poincaré algebra introduced earlier in [9–11]. This algebra naturally appeared in the high-spin extension of the Yang Mills theory suggested in [10, 12, 13]. The non-Abelian tensor gauge fields were defined as rank-$(s+1)$ tensors [12, 13]

$$
A^{a}_{\mu\lambda_1 \ldots \lambda_s}(x), \qquad s = 0, 1, 2, \ldots
$$

These tensor fields are totally symmetric with respect to the indices $\lambda_1, \ldots, \lambda_s$ and had no symmetries with respect to the first index μ. The index a numerates the generators L^a of the Lie algebra of a compact Lie group G. These tensor fields appear in the expansion of the gauge field $\mathcal{A}_{\mu}(x, e)$ over the unit polarization vector e_{λ} [14, 15] :

$$
\mathcal{A}_{\mu}(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!}\ A^{a}_{\mu\lambda_1 \ldots \lambda_s}(x)\ L_a^{\lambda_1 \ldots \lambda_s}, \tag{5}
$$

where $L_a^{\lambda_1 ... \lambda_s} = e^{\lambda_1}...e^{\lambda_s} \otimes L_a$ are the "gauge generators" of the following current algebra [9–11]:

$$[L_a^{\lambda_1 ... \lambda_i}, L_b^{\lambda_{i+1} ... \lambda_s}] = if_{abc}\, L_c^{\lambda_1 ... \lambda_s}, \qquad s = 0, 1, 2, ... \tag{6}$$

This current algebra has infinitely many "gauge generators" $L_a^{\lambda_1 ... \lambda_s}$. They are gauge generators because they carry the isospin and Lorentz indices. The generators $L_a^{\lambda_1 ... \lambda_s}$ are symmetric spacetime tensors, and the full algebra includes the Poincaré generators $P_\mu,\ M_{\mu\nu}$. Algebra $L_G(\mathcal{P})$ has the following form [9–11] :

$$\begin{aligned}
&[P^\mu,\ P^\nu] = 0,\\
&[P^\lambda,\ M^{\mu\nu}] = i\hbar(\eta^{\lambda\nu}\ P^\mu - \eta^{\lambda\mu}\ P^\nu),\\
&[M^{\lambda\rho},\ M^{\mu\nu}] = i\hbar(\eta^{\mu\rho}\ M^{\nu\lambda} - \eta^{\mu\lambda}\ M^{\nu\rho} + \eta^{\nu\lambda}\ M^{\mu\rho} - \eta^{\nu\rho}\ M^{\mu\lambda}),\\
&[P^\mu,\ L_a^{\lambda_1 ... \lambda_s}] = 0,\\
&[L_a^{\lambda_1 ... \lambda_s},\ M^{\mu\nu}]\\
&\ = i\hbar(\eta^{\lambda_1\nu} L_a^{\mu\lambda_2 ... \lambda_s} - \eta^{\lambda_1\mu} L_a^{\nu\lambda_2 ... \lambda_s} + ... + \eta^{\lambda_s\nu} L_a^{\lambda_1 ... \lambda_{s-1}\mu} - \eta^{\lambda_s\mu} L_a^{\lambda_1 ... \lambda_{s-1}\nu}),\\
&[L_a^{\lambda_1 ... \lambda_i}, L_b^{\lambda_{i+1} ... \lambda_s}] = if_{abc} L_c^{\lambda_1 ... \lambda_s} \quad (\mu, \nu, \rho, \lambda = 0, 1, 2, 3; s = 0, 1, 2, ...)
\end{aligned} \tag{7}$$

It is an extension of the Poincaré algebra by "gauge generators" $L_a^{\lambda_1 ... \lambda_s}$, which are the elements of the current algebra (6). Algebra (7) can be extended to a supersymmetric case as well [1]. The supersymmetric generalisations of high-spin de Sitter and conformal groups were also suggested in [1]. The algebra $L_G(\mathcal{P})$ has a representation in terms of differential operators of the following general form:

$$\begin{aligned}
&P^\mu = k^\mu,\\
&M^{\mu\nu} = i\hbar(k^\nu\ \frac{\partial}{\partial k_\mu} - k^\mu\ \frac{\partial}{\partial k_\nu}) + i\hbar(e^\nu\ \frac{\partial}{\partial e_\mu} - e^\mu\ \frac{\partial}{\partial e_\nu}),\\
&L_a^{\lambda_1 ... \lambda_s} = e^{\lambda_1}...e^{\lambda_s} \otimes L_a,
\end{aligned} \tag{8}$$

where e^λ is a translationally invariant space-like unit vector. The vector space of a representation is parameterised by the momentum k^μ and vector variables e^λ: $\Psi(k^\mu, e^\lambda)$. Irreducible representations can be obtained from (8) by imposing invariant constraints on the vector space of the following form:

$$k^2 = 0, \quad k^\mu e_\mu = 0, \quad e^2 = 1\ . \tag{9}$$

These equations have a unique solution $e^\mu = \chi k^\mu + e_1^\mu \cos\varphi + e_2^\mu \sin\varphi$, where $e_1^\mu = (0, 1, 0, 0),\ e_2^\mu = (0, 0, 1, 0)$ when $k^\mu = \omega(1, 0, 0, 1)$. The χ and φ remain as independent variables on the cylinder $\varphi \in S^1, \chi \in R^1$ and the invariant subspace of functions is:

$$\Psi(k^\mu, e^\nu)\ \delta(k^2)\ \delta(k \cdot e)\ \delta(e^2 - 1) = \Phi(k^\mu, \varphi, \chi).$$

The generators $L_a^{\lambda_1 ... \lambda_s}$ take the following form:

$$L_a^{\perp \ \lambda_1 ... \lambda_s} = \prod_{n=1}^{s} (\chi k^{\lambda_n} + e_1^{\lambda_n} \cos\varphi + e_2^{\lambda_n} \sin\varphi) \otimes L_a. \tag{10}$$

This is a purely transversal representation because of (9):

$$k_{\lambda_1} L_a^{\perp \lambda_1 ... \lambda_s} = 0, \qquad s = 1, 2, ...$$

and the generators $L_a^{\perp \ \lambda_1 ... \lambda_s}$ are carry the helicities in the following range:

$$\lambda = (s, s-2, ..., -s+2, -s), \tag{11}$$

in total $s + 1$ states. This can be deduced from the explicit representation (10) by using helicity polarisation vectors $e_\pm^\lambda = (e_1^\lambda \mp i e_2^\lambda)/2$:

$$L_a^{\perp \ \lambda_1 ... \lambda_s} = \prod_{n=1}^{s} (e^{i\varphi} e_+^{\lambda_n} + e^{-i\varphi} e_-^{\lambda_n} + \chi k^{\lambda_n}) \otimes L_a. \tag{12}$$

After performing the multiplication in (12) and collecting the terms of a given power of momentum we will get the following expression:

$$\begin{aligned} L_a^{\perp \ \mu_1 ... \mu_s} &= \prod_{n=1}^{s} (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \otimes L_a + \\ &+ \sum_1 \chi k^{\mu_1} \prod_{n=2}^{s} (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \otimes L_a + ... + \chi k^{\mu_1} ... \chi k^{\mu_s} \otimes L_a, \end{aligned} \tag{13}$$

where the first term $\prod_{n=1}^{s} (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n})$ represents the *helicity generators* $(L_a^{+\cdots+}, ..., L_a^{-\cdots-})$, while their helicity spectrum is described by formula (11).

The transversal representation $L_a^{\perp \lambda_1 ... \lambda_s}$ plays an important role in the definition of the gauge field $\mathcal{A}_\mu(x, e)$ in (5). By substituting the transversal representation (13) of the generators $L_a^{\perp \lambda_1 ... \lambda_s}$ into the expansion (5) and collecting the terms in front of the helicity generators $(L_a^{+\cdots+}, ..., L_a^{-\cdots-})$, we will get

$$\begin{aligned} \mathcal{A}_\mu(x, e) &= \sum_{s=0}^{\infty} \frac{1}{s!} \, (\tilde{A}^a_{\mu\lambda_1 ... \lambda_s} \ e_+^{\lambda_1} ... e_+^{\lambda_s} \otimes L_a + ... + \tilde{A}^a_{\mu\lambda_1 ... \lambda_s} e_-^{\lambda_1} ... e_-^{\lambda_s} \otimes L_a) \\ &= \sum_{s=0}^{\infty} \frac{1}{s!} \, (\tilde{A}^a_{\mu + \cdots +} \ \otimes L_a^{+\cdots+} + ... + \tilde{A}^a_{\mu - \cdots -} \ \otimes L_a^{-\cdots-}), \end{aligned}$$

where s is the number of negative indices. This formula represents the projection $\tilde{A}^a_{\mu\lambda_1 ... \lambda_s}$ of the components of the non-Abelian tensor gauge field

$A^a_{\mu\lambda_1...\lambda_s}$ into the plane transversal to the momentum. The projection contains only positive-definite space-like components of the helicities [1, 10, 11]:

$$\lambda \;\;=\;\; \pm\,(s+1), \quad \begin{matrix}\pm(s-1)\\ \pm(s-1)\end{matrix}, \quad \begin{matrix}\pm(s-3)\\ \pm(s-3)\end{matrix}, \;\; ...., \tag{14}$$

where the lower helicity states have double degeneracy. In summary, we have seen that the spectrum of the high-spin extension of Poincaré algebra is massless and our intention is to analyze the special physical properties of these states of high helicities (14).

2 Casimir operators of Poincaré algebra

The fully invariant vacuum state $|0\rangle$ is defined as

$$P^\mu|0\rangle = 0, \qquad M^{\mu\nu}|0\rangle = 0.$$

The excited states will have a positive value of $P_0 > 0$. Let us consider the first invariant quantity

$$-\,P^\mu P_\mu = (P^0)^2 - \vec{P}^2 = M^2c^2, \tag{15}$$

which can be strictly positive $M^2 > 0$, and therefore $P^0 = +(\vec{P}^2{+}M^2c^2)^{1/2} > 0$, or equal to zero $M^2 = 0$, and then $P^0 = +|\vec{P}| > 0$. The second invariant can be constructed by using the Pauli-Lubanski pseudo-vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}M_{\nu\lambda}P_\rho. \tag{16}$$

It is a translationally invariant pseudo-vector

$$[W^\mu, P^\nu\delta\epsilon_\nu,] = \hbar\epsilon^{\mu\nu\lambda\rho}\delta\epsilon_\lambda P_\rho P_\nu = 0,$$

and it has the following properties:

$$[P^\mu,\ W^\nu] = 0\ , \quad [W^\mu,\ W^\nu] = i\hbar\epsilon^{\mu\nu\lambda\rho}\ W_\lambda P_\rho\ , \qquad P_\mu W^\mu = 0.$$

That is, the Pauli-Lubanski pseudo-vector is translationally invariant, noncommutative pseudo-vector, and because it is orthogonal to the time-like vector P^μ (15), it is a space-like pseudo-vector:

$$W^\mu W_\mu = \varrho^2 \geq 0.$$

The above two Casimir invariants can be used to characterise the unitary irreducible representations of the Poincaré algebra.

3 Representations of Poincaré algebra

We are interested in describing the representations of the Poincaré algebra in terms of operators acting in an appropriate Hilbert space. It follows from (2) and (3) that the angular momentum and boost operators transform under coordinate translations as follows:

$$\begin{aligned} \delta_\epsilon \vec{J} &= \frac{1}{i\hbar}[\vec{J},\ \vec{P}\cdot\delta\vec{\epsilon}] = \delta\vec{\epsilon}\times\vec{P} \\ \delta_\epsilon \vec{N} &= \frac{1}{i\hbar}[\vec{N},\ \vec{P}\cdot\delta\vec{\epsilon}] = -\delta\vec{\epsilon}\,\frac{P^0}{c}. \end{aligned}$$

The Poincaré algebra has the operators $(P^0, \vec{P})$, the boost $\vec{N}$ and angular momentum $\vec{J}$ operators, but there are no explicit coordinate operators in the algebra. The response of the angular momentum to translation is in accordance with the nature of angular momentum and indicates the existence of a position vector operator $\vec{R}$:

$$\delta_\epsilon \vec{R} = \frac{1}{i\hbar}[\vec{R},\ \vec{P}\cdot\delta\vec{\epsilon}] = \delta\vec{\epsilon},$$

thus having the Heisenberg commutator with momentum operators $\vec{P}$:

$$[R_i, P_k] = i\hbar\delta_{ik}.$$

The next step is to construct the representation of the $\vec{J}$ and $\vec{N}$ operators in terms of coordinate and momentum operators. The angular momentum and boost operators can be defined as

$$\vec{J} = \vec{R}\times\vec{P}, \qquad \vec{N} = \vec{P}x^0 - \frac{1}{c}\{P^0, \vec{R}\}, \tag{17}$$

where the product $P^0\vec{R}$ is symmetrised because these operators are not commuting:

$$\frac{1}{i\hbar}[\vec{R}, P^0] = \frac{\partial P^0}{\partial \vec{P}} = \frac{\vec{P}}{P^0}.$$

Importantly, only when the coordinate operators are commuting operators:

$$[R_i, R_j] = 0, \tag{18}$$

then all constructed operators are correctly transforming with respect to the rotations. The most characteristic commutator to be checked is (2). One can derive that

$$\frac{i}{\hbar}[\{P^0, R_n\}, \{P^0, R_m\}] = R_n P_m - R_m P_n$$

and then get convinced that

$$i\vec{N}\times\vec{N} = i(\vec{P}x^0 - \frac{1}{c}\{P^0,\vec{R}\})\times(\vec{P}x^0 - \frac{1}{c}\{P^0,\vec{R}\}) = \frac{\hbar}{c^2}\vec{R}\times\vec{P} = \frac{\hbar}{c^2}\vec{J}. \quad (19)$$

Representation (17) fulfils all commutation relations of the Poincaré algebra (1-4) and completes the construction of the representation of the Poincaré algebra for particles without spin, the internal angular momentum $\vec{S}$.

4 Massive representation with spin

Let us now describe the representation that contains the internal angular momentum, the spin of particles, by adding a new term to the angular momentum $\vec{J}$ (17):

$$\vec{J} = \vec{R}\times\vec{P} \qquad \rightarrow \qquad \vec{J} = \vec{R}\times\vec{P} + \vec{S}.$$

To separate these two terms in the total angular momentum $\vec{J}$ one should request that the spin operator commutes with the coordinate and momentum operators:

$$[S_n, R_m] = [S_n, P_m] = 0,$$

and to keep intact the commutator of angular momentum operators (1) one should have

$$[S_i,\ S_l] = i\hbar\epsilon_{ilk}S_k$$

with all of the $2s+1$ spin magnetic quantum number states. As soon as the angular momentum operator changes, the boost operator $\vec{N}$ should be redefined so that the commutator (2), (19) remains intact:

$$i\frac{c^2}{\hbar}\vec{N}\times\vec{N} = \vec{J} = \vec{R}\times\vec{P} + \vec{S}.$$

One can add the term $f(P^0)\ \vec{S}\times\vec{P}$ with an unknown function $f(P^0)$ and then find out that $f(P^0) = 1/(P^0 + Mc)$, thus the modified boost operator $\vec{N}$ should have the following form:

$$\vec{N} = \vec{P}x^0 - \frac{1}{c}\{P^0\vec{R}\} + \frac{1}{c^2}\frac{1}{P^0 + Mc}\vec{S}\times\vec{P}.$$

In summary, the massive representation that contains the spin operator $\vec{S}$ has the following form:

$$\begin{aligned} \vec{J} &= \vec{R}\times\vec{P} + \vec{S}, \\ \vec{N} &= \vec{P}x^0 - \frac{1}{c}\{P^0\vec{R}\} + \frac{1}{c}\frac{1}{P^0 + Mc}\vec{S}\times\vec{P}. \end{aligned} \quad (20)$$

Representation (20) fulfils all commutation relations of the Poincaré algebra (1-4) and completes the construction of the representation of the Poincaré algebra for particles with spin in terms of the coordinate, momentum and spin operators.

It is possible to invert these formulas and express the coordinate and spin operators in terms of the momentum, angular momentum, and boost operators. By calculating the products $\vec{J} \cdot \vec{P}$, $(\vec{J} \times \vec{P})$, $\vec{N} \cdot \vec{P}$ and $(\vec{N} \times \vec{P})$ one can represent the coordinate and spin operators in the following form $(x^0 = 0)$:

$$\begin{aligned} M\vec{R} &= -\vec{N} + \frac{1}{P^0(P^0 + Mc)}\vec{P}(\vec{P} \cdot \vec{N}) + \frac{1}{P^0 + Mc}\vec{J} \times \vec{P}, \\ M\vec{S} &= \frac{1}{c}P^0\vec{J} - \frac{1}{c}\frac{1}{P^0 + Mc}\vec{P}(\vec{P} \cdot \vec{J}) + \vec{N} \times \vec{P}. \end{aligned} \tag{21}$$

The components of the Pauli-Lubanski vector (16) in this representation are:

$$\begin{aligned} W^0 &= \vec{P} \cdot \vec{J} = \vec{P} \cdot \vec{S}, \\ \vec{W} &= P^0\vec{J} - c\vec{P} \times \vec{N} = cM\vec{S} + \frac{(\vec{P} \cdot \vec{S})}{P^0 + Mc}\vec{P}. \end{aligned} \tag{22}$$

It follows that the second invariant of the Poincaré algebra is proportional to the square of the particles spin:

$$W^2 = c^2 M^2 \vec{S}^2, \tag{23}$$

and that $\vec{S}^2$ is a Lorentz invariant. In summary, the representation is characterised by two invariants, the mass of the particles $-P^2 = M^2c^2$ (15) and their spin (23). This discussion refers to a strictly massive case $M^2 > 0$.

5 Massless representations and helicity states

Our main interest is to consider the massless particles, such as photons, gravitons, and high helicity states (14). We will consider the limit $M^2 \to 0$ of the massive representation described above when $\vec{S}^2$ is kept fixed. In this limit when $M^2 = -P^\mu P_\mu = 0$, from (22) we will have

$$W^0 = \vec{P} \cdot \vec{S}, \quad \vec{W} = \frac{(\vec{P} \cdot \vec{S})}{P}\vec{P}, \quad W^2 = 0.$$

The above relations can be written in the following form:

$$W^\mu = \lambda P^\mu,$$

where λ is the Lorentz invariant helicity operator

$$\lambda = \frac{\vec{P} \cdot \vec{S}}{P}.$$

It is equal to the components of the spin along the direction of the motion, and as far as it is Lorentz invariant, the system exhibits only two values of helicity $\pm s$. *This means that not all of the* $2s + 1$ *spin magnetic quantum number states exist in the massless limit.*

Thus in the limit $M^2 \to 0$ the spin operator ceased to be defined, with two exceptions, and one should *introduce new variables* for this circumstance. In order to eliminate the operator $\vec{S}$ from a massless representation, Schwinger suggested that the new coordinates can be defined [19] in the following form:

$$\hat{\vec{R}} = \vec{R} - \frac{\vec{S} \times \vec{P}}{P^2}.$$

In that case the commutation relation of the coordinate $\hat{\vec{R}}$ and the momentum operator $\vec{P}$ remains intact:

$$[P_n, P_m] = 0, \qquad [\hat{R}_n, P_m] = i\hbar\delta_{nm},$$

but the commutation relation (18) between the coordinates will change and take the following form:

$$[\hat{R}_n, \hat{R}_m] = -i\hbar\lambda\epsilon_{nmk}\frac{P_k}{P^3}, \tag{24}$$

which implies that the Jacobi identity is obstructed by zero-momentum particles and will takes the following form:

$$[[\hat{R}_1, \hat{R}_2], \hat{R}_3] + cycl.perm. = \lambda\hbar^2\triangle_P(\frac{1}{P}) = -4\pi\lambda\hbar^2\delta^{(3)}(\vec{P}). \tag{25}$$

The Jacobi identity (25) imposes the restriction $\vec{P} \neq 0$ validating the Lorentz invariant energy property $P^0 = |\vec{P}| > 0$. The absence of certain values of spin magnetic quantum number states is now manifested by the noncommutativity of $\hat{\vec{R}}$ components (24).

As one can see, the new coordinates are not commuting between themselves, and this is opposite to the massive case where the coordinate operators are commuting (18). The vector product of the new coordinates with the momentum operator is such that

$$\hat{\vec{R}} \times \vec{P} = \vec{R} \times \vec{P} + \vec{S} - \frac{\vec{P} \cdot \vec{S}}{(P^0)^2}\vec{P} = \vec{J} - \lambda\frac{\vec{P}}{P},$$

and we can express the angular momentum operator in terms of new coordinates and helicity operators:

$$\vec{J} = \hat{\vec{R}} \times \vec{P} + \lambda \frac{\vec{P}}{P}, \tag{26}$$

The other commutation relations are given in the Appendix. It follows from this equation that in terms of the new coordinate operators the massless representation of the angular operator $\vec{J}$ can be written only in terms of the helicity operator without any reference to spin operators. From the equation (20) it follows that the boost operator $\vec{N}$ will take the following form:

$$\vec{N} = \vec{P} x^0 - \frac{1}{c} \{ P^0 \hat{\vec{R}} \}. \tag{27}$$

Expressions (24), (26) and (27) completely define the massless representation of the Poincaré algebra in terms of new non-commuting coordinates $\hat{\vec{R}}$, momentum $\vec{P}$, and helicity operator λ. If the explicit expresions of $\vec{J}$ and $\vec{N}$ are inserted in the formulas for $M\vec{R}$ and $M\vec{S}$, the expressions (21) do vanish, as does $M\hat{\vec{R}}$.

The intrinsic non-locality of massless particles (24) is described by the alternative Heisenberg uncertainty relation:

$$\triangle \hat{R}_n \; \triangle \hat{R}_m \geq \hbar \frac{|\lambda|}{2} \Big| \langle \frac{P_k}{P^3} \rangle \Big|, \qquad n \neq m \neq k, \tag{28}$$

where $\triangle \hat{R}_n^{\;2} = \langle \; (\hat{R}_n - \langle \hat{R}_n \rangle)^2 \; \rangle$ is the mean square of the deviation of $\hat{R}_n$ from its mean value $\langle \hat{R}_n \rangle$ and $\langle P_k / P^3 \rangle$ is the mean value of the operator in the state under consideration. For a momentum state that has a direction along a given axis, from the new uncertainty relation (28) we will get

$$(\triangle \hat{\mathbf{R}})^2 \geq \hbar |\lambda| \langle \frac{1}{P^2} \rangle = \frac{1}{4\pi^2} \frac{|\lambda|}{\hbar} \Lambda^2, \tag{29}$$

where $2\pi\hbar/P = \Lambda$ is the wavelength of the massless particle indicating that the wavelength sets the scale of coordinate uncertainty. The relation (29) together with the unbounded nature of the λ spectrum (14), ranging over all integers, indicates that physically accessible states would exist, for which the location uncertainty $(\triangle \hat{\mathbf{R}})^2 = \triangle \hat{R}_1^{\;2} + \triangle \hat{R}_2^{\;2} + \triangle \hat{R}_3^{\;2}$ is arbitrarily large.

6 Note added

The extension of the Poincaré algebra considered in this article (7) uniquely *unifies internal Lie algebra G, in particular, the $SU(N)$ algebra and Poincaré space-time algebra $\mathcal{P}$ into what was defined as $L_G(\mathcal{P})$ algebra* [9].

The mathematical consistency of the algebra (7) can be checked by investigating the corresponding Jacobi identities. This check of the consistency was performed in the articles [9–11, 13], where the author (G.S.) also discussed the reasons why the Coleman-Mandula theorem [4] is not applicable to this extension. Indeed, there is no conflict with the Coleman-Mandula theorem, because the theorem applies to the symmetries that act on S-matrix elements and not on all the other symmetries that occur in quantum field theory. The $L_G(\mathcal{P})$ algebra (7) is the symmetry which acts on the gauge field $A_\mu(x,e)$, and it is not a symmetry of the S-matrix. The theorem assumes among other things that the vacuum is non-degenerate, that there are no massless particles in the spectrum and that the number of generators is finite. In contrary to that, the spectrum of the extended Yang Mills theory is massless 9 and the number of generators is infinite (6), (7).

The other important No-Go theorem of Weinberg and Witten [21] forbids the existence of massless particles with helicity $|h| > 1/2$ if there is a Lorentz-covariant conserved current $\partial_\mu J^\mu = 0$ and massless particles with helicity $|h| > 1$ if there is a conserved Lorentz covariant energy-momentum tensor $\partial_\mu \theta^{\mu\nu} = 0$. Both theorems are not applicable to Yang Mills theory and to general relativity. In the Yang Mills case the current is not conserved because it satisfies the equation $\nabla_\mu J^\mu = 0$. Similarly, the energy-momentum pseudotensor in the general relativity is not a Lorentz invariant tensor. For that reason in nature we observe the helicity $|h| = 1$ massless photons and gluons and the massive W-Z bosons in the spectrum of the Standard Model. The helicity $|h| = 2$ graviton in the general relativity was predicted to exists and was observed indirectly in the form of gravitational waves. In short, the theorem is not applicable to gauge/diffeomorphism invariant field theories. For the same reason the theorem is not applicable to the gauge invariant theory of tensorgluons [9–11, 13]. In summary, the conditions of the Coleman-Mandula and Weinberg-Witten theorems are not fulfilled here, and therefore the theorems are not applicable to the case of this Poincaré algebra.

In short, the results obtained in this theory in the last decade can be summarised in the following form:
a) The calculation of the Callan-Symanzik beta function arising from the quantisation of the tensor theory at the one-loop level and is due to the vacuum polarisation by high-spin massless particles. It has the following form [16]:

$$\beta(g) = -\frac{12s^2 - 1}{48\pi^2} g^3 C_2(G), \qquad s = 1, 2, 3, ...$$

where $C_2(G) = N$ for the $SU(N)$ group. For the vector bosons $s = 1$ it reduces to the Gross-Wilczek-Politzer result for which they obtained the Nobel prise.

b) The calculation of the tensorgluon-gluon splitting amplitudes [6, 16]

$$M_n^{tree}(..., a^{\lambda_a}, b^{\lambda_b}, ...) \quad \stackrel{a \parallel b}{\longrightarrow} \quad \sum_{\lambda=\pm 1} Split_{-\lambda}^{tree}(a^{\lambda_a}, b^{\lambda_b}) \ \times \ M_{n-1}^{tree}(..., P^{\lambda}, ...),$$

where $Split_{-\lambda}^{tree}(a^{\lambda_a}, b^{\lambda_b})$ denotes the splitting amplitude, P denotes the intermediate state of the momentum $k_P = k_a + k_b$ and helicity λ. The splitting probabilities

$$P(z) = C_2(G) \sum_{h_P, h_a, h_b} |Split_{-h_P}(a^{h_a}, b^{h_b})|^2 \ s_{ab}, \quad s_{ab} = 2k_a \cdot k_b,$$

have been obtained by using the spinor representation of the scattering amplitudes [3] The splitting probabilities in the maximally symmetric representation have the following form [6, 16]:

$$P_{\lambda_B \lambda_A}^{\lambda_C}(z) = \frac{1}{z^{2\eta\lambda_B - 1}(1-z)^{2\eta\lambda_C - 1}}, \quad \lambda_C + \lambda_B + \lambda_A = \eta = \pm 1.$$

The formula describes all known splitting probabilities found earlier in QFT as well as the splitting probabilities for high-spin particles.
c) The extension of the Altarelli-Parisi equations that describe the creation of additional tensorgluons in the high-energy scattering experiments [16].
d) The calculation of the contribution of tensorgluons to a proton spin in an attempt to resolve the "proton-spin crisis" [17].
e) The contribution and influence of the tensorgluons vacuum polarisation on the physics of Grand Unified Theory and on its unification scale that merges the electromagnetic, weak, and strong forces into a single force at high energies [13].

The presentation of these results partially can be found in the review article devoted to the celebration of the 60th anniversary of the discovery of Yang Mills gauge theory [13]. The supersymmetric extension of the $L_G(\mathcal{P})$ algebra was realised in [1, 18] and the discovery of new high-rank topological invariants in [2, 5, 7, 20].

7 Appendix

The commutators of the new non-commutative coordinates have the following form:

$$[\hat{R}_n, P^0] = i\hbar \tfrac{P_n}{P^0}$$
$$[P^0\hat{R}_n, P^0\hat{R}_m] = -i\hbar\epsilon_{nmk}\tfrac{P_k}{P^0} + i\hbar(\hat{R}_m P_n - \hat{R}_n P_m)$$
$$[\hat{R}_n P^0, \hat{R}_m P^0] = -i\hbar\epsilon_{nmk}\tfrac{P_k}{P^0} + i\hbar(\hat{R}_m P_n - \hat{R}_n P_m)$$
$$[P^0\hat{R}_n, \hat{R}_m P^0] = -i\hbar\epsilon_{nmk}\tfrac{P_k}{P^0} + i\hbar(\hat{R}_m P_n - P_m\hat{R}_n) + \hbar^2\tfrac{P_n P_m}{(P^0)^2}$$
$$[\hat{R}_n P^0, P^0\hat{R}_m] = -i\hbar\epsilon_{nmk}\tfrac{P_k}{P^0} + i\hbar(P_n\hat{R}_m - \hat{R}_n P_m) - \hbar^2\tfrac{P_n P_m}{(P^0)^2}$$
$$[\{P^0, \hat{R}_n\}, \{P^0, \hat{R}_m\}] = -i\hbar\epsilon_{nmk}\tfrac{P_k}{P^0} + i\hbar(P_n\hat{R}_m - P_m\hat{R}_n)$$

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