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A Type of Eneström-Kakeya Theorem for Quaternionic Polynomials Involving Monotonicity with a Reversal

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Abstract. The Eneström-Kakeya theorem states that if $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all zeros of P lie in $|z| \leq 1$ in the complex plane. Motivated by recent results concerning an Eneström-Kakeya "type" condition on the real and imaginary parts of complex coefficients, we give similar results with hypotheses concerning the real and imaginary parts of the coefficients of a quaternionic polynomial. We give bounds on the moduli of quaternionic zeros of such polynomials.

Key Words: Location of Zeros of a Polynomial, Quaternionic Polynomial, Monotone Coefficients
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Introduction

The classical Eneström-Kakeya theorem concerns the location of the complex zeros of a real polynomial with nonnegative monotone coefficients. It was independently proved by Gustav Eneström in 1893 [4] and Sōichi Kakeya in 1912 [9].

Theorem 1 (Eneström-Kakeya) *If $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of P lie in $|z| \leq 1$.*

A corollary to the main theorem in [6] concerns monotonicity of the real and imaginary parts of the coefficients of a polynomial. The monotonicity condition involves a reversal, as follows.

Theorem 2 *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients where $Re(a_\ell) = \alpha_\ell$ and $Im(a_\ell) = \beta_\ell$ for $\ell = 0, 1, \ldots, n$. Suppose that $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_n$ and $\beta_0 \leq \beta_1 \cdots \leq \beta_r \geq \beta_{r+1} \geq \cdots \geq \beta_n$. Then all the zeros of P lie in*

$$\min\left\{|a_0|/\left(2(\alpha_k+\beta_r)-(\alpha_0+\beta_0)-(\alpha_n+\beta_n-|a_n|)\right),1\right\} \leq |z|$$
$$\leq \max\left\{(|a_0|-(\alpha_0+\beta_0)-(\alpha_n+\beta_n)+2(\alpha_k+\beta_r))/|a_n|,1\right\}.$$

By combining more general monotonicity conditions of Aziz and Zargar [1] and Shah et al. [16], the authors of this work recently proved the following [5, Theorem 5].

Theorem 3 *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients. Let $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ for $0 \leq \ell \leq n$. Suppose that for some positive numbers k_R, k_I, ρ_R, ρ_I, p, and q with $k_R \geq 1$, $k_I \geq 1$, $0 < \rho_R \leq 1$, $0 < \rho_I \leq 1$, and $0 \leq q \leq p \leq n$, the coefficients satisfy*

$$\rho_R\alpha_q \leq \alpha_{q+1} \leq \alpha_{q+2} \leq \cdots \leq \alpha_{p-1} \leq k_R\alpha_p$$

and

$$\rho_I\beta_q \leq \beta_{q+1} \leq \beta_{q+2} \leq \cdots \leq \beta_{p-1} \leq k_I\beta_p.$$

Then all the zeros of P lie in the closed annulus

$$\min\left\{1, \frac{|a_0|}{M-|a_0|+|a_n|}\right\} \leq |q| \leq \frac{M}{|a_n|},$$

where

$$\begin{aligned} M = {} & |a_0| + M_r + (1-\rho_R)|\alpha_q| - \rho_R\alpha_q + (1-\rho_I)|\beta_q| \\ & -\rho_I\beta_q + (k_R-1)|\alpha_p| + k_R\alpha_p + (k_I-1)|\beta_p| + k_I\beta_p + M_p, \end{aligned}$$

$M_r = \sum_{\ell=1}^{r} |a_\ell - a_{\ell-1}|$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

The quaternions, $\mathbb{H} = \{\alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$, where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$, are the standard example of a noncommutative division ring. The modulus of $q \in \mathbb{H}$ is $|q| = \sqrt{\alpha^2+\beta^2+\gamma^2+\delta^2}$. The absence of commutivity leads to some surprising behavior of the zeros of a polynomial of a quaternionic variable. For example, the second degree polynomial q^2+1 has set of zeros $\{\beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k} \mid \beta^2+\gamma^2+\delta^2 = 1\}$.

The Eneström-Kakeya theorem has been extended to polynomials of a quaternionic variable as follows [2].

Theorem 4 *If $p(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying $0 \leq a_0 \leq \cdots \leq a_n$, then all the zeros of p lie in $|q| \leq 1$.*

In addition, a number of related results have recently appeared [7, 10, 12, 11, 13, 17]. These involve various modifications of the monotonicity assumption of the original version of Theorem 4.

By giving results on the location of the quaternionic zeros of a polynomial, we include all (finitely many) complex zeros and potentially infinitely many more quaternionic zeros, as illustrated for polynomial q^2+1. The purpose of this paper is to extend Theorem 3 to quaternionic polynomials and, in the process, to introduce a reversal in the monotonicity condition on the real and imaginary parts of the quaternionic coefficients.

1 The results

In this section, we formulate our main results.

Theorem 5 *Let $P(q)=\sum_{\ell=0}^{n} q^{\ell} a_{\ell}$ be a polynomial of degree n with quaternionic coefficients, that is, $a_{\ell}=\alpha_{\ell}+\beta_{\ell} i+\gamma_{\ell} j+\delta_{\ell} k$, where for positive real $\rho_{R_1}, \rho_{R_2}, \rho_{I_1}, \rho_{I_2}, \rho_{J_1}, \rho_{J_2}, \rho_{K_1}$, ρ_{K_2} each less than or equal to 1 and for k_R, k_I, k_J, k_K each at least 1, we have*

$$\rho_{R_1}\alpha_r \leq \alpha_{r+1} \leq \cdots \leq \alpha_{\eta-1} \leq k_R\alpha_\eta \geq \alpha_{\eta+1} \geq \cdots \geq \rho_{R_2}\alpha_p,$$

$$\rho_{I_1}\beta_r \leq \beta_{r+1} \leq \cdots \leq \beta_{\eta-1} \leq k_I\beta_\eta \geq \beta_{\eta+1} \geq \cdots \geq \rho_{I_2}\beta_p,$$

$$\rho_{J_1}\gamma_r \leq \gamma_{r+1} \leq \cdots \leq \gamma_{\eta-1} \leq k_J\gamma_\eta \geq \gamma_{\eta+1} \geq \cdots \geq \rho_{J_2}\gamma_p,$$

$$\rho_{K_1}\delta_r \leq \delta_{r+1} \leq \cdots \leq \delta_{\eta-1} \leq k_K\delta_\eta \geq \delta_{\eta+1} \geq \cdots \geq \rho_{K_2}\delta_p.$$

Then all the zeros of $P(q)$ lie in

$$\min\left\{1, \frac{|a_0|}{M-|a_0|+|a_n|}\right\} \leq |q| \leq \frac{M}{|a_n|},$$

where

$$\begin{aligned} M \;=\; & |a_0| + M_r - \rho_{R_1}\alpha_r + |\alpha_r|(1-\rho_{R_1}) + 2|\alpha_\eta|(k_R-1) + 2k_R\alpha_\eta \\ & +|\alpha_p|(1-\rho_{R_2}) - \rho_{R_2}\alpha_p - \rho_{I_1}\beta_r + |\beta_r|(1-\rho_{I_1}) + 2|\beta_\eta|(k_I-1) \\ & +2k_I\beta_\eta + |\beta_p|(1-\rho_{I_2}) - \rho_{I_2}\beta_p - \rho_{J_1}\gamma_r + |\gamma_r|(1-\rho_{J_1}) \\ & +2|\gamma_\eta|(k_J-1) + 2k_J\gamma_\eta + |\gamma_p|(1-\rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_r \\ & +|\delta_r|(1-\rho_{K_1}) + 2|\delta_\eta|(k_K-1) + 2k_K\delta_\eta + |\delta_p|(1-\rho_{K_2}) - \rho_{K_2}\delta_p + M_p, \end{aligned}$$

$M_r=\sum_{\ell=1}^{r}|a_\ell-a_{\ell-1}|$, and $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$.

With $\rho_{R_1}=\rho_{I_1}=\rho_{J_1}=\rho_{K_1}=1$, $k_R=k_I=k_J=k_K=1$, and $\rho_{R_2}=\rho_{I_2}=\rho_{J_2}=\rho_{K_2}=1$ in Theorem 5, we get the following corollary.

Corollary 1 *If $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a polynomial of degree n with quaternionic coefficients, that is, $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$, satisfying*

$$\alpha_r \le \alpha_{r+1} \le \cdots \le \alpha_{\eta-1} \le \alpha_\eta \ge \alpha_{\eta+1} \ge \cdots \ge \alpha_p,$$

$$\beta_r \le \beta_{r+1} \le \cdots \le \beta_{\eta-1} \le \beta_\eta \ge \beta_{\eta+1} \ge \cdots \ge \beta_p,$$

$$\gamma_r \le \gamma_{r+1} \le \cdots \le \gamma_{\eta-1} \le \gamma_\eta \ge \gamma_{\eta+1} \ge \cdots \ge \gamma_p,$$

$$\delta_r \le \delta_{r+1} \le \cdots \le \delta_{\eta-1} \le \delta_\eta \ge \delta_{\eta+1} \ge \cdots \ge \delta_p,$$

then all the zeros of $P(q)$ lie in

$$\min\left\{1, \frac{|a_0|}{M - |a_0| + |a_n|}\right\} \le |q| \le \frac{M}{|a_n|},$$

where

$$M = |a_0| + M_r - \alpha_r + 2\alpha_\eta - \alpha_p - \beta_r + 2\beta_\eta - \beta_p - \gamma_r + 2\gamma_\eta - \gamma_p - \delta_r + 2\delta_\eta - \delta_p + M_p,$$

$M_r = \sum_{\ell=1}^{r} |a_\ell - a_{\ell-1}|$, *and* $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

With $r = l$ and $\eta = p = n$, Corollary 1 reduces to the following.

Corollary 2 *If $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is a polynomial of degree n with quaternionic coefficients, that is, $a_\ell = \alpha_\ell + \beta_\ell i + \gamma_\ell j + \delta_\ell k$, satisfying*

$$\alpha_l \le \alpha_{l+1} \le \cdots \le \alpha_{n-1} \le \alpha_n, \ \ \beta_l \le \beta_{l+1} \le \cdots \le \beta_{n-1} \le \beta_n,$$

$$\gamma_l \le \gamma_{l+1} \le \cdots \le \gamma_{n-1} \le \gamma_n, \delta_l \le \delta_{l+1} \le \cdots \le \delta_{n-1} \le \delta_n,$$

then all the zeros of $P(q)$ lie in

$$\min\left\{1, \frac{|a_0|}{M - |a_0| + |a_n|}\right\} \le |q| \le \frac{M}{|a_n|},$$

where $M = |a_0| + M_l - \alpha_l + \alpha_n - \beta_l + \beta_n - \gamma_l + \gamma_n - \delta_l + \delta_n$ *and* $M_l = \displaystyle\sum_{\ell=1}^{l} |a_\ell - a_{\ell-1}|$.

Corollary 2 is a slight refinement of a result of Tripathi [17, Theorem 3.1]. Corollary 2 implies Theorem 9 of [2] when $l = 0$.

In connection with Bernstein inequalities, Chan and Malik [3] (and, independently, Qazi [15]) considered the class of polynomials of a complex variable of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$. Inspired by this, the current authors considered complex polynomials of the form $P(z) = a_0 + \sum_{r}^{p} a_\ell z^\ell + a_n z^n$ in connection to locations of zeros [5]. An additional result follows from Corollary 1 by applying it to a quaternionic polynomial of the form $P(q) = a_0 + \sum_{\ell=r}^{p} q^\ell a_\ell + q^n a_n$ (with the coefficients satisfying the hypotheses of Corollary 1). This result gives the location of the zeros of P as stated in Corollary 1, where $M_r = |a_0| + |a_r|$ and $M_p = |a_p| + |a_n|$.

2 Proof of Theorem 5

We adopt the standard that polynomials have the indeterminate on the left and the coefficients on the right, so that we have quaternionic polynomials of the form $P_1(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$. With $P_2(q) = \sum_{\ell=0}^{m} q^\ell b_\ell$, we have the *regular product*

$$(P_1 * P_2)(q) = \sum_{i=0,1,\dots,n;\, j=0,1,\dots,m} q^{i+j} a_i b_j.$$

Zeros of regular products of quaternionic polynomials behave as follows (see [14]).

Theorem 6 *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies*

$$g(f(q_0)^{-1} q_0 f(q_0)) = 0.$$

Gentili and Struppa [8] introduced a Maximum Modulus theorem for regular functions.

Theorem 7 *Let $B = B(0, r)$ be an open ball in $\mathbb{H}$ with center 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is constant on B.*

Now we give the proof of our main result.

Proof of Theorem 5 Define $f(q)$ with the equation

$$\begin{aligned} P(q) * (1 - q) &= \left(\sum_{\ell=0}^{n} q^\ell a_\ell \right) * (1 - q) \\ &= a_0 + \sum_{\ell=1}^{n} q^\ell (a_\ell - a_{\ell-1}) - q^{n+1} a_n \\ &= f(q) - q^{n+1} a_n. \end{aligned}$$

By Theorem 6, $P(q) * (1 - q) = 0$ if and only if either $P(q) = 0$ or $P(q) \neq 0$ implies $1 - P(q)^{-1} q P(q) = 0$. Note that $1 - P(q)^{-1} q P(q) = 0$ implies $q = 1$. Hence, the only zeros of $P(q) * (1 - q)$ are $q = 1$ and the zeros of $P(q)$. Thus, for $|q| = 1$,

$$\begin{aligned} |f(q)| &= \left| a_0 + \sum_{\ell=1}^{n} q^\ell (a_\ell - a_{\ell-1}) \right| \leq |a_0| + \sum_{\ell=1}^{n} |q|^\ell |a_\ell - a_{\ell-1}| \\ &= |a_0| + \sum_{\ell=1}^{n} |a_\ell - a_{\ell-1}| = |a_0| + M_r + \sum_{\ell=r+1}^{p} |a_\ell - a_{\ell-1}| + M_p \end{aligned}$$

$$
\begin{aligned}
&\leq |a_0| + M_r + \sum_{\ell=r+1}^{p} (|\alpha_\ell - \alpha_{\ell-1}| + |\beta_\ell - \beta_{\ell-1}| + |\gamma_\ell - \gamma_{\ell-1}| \\
&+ |\delta_\ell - \delta_{\ell-1}|) + M_p \\
&\leq |a_0| + M_r + |\alpha_{r+1} - \rho_{R_1}\alpha_r| + |\rho_{R_1}\alpha_r - \alpha_r| - \alpha_{r+1} + \alpha_{\eta-1} \\
&+ |\alpha_\eta - k_R\alpha_\eta| + |k_R\alpha_\eta - \alpha_{\eta-1}| + |\alpha_{\eta+1} - k_R\alpha_\eta| + |k_R\alpha_\eta - \alpha_\eta| \\
&+ \alpha_{\eta+1} - \alpha_{p-1} + |\alpha_p - \rho_{R_2}\alpha_p| + |\rho_{R_2}\alpha_p - \alpha_{p-1}| + |\beta_{r+1} - \rho_{I_1}\beta_r| \\
&+ |\rho_{I_1}\beta_r - \beta_r| - \beta_{r+1} + \beta_{\eta-1} + |\beta_\eta - k_I\beta_\eta| + |k_I\beta_\eta - \beta_{\eta-1}| \\
&+ |\beta_{\eta+1} - k_I\beta_\eta| + |k_I\beta_\eta - \beta_\eta| + \beta_{\eta+1} - \beta_{p-1} + |\beta_p - \rho_{I_2}\beta_p| \\
&+ |\rho_{I_2}\beta_p - \beta_{p-1}| + |\gamma_{r+1} - \rho_{J_1}\gamma_r| + |\rho_{J_1}\gamma_r - \gamma_r| - \gamma_{r+1} + \gamma_{\eta-1} \\
&+ |\gamma_\eta - k_J\gamma_\eta| + |k_J\gamma_\eta - \gamma_{\eta-1}| + |\gamma_{\eta+1} - k_J\gamma_\eta| + |k_J\gamma_\eta - \gamma_\eta| + \gamma_{\eta+1} \\
&- \gamma_{p-1} + |\gamma_p - \rho_{J_2}\gamma_p| + |\rho_{J_2}\gamma_p - \gamma_{p-1}| + |\delta_{r+1} - \rho_{K_1}\delta_r| + |\rho_{K_1}\delta_r - \delta_r| \\
&- \delta_{r+1} + \delta_{\eta-1} + |\delta_\eta - k_K\delta_\eta| + |k_K\delta_\eta - \delta_{\eta-1}| + |\delta_{\eta+1} - k_K\delta_\eta| \\
&+ |k_K\delta_\eta - \delta_\eta| + \delta_{\eta+1} - \delta_{p-1} + |\delta_p - \rho_{K_2}\delta_p| + |\rho_{K_2}\delta_p - \delta_{p-1}| + M_p \\
&= |a_0| + M_r - \rho_{R_1}\alpha_r + |\alpha_r|(1 - \rho_{R_1}) + 2|\alpha_\eta|(k_R - 1) + 2k_R\alpha_\eta \\
&+ |a_p|(1 - \rho_{R_2}) - \rho_{R_2}\alpha_p - \rho_{I_1}\beta_r + |\beta_r|(1 - \rho_{I_1}) + 2|\beta_\eta|(k_I - 1) \\
&+ 2k_I\beta_\eta + |a_p|(1 - \rho_{I_2}) - \rho_{I_2}\beta_p - \rho_{J_1}\gamma_r + |\gamma_r|(1 - \rho_{J_1}) + 2|\gamma_\eta|(k_J - 1) \\
&+ 2k_J\gamma_\eta + |a_p|(1 - \rho_{J_2}) - \rho_{J_2}\gamma_p - \rho_{K_1}\delta_r + |\delta_r|(1 - \rho_{K_1}) \\
&+ 2|\delta_\eta|(k_K - 1) + 2k_K\delta_\eta + |a_p|(1 - \rho_{K_2}) - \rho_{K_2}\delta_p + M_p \\
&= M.
\end{aligned}
$$

Note that $q^n f(1/q)$ has the same bound on $|q| = 1$ as $f(q)$. Thus, by Theorem 7, for $|q| \leq 1$, we have $|q^n f(1/q)| \leq M$, and hence, $|f(1/q)| \leq M/|q|^n$. Replacing q with $1/q$ we have $|f(q)| \leq M|q|^n$ for $|q| \geq 1$. Hence, for $|q| \geq 1$,

$$
\begin{aligned}
|P(q) * (1 - q)| = |f(q) - q^{n+1}a_n| &\geq |q^{n+1}||a_n| - |f(q)| \\
&\geq |q^{n+1}||a_n| - M|q|^n = |q|^n(|q||a_n| - M).
\end{aligned}
$$

Thus, if $|q| > M/|a_n|$, then $P(q) * (1 - q) \neq 0$. Therefore, all the zeros of $P(q)$ lie in $|q| \leq M/|a_n|$, as claimed.

Next, consider $S(q) = q^n * P(1/q) = \sum_{\ell=0}^{n} q^{n-\ell}a_\ell$, and let

$$H(q) = S(q) * (1 - q) = -a_0q^{n+1} + \sum_{\ell=1}^{n} q^{n+1-\ell}(a_{\ell-1} - a_\ell) + a_n.$$

Then

$$|H(q)| \geq |q|^{n+1}|a_0| - \left\{ \sum_{\ell=1}^{n} |q|^{n+1-\ell}|a_{\ell-1} - a_\ell| + |a_n| \right\}$$

$$\geq |q|^{n+1}|a_0| - \Bigg\{ \sum_{\ell=1}^{r} |q|^{n+1-\ell}|a_{\ell-1} - a_\ell| + |q|^{n-r}|\alpha_r|(1-\rho_{R_1})$$
$$+ |q|^{n-r}(\alpha_{r+1} - \rho_{R_1}\alpha_r) + |q|^{n-r}|\beta_r|(1-\rho_{I_1}) + |q|^{n-r}(\beta_{r+1} - \rho_{I_1}\beta_r)$$
$$+ |q|^{n-r}|\gamma_r|(1-\rho_{J_1}) + |q|^{n-r}(\gamma_{r+1} - \rho_{J_1}\gamma_r) + |q|^{n-r}|\delta_r|(1-\rho_{K_1})$$
$$+ |q|^{n-r}(\delta_{r+1} - \rho_{K_1}\delta_r) + \sum_{\ell=r+2}^{\eta-1} \big(|q|^{n+1-\ell}|\alpha_{\ell-1} - \alpha_\ell| + |q|^{n+1-\ell}|\beta_{\ell-1} - \beta_\ell|$$
$$+ |q|^{n+1-\ell}|\gamma_{\ell-1} - \gamma_\ell| + |q|^{n+1-\ell}|\delta_{\ell-1} - \delta_\ell|\big) + |q|^{n+1-\eta}(k_R\alpha_\eta - \alpha_{\eta-1})$$
$$+ |q|^{n+1-\eta}|\alpha_\eta|(k_R - 1) + |q|^{n+1-\eta}(k_I\beta_\eta - \beta_{\eta-1}) + |q|^{n+1-\eta}|\beta_\eta|(k_I - 1)$$
$$+ |q|^{n+1-\eta}(k_J\gamma_\eta - \gamma_{\eta-1}) + |q|^{n+1-\eta}|\gamma_\eta|(k_J - 1) + |q|^{n+1-\eta}(k_K\delta_\eta - \delta_{\eta-1})$$
$$+ |q|^{n+1-\eta}|\delta_\eta|(k_K - 1) + |q|^{n-\eta}|\alpha_\eta|(k_R - 1) + |q|^{n-\eta}(k_R\alpha_\eta - \alpha_{\eta+1})$$
$$+ |q|^{n-\eta}|\beta_\eta|(k_I - 1) + |q|^{n-\eta}(k_I\beta_\eta - \beta_{\eta+1}) + |q|^{n-\eta}|\gamma_\eta|(k_J - 1)$$
$$+ |q|^{n-\eta}(k_J\gamma_\eta - \gamma_{\eta+1}) + |q|^{n-\eta}|\delta_\eta|(k_K - 1) + |q|^{n-\eta}(k_K\delta_\eta - \delta_{\eta+1})$$
$$+ \sum_{\ell=\eta+2}^{p-1} \big(|q|^{n+1-\ell}|\alpha_{\ell-1} - \alpha_\ell| + |q|^{n+1-\ell}|\beta_{\ell-1} - \beta_\ell| + |q|^{n+1-\ell}|\gamma_{\ell-1} - \gamma_\ell|$$
$$+ |q|^{n+1-\ell}|\delta_{\ell-1} - \delta_\ell|\big) + |q|^{n+1-p}(\alpha_{p-1} - \rho_{R_2}\alpha_p) + |q|^{n+1-p}|\alpha_p|(1-\rho_{R_2}\big)$$
$$+ |q|^{n+1-p}(\beta_{p-1} - \rho_{I_2}\beta_p) + |q|^{n+1-p}|\beta_p|(1-\rho_{I_2}) + |q|^{n+1-p}(\gamma_{p-1} - \rho_{J_2}\gamma_p)$$
$$+ |q|^{n+1-p}|\gamma_p|(1-\rho_{J_2}) + |q|^{n+1-p}(\delta_{p-1} - \rho_{K_2}\delta_p)|q|^{n+1-p}|\delta_p|(1-\rho_{K_2}) +$$
$$+ \sum_{\ell=p+1}^{n} |q|^{n+1-\ell}|a_{\ell-1} - a_\ell| + |a_n|\Bigg\}.$$

Thus,

$$|H(q)| \geq |q|^n \Bigg[|q||a_0| - \Bigg\{ \sum_{\ell=1}^{r} |q|^{1-\ell}|a_{\ell-1} - a_\ell| + |q|^{-r}|\alpha_r|(1-\rho_{R_1})$$
$$+ |q|^{-r}(\alpha_{r+1} - \rho_{R_1}\alpha_r) + |q|^{-r}|\beta_r|(1-\rho_{I_1}) + |q|^{-r}(\beta_{r+1} - \rho_{I_1}\beta_r)$$
$$+ |q|^{-r}|\gamma_r|(1-\rho_{J_1}) + |q|^{-r}(\gamma_{r+1} - \rho_{J_1}\gamma_r) + |q|^{-r}|\delta_r|(1-\rho_{K_1})$$
$$+ |q|^{-r}(\delta_{r+1} - \rho_{K_1}\delta_r) + \sum_{\ell=r+2}^{\eta-1} \big(|q|^{1-\ell}|\alpha_{\ell-1} - \alpha_\ell| + |q|^{1-\ell}|\beta_{\ell-1} - \beta_\ell|$$
$$+ |q|^{1-\ell}|\gamma_{\ell-1} - \gamma_\ell| + |q|^{1-\ell}|\delta_{\ell-1} - \delta_\ell|\big) + |q|^{1-\eta}(k_R\alpha_\eta - \alpha_{\eta-1})$$
$$+ |q|^{1-\eta}|\alpha_\eta|(k_R - 1) + |q|^{1-\eta}(k_I\beta_\eta - \beta_{\eta-1}) + |q|^{1-\eta}|\beta_\eta|(k_I - 1)$$
$$+ |q|^{1-\eta}(k_J\gamma_\eta - \gamma_{\eta-1}) + |q|^{1-\eta}|\gamma_\eta|(k_J - 1) + |q|^{1-\eta}(k_K\delta_\eta - \delta_{\eta-1})$$
$$+ |q|^{1-\eta}|\delta_\eta|(k_K - 1) + |q|^{-\eta}|\alpha_\eta|(k_R - 1) + |q|^{-\eta}(k_R\alpha_\eta - \alpha_{\eta+1})$$
$$+ |q|^{-\eta}|\beta_\eta|(k_I - 1) + |q|^{-\eta}(k_I\beta_\eta - \beta_{\eta+1}) + |q|^{-\eta}|\gamma_\eta|(k_J - 1)$$
$$+ |q|^{-\eta}(k_J\gamma_\eta - \gamma_{\eta+1}) + |q|^{-\eta}|\delta_\eta|(k_K - 1) + |q|^{-\eta}(k_K\delta_\eta - \delta_{\eta+1})$$

$$
\begin{aligned}
&+\sum_{\ell=\eta+2}^{p-1}\big(|q|^{1-\ell}|\alpha_{\ell-1}-\alpha_\ell|+|q|^{1-\ell}|\beta_{\ell-1}-\beta_\ell|+|q|^{1-\ell}|\gamma_{\ell-1}-\gamma_\ell|\\
&+|q|^{1-\ell}|\delta_{\ell-1}-\delta_\ell|\big)+|q|^{1-p}(\alpha_{p-1}-\rho_{R_2}\alpha_p)+|q|^{1-p}|\alpha_p|(1-\rho_{R_2})\\
&+|q|^{1-p}(\beta_{p-1}-\rho_{I_2}\beta_p)+|q|^{1-p}|\beta_p|(1-\rho_{I_2})+|q|^{1-p}(\gamma_{p-1}-\rho_{J_2}\gamma_p)\\
&+|q|^{1-p}|\gamma_p|(1-\rho_{J_2})+|q|^{1-p}(\delta_{p-1}-\rho_{K_2}\delta_p)+|q|^{1-p}|\delta_p|(1-\rho_{K_2})\\
&+\sum_{\ell=p+1}^{n}|q|^{1-\ell}|a_{\ell-1}-a_\ell|+|a_n|/|q|^n\Bigg\}\Bigg].
\end{aligned}
$$

For $|q|>1$, and hence, $1/(|q|^{n-\ell})\le 1$ for $0\le\ell<n$, we have

$$
\begin{aligned}
|H(q)|\ge|q|^n\Bigg[&|q||a_0|-\Bigg\{M_r+|\alpha_r|(1-\rho_{R_1})-\rho_{R_1}\alpha_r+|\beta_r|(1-\rho_{I_1})-\rho_{I_2}\beta_r\\
&+|\gamma_r|(1-\rho_{J_1})-\rho_{J_1}\gamma_r+|\delta_r|(1-\rho_{K_1})-\rho_{K_1}\delta_r+2k_R\alpha_\eta\\
&+2|\alpha_\eta|(k_R-1)+2k_I\beta_\eta+2|\beta_\eta|(k_I-1)+2k_J\gamma_\eta+2|\gamma_\eta|(k_J-1)\\
&+2k_K\delta_\eta+2|\delta_\eta|(k_K-1)-\rho_{R_2}\alpha_p+|\alpha_p|(1-\rho_{R_2})-\rho_{I_2}\beta_p+|\beta_p|(1-\rho_{R_2})\\
&-\rho_{J_2}\gamma_p+|\gamma_p|(1-\rho_{J_2})-\rho_{K_2}\delta_p+|\delta_p|(1-\rho_{K_2})+M_p+|a_n|\Bigg\}\Bigg]\\
&=|q|^n(|q||a_0|-(M-|a_0|+|a_n|)).
\end{aligned}
$$

Note that $|H(q)|\ge|q|^n(|q||a_0|-(M-|a_0|+|a_n|))>0$ if $|q|>(M-|a_0|+|a_n|)/|a_0|$. Thus, all the zeros of $H(q)$ whose modulus is greater than 1 lie in $|q|\le(M-|a_0|+|a_n|)/|a_0|$. Hence, all the zeros of $H(q)$ and thus, of $S(q)$ lie in $|q|\le\max\{1,(M-|a_0|+|a_n|)/|a_0|\}$. Therefore, all the zeros of $P(q)$ lie in $|q|\ge\min\{1,|a_0|/(M-|a_0|+|a_n|)$, as claimed. □

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