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A Constructive Approach to Bivariate Hyperbolic Box Spline Functions

H. Jena

Abstract. This article is based on the construction procedure of bivariate hyperbolic box spline functions. Generally, box splines are considered as the multivariate generalizations of univariate B-splines. Both B-splines and box splines are refinable functions. Two different kinds of box splines like the polynomial box splines and the trigonometric box splines along with their usefulness are well studied in literature. However, another variant of box splines named as the class of hyperbolic box spline functions, has not gained much attention. This article focuses on the construction of bivariate hyperbolic box spline functions from univariate hyperbolic B-spline functions through directional convolution method. Also, the importance and usefulness of such functions are discussed.

Key Words: Hyperbolic B-Spline, Box Spline, Directional Convolution, Direction Matrix
Mathematics Subject Classification 2020: 65D07, 65D17

Introduction

Computer-Aided Geometric Design (CAGD) focuses on the construction and manipulation of geometric shapes. Describing a complex geometric shape can be a daunting task, often requiring weeks or even months for artists and designers to describe a single geometric shape. While various mathematical tools are available to represent curves and surfaces, piecewise polynomial functions act as the simplest and one of the most effective methods in CAGD. Their ease of representation makes them a popular choice for defining geometric shapes. Among the different types of piecewise polynomial functions, the classes of spline functions hold a particularly significant role. The term *spline* originally is referred to a flexible ruler used for drawing curves, especially in the aircraft and shipbuilding industries. According to [17], the concept of splines dates back to the nineteenth century with the work of Lobachevsky, who utilized convolution methods for spline construction. However, it was not until 1946 that the work of Schoenberg [26] marked the advent of modern spline approximation theory. Over recent decades, extensive research into spline theories has flourished, driven by their crucial applications in contemporary numerical mathematics.

Among the different types of splines [28], the two important classes of spline functions, such as the polynomial and the trigonometric spline functions have attracted the most attention, primarily due to their extensive applications in numerical computations. This popularity stems largely from the fact that polynomial splines utilize a basis of B-splines, which can be computed both efficiently and accurately through methods such as recurrence relations and convolution. In recent past years, several new kinds of splines and their basis functions have been designed for geometric modeling in CAGD. For instance, Schoenberg introduces trigonometric splines in 1964 [27]. In 1996, Zhang [29] introduces C-B splines by extending cubic uniform B-splines. The following year, he further advanced this work by presenting a new parameterized form of C-B splines and explaining their variation-diminishing properties [30]. The authors of [23] designed a new kind of uniform splines named *trigonometric polynomial B-splines* over the space $span\{\sin t, \cos t, t^{k-3}, t^{k-4}, \dots, t, 1\}$ where k is an arbitrary integer greater than or equal to 3. Again, the same authors introduced another kind of splines named *hyperbolic polynomial B-splines* over the space of $span\{\sinh t, \cosh t, t^{k-3}, t^{k-4}, \dots, t, 1\}$ where k is an arbitrary integer greater than or equal to 3 [22]. The incorporation of hyperbolic functions allows for exact representations of critical curves like the catenary and hyperbola and simplifies the computation of derivatives and integrals. Hyperbolic splines are widely used in data regression. For instance, *HP-splines* ([1,2]) are useful when the data has an exponential trend. They are particularly effective in fitting exponential data with perfect accuracy, regardless of the parameter values. The paper [3] outlines a methodology based on linear algebra for selecting the frequency parameter of HP-splines within an exponential polynomial space. Additionally, the authors in [4] introduce a natural smoothing exponential-polynomial spline to model data that decays exponentially towards zero. For further insights on hyperbolic splines, refer to [13], [10], and [9], among other.

In contrast, box splines are piecewise polynomials that are locally supported on uniform grids, similar to B-splines but in multivariate contexts. Out of many constructive ways, one straightforward method for constructing box splines involves using the directional convolution of B-splines, which grants them properties analogous to those of B-splines. It is a well-known fact that using the directional convolution approach one can choose various directions to have a box spline function with desired order of smoothness.

Among the various types of box splines, while the polynomial and the trigonometric box splines have been extensively studied, the hyperbolic box splines have not been as thoroughly explored, partly due to a lack of systematic analysis. This article aims to address this gap by describing a comprehensive theory for constructing and systematically analyzing a class of hyperbolic box splines of various orders. Our work is motivated by previous studies on hyperbolic polynomial B-splines in [22]. We use them to construct hyperbolic box spline functions through the directional convolution method [11,12,14].

The remainder of this article is organized as follows: Section 1 provides a review of essential preliminaries necessary for understanding the main content. Section 2 describes the construction of hyperbolic box spline functions of various orders. In Section 3, we explore fundamental properties of these hyperbolic box spline functions. Section 4 presents a discussion of the results, and Section 5 concludes with a summary of the findings.

1 Basic Preliminaries

This section recalls some basic preliminaries.

First we give the definition of the convolution product of two functions [15] and then we define the univariate cardinal B-splines [25] through convolution.

Definition 1 *Let f and g be two sufficiently smooth real functions. Then their convolution product $f * g$ is defined by*

$$(f * g)(u) = \int_{-\infty}^{\infty} f(u-t)g(t)dt, \tag{1}$$

where '$*$' denotes the convolution product operator.

Definition 2 *The cardinal B-spline function $\mathbf{B}_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}$, is defined recursively by*

$$\mathbf{B}_n(x) = (\mathbf{B}_{n-1} * \mathbf{B}_1)(x) = \int_0^1 \mathbf{B}_{n-1}(x-t)\, dt \tag{2}$$

where

$$\mathbf{B}_1(x) := \chi_{[0,1)}(x) = \begin{cases} 1, & \text{if } 0 \le x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The repeated use of the convolution product along with the commutativity and associativity properties of convolution product of two functions leads to the following result.

Lemma 1 *For any $l \in \mathbb{N}$ with $l < n$, we have*

$$\mathbf{B}_n = \mathbf{B}_{n-1} * \mathbf{B}_1 = \ldots = \underbrace{\mathbf{B}_1 * \mathbf{B}_1 * \ldots * \mathbf{B}_1}_{n\ times} = \mathbf{B}_l * \mathbf{B}_{n-l}. \tag{4}$$

The univariate cardinal B-splines can be used to define bivariate polynomial box spline functions. Before defining them, we first introduce the concept of a *polygonal mesh*.

A polygonal mesh or simply a mesh is a set of connected polygons. Generally, a mesh is a collection of polygonal faces that include vertices and edges. If all the polygonal faces in a mesh are quadrilaterals, the mesh is referred to as a *two direction mesh* or a *quadrilateral mesh*. On the other hand, if all the polygonal faces are triangles, the mesh is called a *three direction mesh* or a *type-I triangular mesh*. This triangulation is obtained by drawing the north-east diagonals in the bi-infinity grid, where the grid lines are positioned at integer values. A three direction mesh is associated with the three directions: $\mathbf{d}_1 := [1,0]^T$, $\mathbf{d}_2 := [0,1]^T$ and $\mathbf{d}_3 := [1,1]^T$ whereas a two direction mesh is only associated with $\mathbf{d}_1$ and $\mathbf{d}_2$ (see Figure 1).

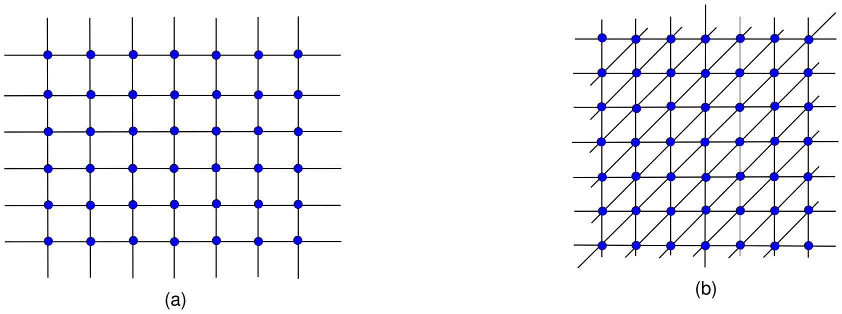

Figure 1: (a) Two direction mesh. (b) Three direction mesh.

Definition 3 *Let $i, j \in \mathbb{N}$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbf{B}_n(x)$, $n \in \{i,j,k\}$ be the univariate cardinal B-splines. Then a bivariate box spline function on a three direction mesh $\mathbf{B}_{i,j,k}(x,y)$ can be constructed as follows:*

$$\mathbf{B}_{i,j,k}(x,y) := \int_{\mathbb{R}} \mathbf{B}_i(x-\tau)\mathbf{B}_j(y-\tau)\mathbf{B}_k(\tau)d\tau, \qquad (x,y) \in \mathbb{R}^2. \tag{5}$$

Bivariate box spline functions defined on two direction meshes are referred to as tensor product box spline functions. While the tensor product representation is highly efficient for computation, it struggles with complex shape modeling [24]. In contrast, three direction meshes offer greater flexibility, making them better suited for adaptation to various topologies. Box splines on three direction meshes can handle complex domains better than their tensor product counterparts [24]. The focus of this paper is on bivariate box splines defined on three-direction meshes.

Next, we recall some basic preliminaries on hyperbolic polynomial B-splines.

Let the parameter axis-x be uniformly partitioned into a set of knots $x_i = i\lambda$ $(i = 0, \pm 1, \pm 2, \ldots)$ with $\lambda \geq 0$ as the interval length. By $\Omega_{n,\lambda}$, we denote the set of all piecewise hyperbolic polynomial splines of order n defined on $[x_i, x_{i+1}]$, $i = 0, \pm 1, \pm 2, \ldots$, in which for all i, each function is a hyperbolic polynomial of order n on the interval $[x_i, x_{i+1}]$ and $(n-2)$-times continuously differentiable at the knot x_i. Being closed under the operations of addition and scalar multiplication of functions, it is not very hard to observe that $\Omega_{n,\lambda}$ is a linear space. There does not exist any hyperbolic polynomial B-spline basis over $\Omega_{2,\lambda}$ (see [22], Theorem 1). However, for $n \geq 3$, there exists a set of basis functions defined over $\Omega_{n,\lambda}$ which is called as the hyperbolic polynomial B-spline basis functions over the space $\Omega_{n,\lambda}$. For basis construction, first, we need to define a set of functions over $\Omega_{2,\lambda}$. Let

$$\mathbf{H}_{0,2}(x;\lambda) = \begin{cases} \dfrac{\lambda \sinh x}{4\sinh^2(\lambda/2)}, & 0 \leq x \leq \lambda, \\[2ex] \dfrac{\lambda \sinh(2\lambda - x)}{4\sinh^2(\lambda/2)}, & \lambda \leq x \leq 2\lambda, \\[2ex] 0, & \text{elsewhere}, \end{cases} \tag{6}$$

and

$$\mathbf{H}_{i,2}(x;\lambda) = \mathbf{H}_{0,2}(x - i\lambda;\lambda), \qquad i = 0, \pm 1, \pm 2, \ldots \tag{7}$$

Then for $n \geq 3$,

$$\mathbf{H}_{0,n}(x;\lambda) = \frac{1}{\lambda}\int_{x-\lambda}^{x} \mathbf{H}_{0,n-1}(t;\lambda)dt, \tag{8}$$

and

$$\mathbf{H}_{i,n}(x;\lambda) = \mathbf{H}_{0,n}(x - i\lambda;\lambda), \qquad i = 0, \pm 1, \pm 2, \ldots \tag{9}$$

The right hand side of (8) is, in fact, $(1/\lambda)$-times the convolution product of $\mathbf{H}_{0,n-1}$ and $\mathbf{H}_{0,1}$ where

$$\mathbf{H}_{0,1}(x;\lambda) = \begin{cases} 1, & 0 \leq x \leq \lambda, \\ 0, & \text{elsewhere}. \end{cases} \tag{10}$$

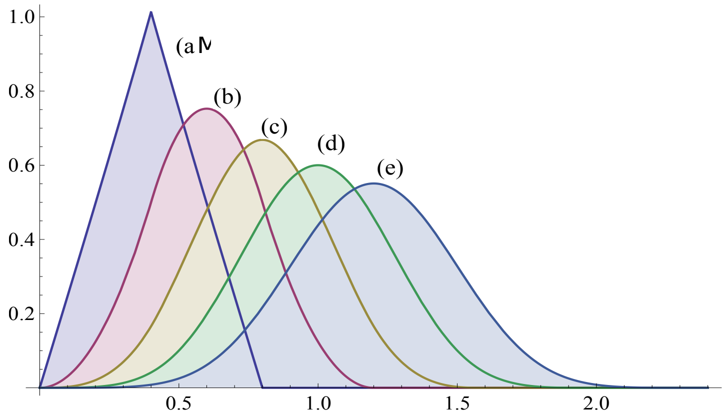

Figure 2: The graphs of (a) $\mathbf{H}_{0,3}(x;\frac{\pi}{4})$, (b) $\mathbf{H}_{0,4}(x;\frac{\pi}{4})$, (c) $\mathbf{H}_{0,5}(x;\frac{\pi}{4})$ and (d) $\mathbf{H}_{0,6}(x;\frac{\pi}{4})$.

The hyperbolic B-spline functions for order $3-6$ are given in Figure 2.

Let us describe some basic properties of this new class of hyperbolic polynomial B-splines of order n. Most of them directly follow from the definition.

Theorem 1 *For $n \geq 3$, we have*

(i) Non-negativity: $\mathbf{H}_{i,n}(x;\lambda) \geq 0$, $x \in (-\infty, \infty)$.

(ii) Local support: $\mathbf{H}_{i,n}(x;\lambda) \begin{cases} > 0, & x \in (i\lambda, (i+n)\lambda), \\ = 0, & \textit{elsewhere}. \end{cases}$

This means, the local support of $\mathbf{H}_{i,n}(x;\lambda)$ is n intervals. This is why we say it is of order n.

(iii) Linear independence: $\mathbf{H}_{i,n}(x;\lambda), \mathbf{H}_{i+1,n}(x;\lambda), \ldots, \mathbf{H}_{i+l,n}(x;\lambda)$ $(l \geq n)$ *are linear independent on interval* $[(i+n-1)\lambda, (i+l+1)\lambda]$.

(iv) Partition of unity: $\lambda^{-1} \sum_i \mathbf{H}_{i,n}(x;\lambda) \equiv 1$.

(v) Continuity: $\mathbf{H}_{i,n}(x;\lambda)$ *is* $(n-2)$*-times differentiable on the whole parameter space.*

(vi) Differentiation: $\mathbf{H}'_{i,n}(x;\lambda) = \lambda^{-1}\left(\mathbf{H}_{i,n-1}(x;\lambda) - \mathbf{H}_{i+1,n-1}(x;\lambda)\right)$.

(vii) Symmetry: $\mathbf{H}_{i,n}((i+n)\lambda - x;\lambda) = \mathbf{H}_{i,n}(x + i\lambda;\lambda)$, $x \in [0,\ n\lambda]$.

Proof. All the basic properties hold true for $\mathbf{H}_{0,3}(x;\lambda)$, and it is also evident in the graph of $\mathbf{H}_{0,3}(x;\lambda)$. Again, for $n > 4$, the properties *(i)*-*(vi)* are apparent since they are obtained by direct integration of $\mathbf{H}_{0,3}(x;\lambda)$. Further,

$$\begin{aligned}\mathbf{H}'_{i,n}(x;\lambda) =& \frac{d}{dx}\left(\frac{1}{\lambda}\int_{x-\lambda}^{x}\mathbf{H}_{i,n-1}(t;\lambda)dt\right)\\ =& \frac{1}{\lambda}\big(\mathbf{H}_{i,n-1}(x;\lambda)-\mathbf{H}_{i,n-1}(x-\lambda;\lambda)\big).\end{aligned}$$

But by following the definition 9 in the right hand side of the above equation, $\mathbf{H}_{i,n-1}(x-\lambda;\lambda)$ is equal with $\mathbf{H}_{i+1,n-1}(x;\lambda)$, and this proves *(vi)*.

The equation (9) implies $\mathbf{H}_{i,n}(i\lambda+x;\lambda)=\mathbf{H}_{0,n}(x;\lambda)$, $x\in[0,n\lambda]$. Hence, to prove *(vii)*, we only need to show that

$$\mathbf{H}_{0,n}(n\lambda-x;\lambda)=\mathbf{H}_{0,n}(x;\lambda). \tag{11}$$

Equation (11) is truly satisfied for $\mathbf{H}_{0,3}(x;\lambda)$. For $n=4$, we can write

$$\begin{aligned}\mathbf{H}_{0,4}(4\lambda-x;\lambda) =& \frac{1}{\lambda}\int_{4\lambda-x-\lambda}^{4\lambda-x}\mathbf{H}_{0,3}(t;\lambda)dt\\ =& \frac{1}{\lambda}\int_{3\lambda-x}^{4\lambda-x}\mathbf{H}_{0,3}(3\lambda-t;\lambda)dt\\ =& \frac{1}{\lambda}\int_{x-\lambda}^{x}\mathbf{H}_{0,3}(y;\lambda)dy\\ =& \mathbf{H}_{0,4}(x;\lambda).\end{aligned}$$

Now, suppose the property holds for $n=l-1$. Then for $n=l$,

$$\begin{aligned}\mathbf{H}_{0,l}(x;\lambda) =& \frac{1}{\lambda}\int_{x-\lambda}^{x}\mathbf{H}_{0,l-1}(t;\lambda)dt\\ =& \frac{1}{\lambda}\int_{x-\lambda}^{x}\mathbf{H}_{0,l-1}((l-1)\lambda-t;\lambda)dt\\ =& \frac{1}{\lambda}\int_{x}^{x+\lambda}\mathbf{H}_{0,l-1}(l\lambda-y;\lambda)dy\\ =& \mathbf{H}_{0,l}(l\lambda-x;\lambda).\end{aligned}$$

Therefore, by method of induction, property *(vii)* is true for all l. This completes the proof. □

Thus, the new class of hyperbolic algebraic B-spline functions, i.e., $\mathbf{H}_{i,n}(x;\lambda)$, $i=0,\pm1,\pm2,\ldots$, constitute a set of bases in $\Omega_{n,\lambda}$ $(n\geq 3)$.

2 Construction of hyperbolic box spline functions

In this section, we construct a family of hyperbolic box spline functions using the *directional convolution* method. Let us recall the definition of the directional convolution method [11].

Definition 4 *Let* $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ *and* $\mathbf{g} : \mathbb{R} \to \mathbb{R}$ *be two refinable functions, bivariate and univariate, respectively. The convolution product between* $\mathbb{F}$ *and* g *along the direction* $\mathbf{d} \in \mathbb{Z}^{\mathbf{2}}$ *is defined as*

$$\mathbf{T}(\mathbf{x}) := (\mathbf{F} *_{\mathbf{d}} \mathbf{g})(\mathbf{x}) := \int_{\mathbb{R}} \mathbf{F}(\mathbf{x} - \mathbf{d}\tau)\mathbf{g}(\tau)\mathbf{d}\tau, \qquad \mathbf{x} \in \mathbb{R}^{\mathbf{2}}. \tag{12}$$

The bivariate function $\mathbf{F}$ *can also be taken as the tensor product of two univariate functions.*

In analogy with the Definition 4, we can consider the tensor product of $\mathbf{H}_{0,i}$ and $\mathbf{H}_{0,j}$ as the bivariate function $\mathbf{F}$ and $\mathbf{H}_{0,k}$ as the univariate function $\mathbf{g}$, respectively, for the construction of the hyperbolic box spline functions as follows.

Definition 5 *Let* $\mathbf{H}_{0,n}(x;\lambda)$*,* $n \in \{i,j,k\}$*, be the univariate hyperbolic B-splines where* $i,j,k \geq 3$*. Then a hyperbolic box spline function,* $\mathbf{H}_{i,j,k}(x,y;\lambda)$*,* $(x,y) \in \mathbb{R}^2$*, can be constructed as follows:*

$$\mathbf{H}_{i,j,k}(x,y;\lambda) := \frac{1}{\lambda}\int_{\mathbb{R}} \mathbf{H}_{0,i}(x - \tau\lambda;\lambda)\mathbf{H}_{0,j}(y - \tau\lambda;\lambda)\mathbf{H}_{0,k}(\tau\lambda;\lambda)d\tau. \tag{13}$$

The hyperbolic box spline $\mathbf{H}_{i,j,k}$ is associated with three directions $\lambda\mathbf{d}_1$, $\lambda\mathbf{d}_2$ and $\lambda\mathbf{d}_3$ each repeated i-times, j-times and k-times, respectively, with i, j and k being all greater than or equal to 3. Hence, the direction matrix Ξ_μ for $\mathbf{H}_{i,j,k}$ is given by

$$\Xi_\mu := \lambda[\underbrace{\mathbf{d}_1,\ldots,\mathbf{d}_1}_{i},\underbrace{\mathbf{d}_2,\ldots,\mathbf{d}_2}_{j},\underbrace{\mathbf{d}_3,\ldots,\mathbf{d}_3}_{k}], \tag{14}$$

with $\mu := i + j + k$ being the total number of entries in Ξ_μ. Generally, it is known that a box spline does not depend on the ordering of the direction vector elements contained in its direction matrix. Thus, the rearrangement of the entries in (14) expresses Ξ_μ as a sequence of direction vector elements with three or more identical entries, i.e., $\Xi_\mu := \lambda[\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mu]$, $\mathbf{e}_l \in \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $l = 1, 2, \ldots, \mu$.

Theorem 2 *The hyperbolic box spline functions* $\mathbf{H}_{i,j,k}$*,* $i,j,k \geq 3$*, depend on their corresponding direction matrices.*

Proof. Let Ξ_μ be the direction matrix associated with the hyperbolic box spline functions $\mathbf{H}_{i,j,k}$. The hyperbolic box spline $\mathbf{H}_{i+1,j,k}$ can be computed from $\mathbf{H}_{i,j,k}$ by integrating $\mathbf{H}_{i,j,k}$ in the direction $\lambda\mathbf{d}_1$. Similarly, the hyperbolic box splines $\mathbf{H}_{i,j+1,k}$ and $\mathbf{H}_{i,j,k+1}$, respectively, can be derived from $\mathbf{H}_{i,j,k}$ by integrating $\mathbf{H}_{i,j,k}$ in the directions $\lambda\mathbf{d}_2$ and $\lambda\mathbf{d}_3$. Then for $\mathbf{H}_{i+1,j,k}$

the direction matrix will be $\Xi_\mu \cup \{\lambda \mathbf{d}_1\}$. Similarly, the direction matrices $\Xi_\mu \cup \{\lambda \mathbf{d}_2\}$ and $\Xi_\mu \cup \{\lambda \mathbf{d}_3\}$ will be for $\mathbf{H}_{i,j+1,k}$ and $\mathbf{H}_{i,j,k+1}$, respectively.

Thus, the addition of any of the above direction vectors to Ξ_μ results in a new direction matrix upon which the newly formed hyperbolic box spline depends. □

To avoid the notational ambiguity, we denote $\mathbf{H}_{\Xi_\mu}$ by the box spline $\mathbf{H}_{i,j,k}$ associated with Ξ_μ. Later if any arbitrary direction vector $\lambda \mathbf{e}_{\mu+1} := \lambda[e^1_{\mu+1}, e^2_{\mu+1}]^T, \mathbf{e}_{\mu+1} \in \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$, is added to Ξ_μ, the resulting new direction matrix is denoted by $\Xi_{\mu+1}$, and the corresponding hyperbolic box spline is denoted by $\mathbf{H}_{\Xi_{\mu+1}}$. With this notation, the following definition serves as an equivalent representation of formula (13).

Definition 6 *Let* $\Xi_\mu = \lambda[\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mu]$ *be the direction matrix associated with the hyperbolic box spline* $\mathbf{H}_{\Xi_\mu}$*, with* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *being linearly independent. Let the direction matrix* $\Xi_{\mu+1}$ *be obtained by including the direction vector* $\lambda \mathbf{e}_{\mu+1} = \lambda[e^1_{\mu+1}, e^2_{\mu+1}]^T, \mathbf{e}_{\mu+1} \in \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ *to* Ξ_μ*. Then the corresponding hyperbolic box spline* $\mathbf{H}_{\Xi_{\mu+1}}$ *is obtained by*

$$\mathbf{H}_{\Xi_{\mu+1}}(x, y; \lambda) := \frac{1}{\lambda} \int_0^\lambda \mathbf{H}_{\Xi_\mu}(x - \tau e^1_{\mu+1}, y - \tau e^2_{\mu+1}; \lambda) d\tau. \tag{15}$$

Remark 1 Note that formula (15) is applicable only when all $i, j, k \geq 3$. That means, if we have $\mathbf{H}_{3,3,3}$ in our hand, we can derive all other higher order trigonometric box splines by using (15). Thus, the reader should remember that, first, one has to follow the equation (13) to find out $\mathbf{H}_{3,3,3}$. This is because $\mathbf{H}_{3,3,3}$ is our starting hyperbolic box spline function.

In view of (13), for all $(x, y) \in \mathbb{R}^2$,

$$\mathbf{H}_{3,3,3}(x, y; \lambda) := \frac{1}{\lambda} \int_{\mathbb{R}} \mathbf{H}_{0,3}(x - \tau\lambda; \lambda) \mathbf{H}_{0,3}(y - \tau\lambda; \lambda) \mathbf{H}_{0,3}(\tau\lambda; \lambda) d\tau. \tag{16}$$

where the explicit expression for $\mathbf{H}_{0,3}(x; \lambda)$ in (16) is given by

$$\mathbf{H}_{0,3}(x; \lambda) = \begin{cases} \dfrac{\cosh x - 1}{4 \sinh^2 (\lambda/2)}, & 0 \leq x \leq \lambda, \\[2ex] \dfrac{-\cosh(2\lambda - x) - \cosh(\lambda - x) + 2\cosh\lambda}{4 \sinh^2 (\lambda/2)}, & \lambda \leq x \leq 2\lambda, \\[2ex] \dfrac{\cosh(3\lambda - x) - 1}{4 \sinh^2 (\lambda/2)}, & 2\lambda \leq x \leq 3\lambda, \\[2ex] 0, & \text{elsewhere.} \end{cases} \tag{17}$$

$\mathbf{H}_{3,3,3}$ involves three directions $\lambda\mathbf{d}_1$, $\lambda\mathbf{d}_2$ and $\lambda\mathbf{d}_3$ each with multiplicity 3. Hence, its direction matrix is given by $\lambda[\mathbf{d}_1, \mathbf{d}_1, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_2, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_3, \mathbf{d}_3]$. The following properties hold true for $\mathbf{H}_{3,3,3}$.

Theorem 3 *Let* $\mathbf{H}_{3,3,3}$ *be defined as in* (16). *Then*

P1. *The support of* $\mathbf{H}_{3,3,3}$ *is* $\{\lambda(t_1 + t_3, t_2 + t_3) : 0 \leq t_j \leq 3, 1 \leq j \leq 3\}$.

P2. $\mathbf{H}_{3,3,3}$ *is non-negative in its support.*

P3. $\mathbf{H}_{3,3,3}$ *is symmetric about* $(3\lambda, 3\lambda)$ *which is the center of its support.*

Proof. Following the definition of $\mathbf{H}_{3,3,3}$ in (16) and properties of $\mathbf{H}_{0,3}$, the integrand in (16) is non-zero when $(x - \tau\lambda, y - \tau\lambda) \in [0, 3\lambda]^2$ with $\tau\lambda \in (0, 3\lambda)$. This implies the conditions:

$$0 \leq x - \tau\lambda \leq 3\lambda \quad \text{and} \quad 0 \leq y - \tau\lambda \leq 3\lambda \quad \text{with } \tau\lambda \in (0, 3\lambda).$$

Let $x - \tau\lambda = t_1\lambda$, $y - \tau\lambda = t_2\lambda$, and $\tau\lambda = t_3\lambda$. Then the support of $\mathbf{H}_{3,3,3}$ is given by $(x, y) = \lambda(t_1 + t_3, t_2 + t_3)$ where $0 \leq t_j \leq 3$ for $1 \leq j \leq 3$. This is illustrated in Figure 3(a) confirming that $\mathbf{H}_{3,3,3}$ has compact support. This proves the property ***P1***.

Next, the property ***P2*** is obvious, as $\mathbf{H}_{3,3,3}$ is constructed by convolution of $\mathbf{H}_{0,3}(x; \lambda)$, which is non-negative on its support $[0, 3\lambda]$.

Finally, the property ***P3*** can be proved by showing that $\mathbf{H}_{3,3,3}$ is symmetric with respect to the lines $y = x$ and $y = 6\lambda - x$. It is easy to verify $\mathbf{H}_{3,3,3}$ is symmetric about the line $y = x$, since interchanging the variables x and y in (16) results in the same expression. Similarly, the symmetry of $\mathbf{H}_{3,3,3}$ about the line $y = 6\lambda - x$ follows directly from the definition of $\mathbf{H}_{3,3,3}$ and the symmetric property of $\mathbf{H}_{0,3}$. □

The graphical representation of $\mathbf{H}_{3,3,3}$ is shown in Figure 3(b).

3 Properties

We will now examine some fundamental properties of this new class of hyperbolic box splines. Many of these properties and their proofs are straightforward and closely resemble those of polynomial box splines. Additionally, we will investigate the basic properties of various hyperbolic box splines by focusing on the foundational case of $\mathbf{H}_{3,3,3}$. This is because higher-order hyperbolic box splines are derived from the integration of $\mathbf{H}_{3,3,3}$.

Let us introduce two notions to define the support of the hyperbolic box splines:

$$[\Xi_\mu] := \{\lambda(t_1\mathbf{e_1} + t_2\mathbf{e_2} + \ldots + t_\mu\mathbf{e}_\mu) : 0 \leq t_j \leq 1, 1 \leq j \leq \mu\} \tag{18}$$

and

$$(\Xi_\mu) := \{\lambda(t_1\mathbf{e_1} + t_2\mathbf{e_2} + \ldots + t_\mu\mathbf{e}_\mu) : 0 < t_j < 1, 1 \leq j \leq \mu\}. \tag{19}$$

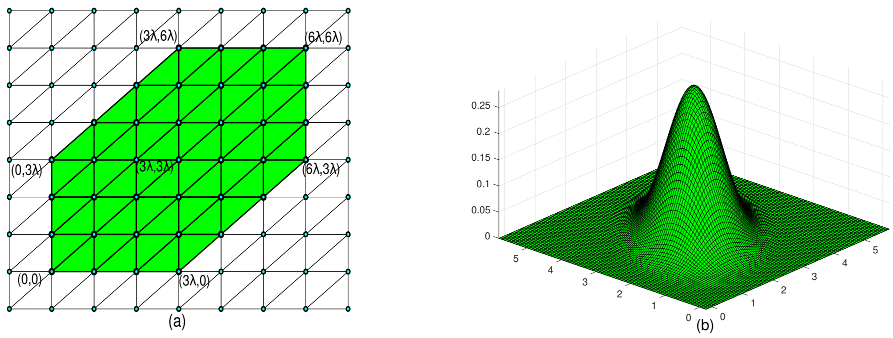

Figure 3: (a) The shaded portion is the support of the $\mathbf{H}_{3,3,3}(x,y;\lambda)$. (b) The hyperbolic box spline $\mathbf{H}_{3,3,3}(x,y;\lambda)$ where $\lambda = 0.4$.

Theorem 4 *Let $\mathbf{H}_{\Xi_\mu}$ and $\mathbf{H}_{\Xi_{\mu+1}}$ be the trigonometric box splines associated with Ξ_μ and $\Xi_{\mu+1}$, respectively, for all $l,m,n \geq 3$. Then*

(i) Local support: $\mathbf{H}_{\Xi_{\mu+1}}(x,y;\lambda) \equiv 0$ *for* $(x,y) \notin [\Xi_{\mu+1}]$.

(ii) Non-negativity: $\mathbf{H}_{\Xi_{\mu+1}}(x,y;\lambda) > 0$ *for* $(x,y) \in (\Xi_{\mu+1})$.

(iii) Ordering: $\mathbf{H}_{\Xi_\mu}$ *does not depend on the ordering of the directions contained in Ξ_μ, provided that the first two vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ are linearly independent.*

(iv) Symmetric: $\mathbf{H}_{\Xi_\mu}$ *is symmetric with respect to the center of its support.*

Proof. From the definition of $\mathbf{H}_{\Xi_\mu}(x,y;\lambda)$ and also from Figure 3, it is easy to see that *(i)* and *(ii)* both hold for $\mu = 9$. Let us assume that they both hold for $\mu = m$, where $m \geq 9$. Then by following the induction hypothesis we need to show that they both hold for $\mu = m+1$. Let $(x,y) \notin [\Xi_{\mu+1}]$. Then $(x - \tau e^1_{\mu+1}, y - \tau e^2_{\mu+1}) \notin [\Xi_\mu]$ for all $\tau \in [0,1]$. But this leads to the integrand in (15) to be zero. Hence, *(i)* is proved.

Next, to prove *(ii)*, take $(x,y) \in (\Xi_{\mu+1})$. Then $(x - \tau e^1_{\mu+1}, y - \tau e^2_{\mu+1}) \in (\Xi_\mu)$ for some $\tau \in (0,1)$. We can find an interval $[a,b]$ with $0 \leq a < \tau < b \leq 1$ such that $(x - \xi e^1_{\mu+1}, y - \xi e^2_{\mu+1}) \in (\Xi_\mu)$ for all $\xi \in [a,b]$. But then the integrand in (15) is positive for $\xi \in [a,b]$.

The properties *(iii)* and *(iv)* are trivial. □

Next theorem establishes linear independence of box splines.

Theorem 5 *The translates of the hyperbolic box spline $\mathbf{H}_{\Xi_\mu}(x,y;\lambda)$ are locally linear independent.*

Proof. For all $(x,y) \in \mathbb{R}^2$, consider the equation

$$\sum_{(p,q)\in\mathbb{Z}^2} \alpha_{p,q} \mathbf{H}_{\Xi_\mu}(x - p\lambda, y - q\lambda; \lambda)$$
$$= \sum_{(p,q)\in\mathbb{Z}^2} \alpha_{p,q} \mathbf{H}_{i,j,k}(x - p\lambda, y - q\lambda; \lambda) = 0. \quad (20)$$

We need to show that $\alpha_{p,q} = 0$ for all $(p, q) \in \mathbb{Z}^2$. By definition of $\mathbf{H}_{i,j,k}$, equation (20) implies

$$\frac{1}{\lambda} \sum_{(p,q)\in\mathbb{Z}^2} \alpha_{p,q} \int_{\mathbb{R}} \mathbf{H}_{0,i}(x - \tau\lambda - p\lambda; \lambda)\mathbf{H}_{0,j}(y - \tau\lambda - q\lambda; \lambda)\mathbf{H}_{0,k}(\tau\lambda; \lambda)d\tau = 0,$$

which becomes

$$\frac{1}{\lambda} \int_0^l \sum_p \sum_q \alpha_{p,q}\mathbf{H}_{0,i}(x - \tau\lambda - p\lambda; \lambda)\mathbf{H}_{0,j}(y - \tau\lambda - q\lambda; \lambda)\mathbf{H}_{0,k}(\tau\lambda; \lambda)d\tau = 0,$$

where $l = \max\{i, j, k\}$. Let us take $x - \tau\lambda - p\lambda = s$. Then $\tau\lambda = x - s - p\lambda$, and $d\tau = -\lambda^{-1}\, ds$. The change in variable in the above equation leads to

$$\frac{1}{\lambda^2} \int_{x-(p+l)\lambda}^{x-p\lambda} \Bigg(\sum_p \sum_q \alpha_{p,q}\mathbf{H}_{0,i}(s; \lambda)\mathbf{H}_{0,j}(y + p\lambda + s - x - q\lambda; \lambda) \mathbf{H}_{0,k}(x - s - p\lambda; \lambda) \Bigg) ds = 0. \tag{21}$$

This is a known fact that for any function $f(t)$, if $\sum_{x-a}^{x-b} f(t)dt = 0$ for any $x \in \mathbb{R}$ then $f(x - b) - f(x - a) = 0$ which leads to $f(x) = 0$ for all x. Using this argument in (21), we get

$$\sum_p \sum_q \alpha_{p,q}\mathbf{H}_{0,i}(s; \lambda)\mathbf{H}_{0,j}(y + p\lambda + s - x - q\lambda; \lambda)\mathbf{H}_{0,k}(x - s - p\lambda; \lambda)ds = 0.$$

Since the set $\{\mathbf{H}_{0,k}(x - s - p\lambda; \lambda), p \in \mathbb{Z}\}$ is linear independent, we get

$$\sum_q \alpha_{p,q}\mathbf{H}_{0,j}(y + p\lambda + s - x - q\lambda; \lambda) = 0 \qquad \text{for all } p \in \mathbb{Z}.$$

Again, by the same reason, $\alpha_{p,q} = 0$ for all $(p, q) \in \mathbb{Z}^2$. Hence, the theorem is proved. □

Theorem 6 *If* $\mathbf{e_1}$ *and* $\mathbf{e_2}$ *are linearly independent and* $\mathbf{e_{\mu+1}} = [e^1_{\mu+1}, e^2_{\mu+1}]^T$, *then*

$$D_{\mathbf{e}_{\mu+1}}\mathbf{H}_{\mathbf{\Xi}_{\mu+1}}(x, y; \lambda) = \frac{1}{\lambda}\left(\mathbf{H}_{\mathbf{\Xi}_\mu}(x, y; \lambda) - \mathbf{H}_{\mathbf{\Xi}_\mu}(x - \lambda e^1_{\mu+1}, y - \lambda e^2_{\mu+1}; \lambda)\right).$$

Proof. By (15), we have

$$\begin{aligned}
\mathbf{H}_{\mathbf{\Xi}_{\mu+1}}(x,y;\lambda) :=& \frac{1}{\lambda}\int_0^{\lambda} \mathbf{H}_{\mathbf{\Xi}_{\mu}}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1};\lambda)d\tau \\
=& \frac{1}{\lambda}\int_0^{\infty} \mathbf{H}_{\mathbf{\Xi}_{\mu}}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1};\lambda)d\tau \\
&- \frac{1}{\lambda}\int_{\lambda}^{\infty} \mathbf{H}_{\mathbf{\Xi}_{\mu}}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1};\lambda)d\tau \\
=& \frac{1}{\lambda}\int_0^{\infty} \big(\mathbf{H}_{\mathbf{\Xi}_{\mu}}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1};\lambda) \\
&- \mathbf{H}_{\mathbf{\Xi}_{\mu}}(x-(\tau+\lambda) e^1_{\mu+1}, y-(\tau+\lambda) e^2_{\mu+1};\lambda)\big)d\tau \\
=& \frac{1}{\lambda}\int_0^{\infty} \mathbf{F}(\mathbf{x}-\tau\mathbf{e}_{\mu+1})\mathbf{d}\tau,
\end{aligned}$$

where $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x,y) = \mathbf{H}_{\mathbf{\Xi}_{\mu}}(x,y;\lambda) - \mathbf{H}_{\mathbf{\Xi}_{\mu}}(x-\lambda e^1_{\mu+1}, y-\lambda e^2_{\mu+1};\lambda)$. Let us take $\epsilon > 0$. Then

$$\begin{aligned}
&\frac{1}{\epsilon}\left(\mathbf{H}_{\mathbf{\Xi}_{\mu+1}}(x-\epsilon e^1_{\mu+1}, y-\epsilon e^2_{\mu+1};\lambda) - \mathbf{H}_{\mathbf{\Xi}_{\mu+1}}(x,y;\lambda)\right) \\
&= \frac{1}{\lambda}\int_0^{\infty} \frac{\mathbf{F}(x-(\tau-\epsilon)e^1_{\mu+1}, y-(\tau-\epsilon)e^2_{\mu+1}) - \mathbf{F}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1})}{\epsilon}d\tau.
\end{aligned}$$

We then make the change of variables $\tau \leftrightarrow \tau-\epsilon$ in the first term in the numerator to obtain

$$\begin{aligned}
&= \frac{1}{\lambda}\left(\int_{-\epsilon}^{\infty} \frac{\mathbf{F}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1})}{\epsilon}d\tau - \int_0^{\infty} \frac{\mathbf{F}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1})}{\epsilon}\, d\tau\right) \\
&= \frac{1}{\lambda}\int_{-\epsilon}^{0} \frac{\mathbf{F}(x-\tau e^1_{\mu+1}, y-\tau e^2_{\mu+1})}{\epsilon}d\tau.
\end{aligned}$$

Now, taking limit as $\epsilon \to 0^+$, we get

$$\begin{aligned}
D_{\mathbf{e}_{\mu+1}}\mathbf{H}_{\mathbf{\Xi}_{\mu+1}}(x,y;\lambda) &= \frac{1}{\lambda}\,\mathbf{F}(x,y) \\
&= \frac{1}{\lambda}\,\left(\mathbf{H}_{\mathbf{\Xi}_{\mu}}(x,y;\lambda) - \mathbf{H}_{\mathbf{\Xi}_{\mu}}(x-\lambda e^1_{\mu+1}, y-\lambda e^2_{\mu+1};\lambda)\right)
\end{aligned}$$

at all points of continuity. □

4 Result Discussion

In the previous sections, we constructed hyperbolic box spline functions and also studied their properties. It is worthwhile to note an important fact that the classes of spline functions are related to *subdivision schemes* [5,16,20,21, 31]. Subdivision schemes are generally iterative algorithms which consist of simple refinement rules to produce finer and finer meshes starting from an initial unrefined mesh. The connection between the splines and subdivision is established on the basis of a remarkable property of B-splines and box splines, i.e., they obey a refinement equation (also called a dilation equation or two scale equation). In the context of subdivision schemes, the *dilation matrix* is a matrix used to refine a set of points or control polygons in order to create smoother curve or surfaces. It indicates how to expand the original data points to create new data points in the next iteration of the subdivision method. The formal definition of a refinable function [7] is provided below.

Definition 7 *Let* $\mathbb{M}$ *be an* $s \times s$ *dilation matrix with integer entries, and* $l_0(\mathbb{Z})$ *denotes the subspace of real valued sequences defined on* $\mathbb{Z}$. *Let also* $\mathbf{F}$ *be a finitely supported real valued* s*-variate function on* $\mathbb{R}^s$. *If* $\mathbf{F}$ *satisfies*

$$\mathbf{F}(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^s} c_{\mathbf{j}}\, \mathbf{F}(\mathbb{M}\mathbf{x} - \mathbf{j}), \qquad \mathbf{x} \in \mathbb{R}^s, \tag{22}$$

then it is said to be a refinable function with respect to the dilation matrix $\mathbb{M}$. *The set of real coefficients* $\{c_{\mathbf{j}}\}$ *associated with* $\mathbf{F}$ *is called the refinement mask and, equation* (22) *is called a refinement equation.*

Following the above definition, for the refinability of bivariate polynomial box splines, s is set to be 2, and the dilation matrix $\mathbb{M}$ is chosen to be $2I_2$ (identity matrix of order 2), which is described below.

Proposition 1 *Let* $\mathbf{B}_{i,j,k}$ *be a bivariate box spline. Then* $\mathbf{B}_{i,j,k}$ *obeys a refinement equation*

$$\mathbf{B}_{i,j,k}(x, y) = \sum_{(m,n)\in\mathbb{Z}^2} c_{m,n}\, \mathbf{B}_{i,j,k}(2x - m, 2y - n), \qquad (x, y) \in \mathbb{R}^2, \tag{23}$$

where the set of real coefficients $\{c_{m,n}|(m, n) \in \mathbb{Z}^2\}$ *is the refinement mask.*

The proof of this property can be found in [7].

It is important to mention here that the refinement mask $\{c_{m,n}|(m, n) \in \mathbb{Z}^2\}$ can be considered as the subdivision mask to develop a subdivision scheme. This subdivision scheme generates the corresponding box spline surfaces when implemented on an initial mesh. In the same way, we can study the two scale equations of the proposed hyperbolic box spline functions in this paper and further derive the associated subdivision schemes. Subdivision schemes are of two types, for e.g., *stationary subdivision scheme* (refinement masks are level independent) [5, 16] and *non-stationary subdivision scheme* (refinement masks are level independent) [20, 31]. The inability of stationary subdivision schemes to reconstruct spirals, conic sections, etc., is a well-known deficiency which motivates the search for non-stationary subdivision schemes with the specific properties. In general, polynomial box spline surfaces are generated from stationary subdivision schemes. For example, the surfaces generated by Loop scheme [21] and the $C2g0$ scheme [6] produce C^2-quartic and C^4 box spline surfaces, respectively, in regular regions. However, trigonometric box spline surfaces are generated from non-stationary subdivision schemes [18,19]. Similarly, an effective bivariate non-stationary subdivision scheme can be derived from the two-scale relationship of the Fourier transform of a particular hyperbolic box spline function proposed here. The idea of anticipating a non-stationary subdivision scheme from the proposed hyperbolic box splines is quite similar to the trigonometric box spline case and is also due to the presence of the real parameter λ in the definition. The non-stationary subdivision scheme will be capable of producing hyperbolic box spline surfaces in regular regions of a triangular mesh.

Since box splines are a well-known class of refinable functions, they provide a foundation for constructing various types of wavelets and frames, as demonstrated in [8]. With the proper definition of hyperbolic box spline functions now established, we can extend these constructions to include wavelets and frames based on hyperbolic box splines.

5 Conclusion

In this article, we have mainly constructed a class of hyperbolic box spline functions and analyzed their properties. For this we took help of hyperbolic algebraic B-splines defined in [22] and followed the directional convolution method. We also discussed some important properties of the proposed hyperbolic box spline functions and most of the properties are quite similar with the properties of polynomial or trigonometric box splines. The study of refinability equation of the proposed box splines and derivation of suitable non-stationary subdivision schemes are two immediate future aspects of this work. In conclusion, hyperbolic box spline functions represent a significant advancement in the study of spline theory and its applications providing a versatile tool for various mathematical and computational tasks.

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Hrushikesh Jena
Department of mathematics,
Institute of Technical Education and Research (ITER)
Siksha 'O' Anusandhan Deemed to be University, 751030, India
hrushikeshjena@soa.ac.in

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