Armenian Journal of Mathematics
Volume 16, Number 14, 2024, 1–12
https://doi.org/10.52737/18291163-2024.16.14-1-12

Sharp Extensions of a Cusa-Huygens Type Inequality

Y. J. Bagul and C. Chesneau

Abstract. In this article, we propose an extension and generalization of a Cusa-Huygens type inequality and thus refine an existing inequality in the literature. As an application, we extend the improved Shafer's inequality for the arctangent function.

Key Words: Cusa-Huygens Type Inequality, Sharp Bounds, l'Hospital's Rule of Monotonicity
Mathematics Subject Classification 2020: 26D05, 26D07, 26D99, 33B10

Introduction

The inequality

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{1}$$

was discovered by the German philosopher and theologian Nicolaus de Cusa (1401–1464) by using a certain geometric construction (see [11, 15]). A rigorous proof of the Cusa inequality was given by Huygens [11] in 1664 while considering the estimation of π. Hence, the inequality (1) is known in the literature as the Cusa-Huygens inequality. This key inequality has been considered by many researchers for various purposes. We refer the reader to [1–9, 12–14, 16–19, 21–28] for the extensions, refinements, generalizations, and applications of this inequality. In particular, in 2019, Bercu [5] obtained the following refinement:

$$\frac{\sin x}{x} < \frac{2+\cos x}{3} - \frac{1}{45}(1-\cos x)^2, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{2}$$

We call this kind of inequality (2) a Cusa-Huygens type inequality.

In 2021, Bagul et al. [1] proved the following result: for $x \in (0, \pi/2)$,

$$\frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right)\phi_1(x) < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right)\phi_2(x), \tag{3}$$

where

$$\phi_1(x) = \frac{x - \sin x}{\pi/2 - 1}, \qquad \phi_2(x) = (\sin x - x \cos x)^2.$$

The graphical and numerical study shows that the upper bound of $(\sin x)/x$ in (2) is sharper than that in (3) for $(0, \lambda)$, where $\lambda \approx 1.332$.

The main focus of our study is to show that the inequality (2) is indeed true in $(0, 2\pi)$ by giving another proof. We also extend it to another side in order to refine both the inequalities (2) and (3).

Section 1 presents the main theorems of the article, together with direct graphical illustrations. To prove these theorems, some series expansions involving Bernoulli numbers are needed. These are recalled in Section 2. The detailed proofs are given in Section 3. Finally, an application to an arctangent inequality is examined in Section 4.

1 Main theorems

Our main results are as stated in the below theorems.

Theorem 1 *Let us define*

$$f(x) = -\frac{1}{(1 - \cos x)^2}\left(\frac{2 + \cos x}{3} - \frac{\sin x}{x}\right), \qquad x \in (0, 2\pi).$$

Then the function $-f(x)$ is absolutely monotonic increasing on $(0, 2\pi)$. Consequently, for $x \in (0, \theta) \subseteq (0, 2\pi)$ and a positive integer n, the double inequality

$$(1 - \cos x)^2 \sum_{k=0}^{n} a_k x^{2k} < \frac{2 + \cos x}{3} - \frac{\sin x}{x} < (1 - \cos x)^2 \sum_{k=0}^{n} a_k^* x^{2k}, \qquad (4)$$

where

$$a_k = \frac{8}{3}\frac{(k + 1)(k + 2)(2k + 3)}{(2k + 4)!}|B_{2k+4}|,$$

$a_k^* = a_k$ *for* $0 \leq k \leq n - 1$, *and*

$$a_n^* = \theta^{-2n}\left[\frac{1}{(1 - \cos x)^2}\left(\frac{2 + \cos \theta}{3} - \frac{\sin \theta}{\theta}\right) - \sum_{k=0}^{n-1} a_k \theta^{2k}\right]$$

holds. Two special cases are highlighted below.

1. *For $x \in (0, \pi/2)$, we have*

$$\frac{2 + \cos x}{3} + \left(\frac{2}{\pi} - \frac{2}{3}\right)(1 - \cos x)^2 < \frac{\sin x}{x} < \frac{2 + \cos x}{3} - \frac{1}{45}(1 - \cos x)^2. \qquad (5)$$

2. For $x \in (0, \pi)$, we have

$$\frac{2+\cos x}{3} - \frac{1}{12}(1-\cos x)^2 < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \frac{1}{45}(1-\cos x)^2. \quad (6)$$

In order to visually capture the sharpness of the particular inequalities in Theorem 1, Figure 1 shows the curves of the following difference functions:

$$g(x) = \frac{\sin x}{x} - \frac{2+\cos x}{3} + \frac{1}{12}(1-\cos x)^2, \qquad x \in (0, \pi)$$

and

$$h(x) = \frac{\sin x}{x} - \frac{2+\cos x}{3} + \frac{1}{45}(1-\cos x)^2, \qquad x \in (0, \pi).$$

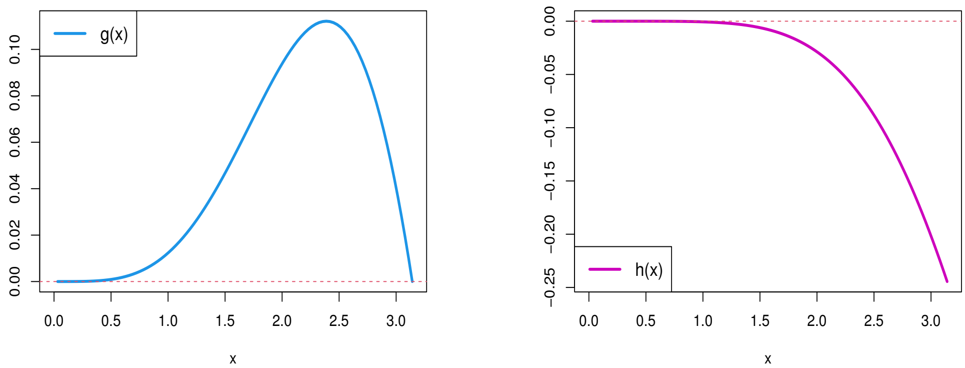

Figure 1: Graphs of $g(x)$ (left) and $h(x)$ (right) for $x \in (0, \pi)$

It is visually clear that $g(x) > 0$ and $h(x) < 0$, with a particular sharpness for $x \in (0, x_*)$, where $x_* \approx 1.5$.

From Figure 2, we illustrate the inequality $(x - \sin x)/(\pi/2 - 1) > (1 - \cos x)^2$, $x \in (0, \pi/2)$, by considering the following difference function:

$$i(x) = \frac{x - \sin x}{\pi/2 - 1} - (1-\cos x)^2, \;\; x \in (0, \pi/2).$$

It is immediate that $i(x) > 0$ for $x \in (0, \pi/2)$. This implies that the lower bound of $(\sin x)/x$ in (5) is sharper than that in (3).

Furthermore, we obtain sharper bounds for $(\sin x)/x$ by putting $n = 1, 2, 3, \cdots$ in (4). For instance, after putting $n = 1$, and $\theta = \pi/2$ and π in (4), we get respectively the following inequalities:

$$\frac{2+\cos x}{3} - \left[\frac{1}{45} + \frac{4}{\pi^2}\left(\frac{29}{45} - \frac{2}{\pi}\right)x^2\right](1-\cos x)^2 < \frac{\sin x}{x}$$
$$< \frac{2+\cos x}{3} - \left(\frac{1}{45} + \frac{1}{378}x^2\right)(1-\cos x)^2, \;\; x \in (0, \pi/2)$$

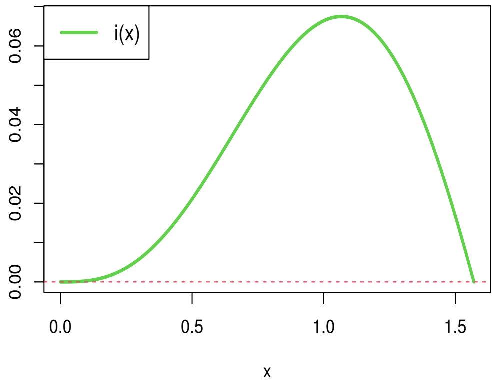

Figure 2: Graphs of $i(x)$ for $x \in (0, \pi/2)$

and

$$\frac{2+\cos x}{3} - \left[\frac{1}{45} + \frac{1}{\pi^2}\left(\frac{1}{12} - \frac{1}{45}\right)x^2\right](1-\cos x)^2 < \frac{\sin x}{x}$$
$$< \frac{2+\cos x}{3} - \left(\frac{1}{45} + \frac{1}{378}x^2\right)(1-\cos x)^2, \ x \in (0, 2\pi).$$

The above inequalities are clear refinements of the inequalities (2), (3), (5) and (6).

In the next theorem, we give two complementary results.

Theorem 2 *For $x \in (0, \theta) \subseteq (0, 2\pi)$, the double inequality*

$$(1-\xi) + \xi \cos x - \frac{(1-\cos x)^2}{45} < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \frac{(1-\cos x)^2}{45},$$

where

$$\xi = \frac{1}{\cos\theta - 1}\left[\frac{\sin\theta}{\theta} + \frac{(1-\cos\theta)^2}{45} - 1\right]$$

holds. Two special cases are highlighted below.

1. *For $x \in (0, \pi/2)$, we have*

$$\left(\frac{1}{45} + \frac{2}{\pi}\right) + \left(\frac{44}{45} - \frac{2}{\pi}\right)\cos x - \frac{1}{45}(1-\cos x)^2$$
$$< \frac{\sin x}{x} < \frac{2+\cos x}{3} - \frac{1}{45}(1-\cos x)^2.$$

2. For $x \in (0, \pi)$, we have

$$\frac{49}{90} + \frac{41}{90}\cos x - \frac{1}{45}(1-\cos x)^2 < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \frac{1}{45}(1-\cos x)^2.$$

In order to visually capture the sharpness of the inequalities in Theorem 2, and the lower bounds in particular since they are the main novelties, Figure 3 displays the curves of the following difference functions:

$$m(x) = \frac{\sin x}{x} - \left(\frac{1}{45} + \frac{2}{\pi}\right) - \left(\frac{44}{45} - \frac{2}{\pi}\right)\cos x + \frac{1}{45}(1-\cos x)^2, \ x \in (0, \pi/2)$$

and

$$n(x) = \frac{\sin x}{x} - \frac{49}{90} - \frac{41}{90}\cos x + \frac{1}{45}(1-\cos x)^2, \ x \in (0, \pi).$$

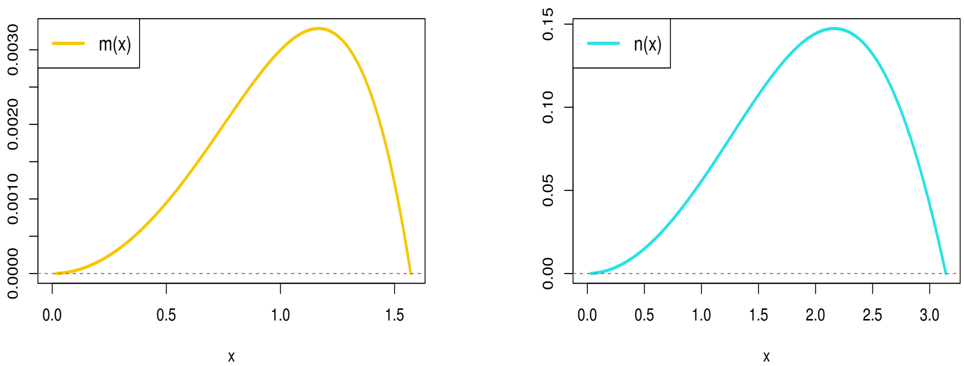

Figure 3: Plots of $m(x)$ for $x \in (0, \pi/2)$ (left) and $n(x)$ for $x \in (0, \pi)$ (right)

It is visually clear that $m(x) > 0$ for $x \in (0, \pi/2)$, and $n(x) < 0$ for $x \in (0, \pi)$. A great precision is observed for the curve of $m(x)$, with a maximum of 0.0033, which illustrates the great sharpness of our theoretical developments.

2 Preliminaries

We plan to use some series expansions recalled in the lemma below.

Lemma 1 *For $t \in (-\pi, \pi)$, we have*

$$\frac{1}{\sin^2 t} = \frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{(2n-1) \cdot 2^{2n}}{(2n)!} |B_{2n}| t^{2n-2}, \tag{7}$$

$$\left(\frac{1}{\sin^2 t}\right)' = -\frac{2}{t^3} + \sum_{n=2}^{\infty} \frac{(2n-1)(2n-2) \cdot 2^{2n}}{(2n)!} |B_{2n}| t^{2n-3}, \tag{8}$$

$$\left(\frac{1}{\sin^2 t}\right)'' = \frac{6}{t^4} + \sum_{n=2}^{\infty} \frac{(2n-1)(2n-2)(2n-3) \cdot 2^{2n}}{(2n)!} |B_{2n}| t^{2n-4} \tag{9}$$

and

$$\left(\frac{1}{\sin^2 t}\right)''' = -\frac{24}{t^5} + \sum_{n=2}^{\infty} \frac{(2n-1)(2n-2)(2n-3)(2n-4) \cdot 2^{2n}}{(2n)!} |B_{2n}| t^{2n-5}. \tag{10}$$

Proof. The following series expansion can be found in [10, 1.411 (7)]:

$$\cot t = \frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| t^{2n-1}, \tag{11}$$

where B_{2n} are even indexed Bernoulli numbers. On differentiating (11) successively, we get the required series (7), (8), (9) and (10). □

3 Proofs of results

This section is devoted to the proofs of our results.

Proof of Theorem 1 Due to half-angle formulas, setting $t = x/2 \in (0, \pi)$, we have

$$\begin{aligned} -f(x) &= \frac{3 - 2\sin^2(x/2)}{12\sin^4(x/2)} - \frac{2\sin(x/2)\cos(x/2)}{4x\sin^4(x/2)} \\ &= \frac{1}{4\sin^4 t} - \frac{1}{6\sin^2 t} - \frac{1}{4t}\frac{\cos t}{\sin^3 t} \\ &= \frac{1}{24}\left(\frac{1}{\sin^2 t}\right)'' + \frac{1}{8t}\left(\frac{1}{\sin^2 t}\right)'. \end{aligned}$$

Using (8) and (9), we get

$$\begin{aligned}-f(x) &= \frac{1}{24}\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)(2n-3)\cdot 2^{2n}}{(2n)!}|B_{2n}|t^{2n-4}\\ &\quad+\frac{1}{8}\sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)\cdot 2^{2n}}{(2n)!}|B_{2n}|t^{2n-4}\\ &= \frac{1}{6}\sum_{n=2}^{\infty}\frac{n(n-1)(2n-1)\cdot 2^{2n}}{(2n)!}|B_{2n}|t^{2n-4}\\ &= \frac{8}{3}\sum_{n=2}^{\infty}\frac{n(n-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-4}.\end{aligned}$$

This shows that $-f(x)$ is absolutely monotonically increasing on $(0,2\pi)$, and the double inequality (4) follows. The inequalities (5) and (6) follow due to the limits

$$\lim_{x\to 0^+} f(x) = -\frac{1}{45}, \qquad \lim_{x\to \pi/2^-} f(x) = \frac{2}{\pi}-\frac{2}{3}, \qquad \lim_{x\to \pi^-} f(x) = -\frac{1}{12}.$$

This ends the proof of Theorem 1. □

Proof of Theorem 2 For $x\in(0,2\pi)$, let us set

$$g(x) = \frac{1}{\cos x - 1}\left[\frac{\sin x}{x}+\frac{(1-\cos x)^2}{45}-1\right].$$

Setting $t = x/2 \in (0,\pi)$, the half-angle formulas yield

$$g(x) = -\frac{1}{2\sin^2 t}\left[\frac{\sin t\cos t}{t}+\frac{4\sin^4 t}{45}-1\right] = \frac{1}{2}\cdot\phi(t),$$

where

$$\phi(t) = \frac{1}{\sin^2 t}-\frac{4}{45}\sin^2 t-\frac{\cot t}{t}.$$

After a differentiation work, we obtain

$$\phi'(t) = \frac{1}{2}\left[-\frac{2\cos t}{\sin^3 t}-\frac{8}{45}\sin t\cdot\cos t+\frac{1}{t\sin^2 t}+\frac{\cot t}{t^2}\right] = \frac{1}{2}\sin^2 t\cdot\psi(t),$$

where

$$\begin{aligned}\psi(t) &= -\frac{2\cos t}{\sin^5 t}-\frac{8}{45}\cot t+\frac{1}{t\sin^4 t}+\frac{\cos t}{t^2\sin^3 t}\\ &= \frac{1}{12}\left(\frac{1}{\sin^2 t}\right)'''+\frac{1}{3}\left(\frac{1}{\sin^2 t}\right)'-\frac{8}{45}\cot t+\frac{1}{6t}\left(\frac{1}{\sin^2 t}\right)''+\frac{2}{3t}\frac{1}{\sin^2 t}\\ &\quad-\frac{2}{t^2}\left(\frac{1}{\sin^2 t}\right)'\end{aligned}$$

Based on series expansions (7), (8), (9) and (10), we can express $\psi(t)$ in the following manner:

$$\begin{aligned}
\psi(t) = & -\frac{2}{t^5} + \sum_{n=2}^{\infty} \frac{(2n-1)(2n-2)(2n-3)(2n-4)\cdot 2^{2n}}{12\cdot(2n)!}|B_{2n}|t^{2n-5} \\
& - \frac{8}{45}\left[\frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}\right] \\
& + \frac{1}{6t}\left[\frac{6}{t^4} + \sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)(2n-3)\cdot 2^{2n}}{(2n)!}|B_{2n}|t^{2n-4}\right] \\
& + \frac{2}{3t}\left[\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{(2n-1)\cdot 2^{2n}}{(2n)!}|B_{2n}|t^{2n-2}\right] + \frac{1}{t^5} \\
& - \sum_{n=2}^{\infty}\frac{(2n-1)(2n-2)\cdot 2^{2n}}{2\cdot(2n)!}|B_{2n}|t^{2n-5} \\
= & \sum_{n=2}^{\infty}\frac{n(2n-1)(2n-2)(2n-5)\cdot 2^{2n}}{6\cdot(2n)!}|B_{2n}|t^{2n-5} \\
& - \frac{8}{45t} + \sum_{n=3}^{\infty}\frac{8}{45}\frac{2^{2n-4}}{(2n-4)!}|B_{2n-4}|t^{2n-5} \\
& + \frac{2}{3t^3} + \sum_{n=2}^{\infty}\frac{(2n-3)\cdot 2^{2n-1}}{3\cdot(2n-2)!}|B_{2n-2}|t^{2n-5} \\
= & \frac{2}{t^3} + \sum_{n=3}^{\infty}\frac{n(2n-1)(2n-2)(2n-5)\cdot 2^{2n-1}}{3\cdot(2n)!}|B_{2n}|t^{2n-5} \\
& + \sum_{n=3}^{\infty}\frac{2^{2n-1}}{45\cdot(2n-4)!}|B_{2n-4}|t^{2n-5} + \sum_{n=3}^{\infty}\frac{(2n-3)\cdot 2^{2n-1}}{3\cdot(2n-2)!}|B_{2n-2}|t^{2n-5} > 0.
\end{aligned}$$

As a result, we have $\phi'(t) > 0$ for $t \in (0, \pi)$. Therefore, $\phi(t)$, and hence $g(x)$, is absolutely monotonically increasing. The desired inequalities follow due to the limits

$$\lim_{x\to 0^+} g(x) = \frac{1}{3}, \qquad \lim_{x\to\pi/2^-} g(x) = \frac{44}{45} - \frac{2}{\pi}, \qquad \lim_{x\to\pi^-} g(x) = \frac{41}{90}.$$

This ends the proof of Theorem 2. □

4 An application

The famous Shafer inequality, established in [20], can be formulated as follows:

$$\frac{3}{1+2\sqrt{t^2+1}} < \frac{\arctan t}{t}, \qquad t > 0.$$

It was recently improved by Bercu [5] as

$$\frac{45\sqrt{t^2+1}}{17\sqrt{t^2+1}+29t^2+28} < \frac{\arctan t}{t}, \qquad t>0, \tag{12}$$

using the inequality (2). Here, by taking $x = \tan t$, $\in (0, \pi/2)$, in the left inequality of (5), we obtain the following extension of (12):

$$\frac{\arctan t}{t} < \frac{3\pi\sqrt{t^2+1}}{6t^2+(5\pi-12)\sqrt{t^2+1}-(2\pi-12)}, \qquad t>0.$$

The sharpness of this inequality is illustrated in Figure 4, with the consideration of the following difference function:

$$p(t) = \frac{\arctan t}{t} - \frac{3\pi\sqrt{t^2+1}}{6t^2+(5\pi-12)\sqrt{t^2+1}-(2\pi-12)}.$$

We take $x \in (0, 15)$, the value of 15 being arbitrary, any other "large" value can be selected.

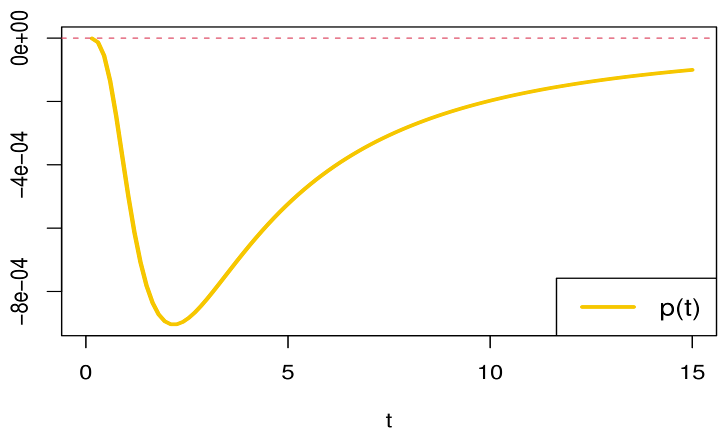

Figure 4: Plots of $p(t)$ for $t \in (0, 15)$

As expected, we see that $p(t) < 0$ for $t > 0$. To the best of our knowledge, this sharp inequality is new in the literature and opens some direction for research on the bounds of arctangent-type functions.

Acknowledgements. The authors are grateful to the anonymous referee for his/her careful reading of the manuscript and thorough comments that helped to improve the article.

References

[1] Y.J. Bagul, B. Banjac, C. Chesneau, M. Kostić and B. Malešević, New refinements of Cusa-Huygens inequality. Results Math., **76** (2021), Article no. 107. https://doi.org/10.1007/s00025-021-01392-8

[2] Y.J. Bagul and C. Chesneau, Refined forms of Oppenheim and Cusa-Huygens type inequalities. Acta Comment. Univ. Tartu. Math., **24** (2020), no. 2, pp. 183–194. https://doi.org/10.12697/ACUTM.2020.24.12

[3] Y.J. Bagul, C. Chesneau and M. Kostić, On the Cusa-Huygens inequality. RACSAM., **115** (2021), Article no. 29. https://doi.org/10.1007/s13398-020-00978-1

[4] Y.J. Bagul, C. Chesneau and M. Kostić, The Cusa-Huygens inequality revisited. Novi Sad J. Math., **50** (2020), no. 2, pp. 149–159. https://doi.org/10.30755/NSJOM.10667

[5] G. Bercu, Fourier series method related to Wilker-Cusa-Huygens inequalities. Math. Inequal. Appl., **22** (2019), no. 4, pp. 1091–1098.

[6] B. Chaouchi, V.E. Fedorov and M. Kostić, Monotonicity of certain classes of functions related with Cusa-Huygens inequality. Chelyab. Fiz.-Mat. Zh., **6** (2021), no. 3, pp. 331–337. https://doi.org/10.47475/2500-0101-2021-16307

[7] C.-P. Chen and W.-S. Cheung, Sharp Cusa and Becker-Stark inequalities. J. Inequal. Appl., **2011** (2011), Article No. 136. https://doi.org/10.1186/1029-242X-2011-136

[8] C.-P. Chen and J. Sándor, Inequality chains for Wilker, Huygens and Lazarević type inequalities. J. Math. Inequal., **8** (2014), no. 1, pp. 55–67. http://dx.doi.org/10.7153/jmi-08-02

[9] R.M. Dhaigude, C. Chesneau and Y.J. Bagul, About trigonometric-polynomial bounds of sinc function. Math. Sci. Appl. E-Notes, **8** (2020), no. 1, pp. 100–104. https://doi.org/10.36753/mathenot.585735

[10] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Elsevier, 2007. https://doi.org/10.1016/C2009-0-22516-5

[11] C. Huygens, *Oeuvres Completes*, Société Hollandaise des Sciences, Haga, 1888–1940.

[12] B. Malešević, M. Nenezić, L. Zhu, B. Banjac and M. Petrović, Some new estimates of precision of Cusa-Huygens and Huygens approximations. Appl. Anal. Discrete Math., **15** (2021), no. 1, pp. 243–259. http://dx.doi.org/10.2298/AADM190904055M

[13] B. Malešević, T. Lutovac, M. Rašajski and C. Mortici, Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities. Adv. Difference Equ., **2018** (2018), Article no. 90. https://doi.org/10.1186/s13662-018-1545-7

[14] B. Malešević and M. Makragić, A method for proving some inequalities on mixed trigonometric polynomial functions. J. Math. Inequal., **10** (2016), no. 3, pp. 849–876. http://dx.doi.org/10.7153/jmi-10-69

[15] D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[16] C. Mortici, The natural approach of Wilker-Cusa-Huygens Inequalities. Math. Inequal. Appl., **14** (2011), no. 3, pp. 535–541. https://dx.doi.org/10.7153/mia-14-46

[17] E. Neuman and J. Sándor, On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities. Math. Inequal. Appl., **13** (2010), no. 4, pp. 715–723. https://dx.doi.org/10.7153/mia-13-50

[18] J. Sándor, Sharp Cusa-Huygens and related inequalities. Notes on Number Theory and Discrete Mathematics, **19** (2013), no. 1, pp. 50–54.

[19] J. Sándor and R. Oláh-Gal, On Cusa-Huygens type trigonometric and hyperbolic inequalities. Acta. Univ. Sapientiae Mathematica, **4** (2012), no. 2, pp. 145–153.

[20] R.E. Shafer, Problem E 1867. Amer. Math. Monthly, **73**(1966), p. 309.

[21] Z.-H. Yang, Refinements of a two-sided inequality for trigonometric functions. J. Math. Inequal., **7** (2013), no. 4, pp. 601–615. http://dx.doi.org/10.7153/jmi-07-57

[22] Z.-H. Yang and Y.-M. Chu, A note on Jordan, Adamović-Mitrinović, and Cusa inequalities. Abst. Appl. Anal., **2014** (2014), Article ID: 364076, 12 pp. http://dx.doi.org/10.1155/2014/364076

[23] Z.-H. Yang, Y.-M. Chu, Y.-Q. Song and Y.-M. Li, A sharp double inequality for trigonometric functions and its applications. Abst. Appl. Anal., **2014** (2014), Article ID: 592085, 9 pp. http://dx.doi.org/10.1155/2014/592085

[24] Z.-H. Yang, Y.-L. Jiang, Y.-Q. Song and Y.-M. Chu, Sharp inequalities for trigonometric functions. Abst. Appl. Anal., **2014** (2014), Article ID: 601839, 18 pp. http://dx.doi.org/10.1155/2014/601839

[25] Z.-H. Yang and Y.-M. Chu, A sharp double inequality involving trigonometric functions and its applications. J. Math. Inequal., **10** (2016), no. 2, pp. 423–432. http://dx.doi.org/10.7153/jmi-10-33

[26] L. Zhu, New inequalities of Cusa-Huygens type. Mathematics, **9** (2021), no. 17, 2101. https://doi.org/10.3390/math9172101

[27] L. Zhu, New bounds for the sine function and tangent function. Mathematics, **9** (2021), no. 19, 2373. https://doi.org/10.3390/math9192373

[28] L. Zhu, New Cusa-Huygens type inequalities. AIMS Mathematics, **5** (2020), no. 5, pp. 5320–5331. https://doi.org/10.3934/math.2020341

Yogesh J. Bagul
Department of mathematics,
K.K.M. College,
Manwath, Dist.: Parbhani (M.S.) - 431505, India.
yjbagul@gmail.com

Christophe Chesneau
Department of Mathematics,
LMNO, University of Caen-Normandie,
14032 Caen, France.
christophe.chesneau@gmail.com

Please, cite to this paper as published in
Armen. J. Math., V. **16**, N. 14(2024), pp. 1–12
https://doi.org/10.52737/18291163-2024.16.14-1-12