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Approximation by Some Singular Operators Type of Functions from a Generalized Zygmund Space

Xh. Z. Krasniqi and R. N. Mohapatra

Abstract. This study continues previous research on the approximation of functions by means of singular integrals. We begin by introducing the truncated Picard singular integral. Subsequently, using this integral along with the classical Picard–Cauchy and Gauss–Weierstrass singular integrals, we establish the orders of approximation for functions belonging to a generalized Zygmund space, both in the L^p-norm and in the corresponding norm of the generalized Zygmund space.

Key Words: Degree of Approximation, Singular Integrals, Zygmund Modulus of Continuity, Generalized Zygmund Class, Generalized Minkowski Inequality
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Introduction

For a long time, there has been some interest in investigating the approximation of functions by the Fourier means in the Hölder spaces. This interest originated from the study of a certain class of integro-differential equations and from applications in error estimations for singular integral equations. Staying behind in this particular topic, the approximation of 2π-periodic and integrable functions by their Fourier series in the Hölder metric has been studied regularly by mathematicians. Among of such results, we mention Das at al. [4] who studied the degree of approximation of functions by matrix means of their Fourier series in the generalized Hölder metric, generalizing some well-known previous results. One more time, Das at al. [3] studied the rate of convergence problem of Fourier series in a new Banach space of functions conceived as a generalization of the spaces introduced by Prössdorf [23]. Later on, Nayak et al. [21] studied the problem of the rate of convergence of Fourier series using delayed arithmetic means in the generalized Hölder metric space, which was previously introduced in [18]. Their main objective was to obtain an estimate of Jackson's order. Very recently, Kim [10] has treated the degree of approximation of functions in the same generalized Hölder metric, but using only the so-called even-type delayed arithmetic mean of Fourier series. On purpose, we don't list all known results here, as they go beyond the topic discussed. Nevertheless, for the reader's interest, recent results on the degree of approximation of functions in the Hölder metric can be found in [16]–[10], [21], and the references therein.

Once more, we return back to [23] which reports the degree of approximation (in the Hölder metric) of functions from H_α $(0<\alpha\leq 1)$ space by Fourier series. To be specific, let $C_{2\pi}$ be the Banach space of 2π-periodic continuous functions defined in $[-\pi,\pi]$ under the sup-norm. For $0<\alpha\leq 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha=\left\{f\in C_{2\pi}:|f(x)-f(y)|\leq K|x-y|^\alpha\right\}.$$

The space H_α is a Banach space with the norm $\|f\|_\alpha$ defined by

$$\|f\|_\alpha=\|f\|_C+\sup_{x,y}\{\Delta^\alpha f(x,y)\},$$

where

$$\|f\|_C=\sup_{-\pi\leq x\leq\pi}|f(x)| \quad \text{and} \quad \Delta^\alpha f(x,y):=|f(x)-f(y)||x-y|^{-\alpha}, \quad x\neq y.$$

At the current stage, we are going to agree with $\Delta^0 f(x,y):=0$ (by convention).

Further, let us recall the well-known Picard, Picard-Cauchy and Gauss–Weierstrass singular integrals given by

$$P_\xi(f;x)=\frac{1}{2\xi}\int_{-\infty}^{\infty}f(x+t)e^{-\frac{|t|}{\xi}}\,dt, \tag{1}$$

$$Q_\xi(f;x)=\frac{\xi}{\pi}\int_{-\pi}^{\pi}\frac{f(x+t)}{t^2+\xi^2}dt, \tag{2}$$

and

$$W_\xi(f;x)=\frac{1}{\sqrt{\pi\xi}}\int_{-\pi}^{\pi}f(x+t)e^{-\frac{t^2}{\xi}}\,dt, \tag{3}$$

respectively, where ξ is a positive parameter which tends to zero.

Everywhere in this paper, we write

$$\varphi_x(t):=\frac{1}{2}[f(x+t)+f(x-t)-2f(x)]$$

and $u=\mathcal{O}(v)$, whenever there exists a positive constant K, not necessarily the same at each occurrence, such that $u\leq Kv$.

Let f be a bounded real valued function defined on the real line $\mathbb{R}$ or $(-\pi,\pi)$. By $\mathbb{B}$ we denote the Banach space of such functions under the sup-norm.

Mohapatra and Rodrigez [20] yielded the error bound of $f \in H_\alpha$ in the norm $\|\cdot\|_\beta$ for $0 \le \beta < \alpha \le 1$. Among others, they proved the following theorems.

Theorem 1 *Let $f \in \mathbb{B}$, $x \in \mathbb{R}$, and*

$$\omega(\delta) = \sup_{|t|\le\delta} |f(x+t) - f(x)|, \quad (\delta > 0)$$

such that $\omega(t)/t$ is a non-increasing function of t. Then, as $\xi \to 0+$, the following hold:

$$\begin{aligned}
\|f - P_\xi(f;\cdot)\|_C &= \mathcal{O}\left(\omega(\xi)\right),\\
\|f - Q_\xi(f;\cdot)\|_C &= \mathcal{O}\left(\omega(\xi)|\ln(1/\xi)|\right),\\
\|f - W_\xi(f;\cdot)\|_C &= \mathcal{O}\left(\omega(\xi)\xi^{-\frac{1}{2}}\right).
\end{aligned}$$

Theorem 2 *Let $0 \le \beta < \alpha \le 1$ and $f \in H_\alpha$. Then, as $\xi \to 0+$,*

$$\begin{aligned}
&\|f - P_\xi(f;\cdot)\|_\beta = \mathcal{O}\left(\xi^{\alpha-\beta}\right), \quad \|f - Q_\xi(f;\cdot)\|_\beta = \mathcal{O}\left(\xi^{\alpha-\beta}|\ln(1/\xi)|\right),\\
&\|f - W_\xi(f;\cdot)\|_\beta = \mathcal{O}\left(\xi^{\alpha-\beta-\frac{1}{2}}\right).
\end{aligned}$$

Let $w_1 : [0,2\pi] \to \mathbb{R}$ be an arbitrary function with $w_1(t) > 0$ for $0 < t \le 2\pi$ and $\lim_{t\to 0+} w_1(t) = w_1(0) = 0$. The generalized Zygmund class (see [22]) is defined by

$$\mathcal{Z}_p^{(w_1)} := \left\{ f \in L^p[0,2\pi] : \sup_{t\neq 0} \frac{\|f(\cdot+t) + f(\cdot-t) - 2f(\cdot)\|_p}{w_1(|t|)} < \infty \right\}$$

and

$$\|f\|_p^{(w_1)} := \|f\|_p + \sup_{t\neq 0} \frac{\|f(\cdot+t) + f(\cdot-t) - 2f(\cdot)\|_p}{w_1(|t|)}$$

for $1 \le p < \infty$, where $L^p[0,2\pi]$ denotes all measurable functions for which the norm

$$\|f\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Note that $\|\cdot\|_p^{(w_1)}$ is a norm in the space $\mathcal{Z}_p^{(w_1)}$. The completeness of the space $\mathcal{Z}_p^{(w_1)}$ can be debated as long as the completeness of L^p space, and thus the space $\mathcal{Z}_p^{(w_1)}$ is a Banach space equipped with the norm $\|f\|_p^{(w_1)}$.

The Zygmund modulus of continuity of f is defined by

$$w(f,h) = \sup_{0\le t\le h,\, x\in\mathbb{R}} |f(x+t) + f(x-t) - 2f(x)| \text{ and } w(f,h) \to 0 \text{ as } h \to 0.$$

Choosing the functions $w(t)$ and $v(t)$ to be two Zygmund moduli of continuity and $w(t)/v(t)$ a non-negative and non-decreasing function, then the following inequality holds

$$\|f\|_p^{(v)} \leq \max\left(1, \frac{w(2\pi)}{v(2\pi)}\right)\|f\|_p^{(w)}$$

which shows that $\mathcal{Z}_p^{(w)} \subseteq \mathcal{Z}_p^{(v)} \subseteq L^p$ for $1 \leq p < \infty$.

Throughout the paper, $w_1(t)$ and $w_2(t)$ denote the two Zygmund moduli of continuity such that $w(t)/v(t)$ positive and non-decreasing in t.

The singular integrals (1), (2), (3), as well as their generalizations, are widely used in issues of approximation of a certain classes of functions.

Such issues for one-dimensional and multidimensional functions, as might be expected, have been studied by Gal [7], Deeba and al. [5], Mezei [19], Anastassiou and Aral [1], Rempulska and Tomczak [24], Firlejy and Rempulska [6], Bogalska et al. [2], Khan and Ram [9], Krasniqi [13], and there are many other results established previously by a numerous researchers. In anticipation to reveal the aim of this paper, we introduce the integral

$$\overline{P}_\xi(f;x) := \frac{1}{2(1-e^{-\frac{\pi}{\xi}})\xi}\int_{-\pi}^{\pi} f(x+t)e^{-\frac{|t|}{\xi}}dt, \tag{4}$$

which we name "the truncated Picard singular integral". The idea of introducing such an integral is that it enables Lemma 1 (see section 2) to be applied in the proof of the main results, and can thus extend, its applicability and usefulness in approximation problems and beyond. As we will see, the application of the truncated Picard singular integral in approximation of a function f provides the degree of approximation that we are going to show in Section 3.

The main objective of this paper is to prove the analogues of Theorems 1 and 2 in the metric $\|\cdot\|_p^{(\cdot)}$ of the space $\mathcal{Z}_p^{(\cdot)}$. As far as we are aware, such results are not reported heretofore. For the proofs of our results, we use the same lines of reasoning as in [20] and [23].

1 Auxiliary Lemmas

We are going to recall the generalized Minkowski's inequality for integrals, which states that the norm of an integral is less or equal to the integral of the corresponding norm. For L^p spaces, it is reproduced below.

Lemma 1 (Generalized Minkowski inequality [8]) *If $z(x,t)$ is a function in two variables defined for $a \leq x \leq b$, $c \leq t \leq d$, then*

$$\left\{\int_a^b\left|\int_c^d z(x,t)dt\right|^p dx\right\}^{\frac{1}{p}} \leq \int_c^d\left\{\int_a^b |z(x,t)|^p dx\right\}^{\frac{1}{p}} dt, \quad p \geq 1.$$

Next lemma (it is given implicitly in [17], see pages 7-9) plays a key role in the proofs of our main results.

Lemma 2 *Let $f \in \mathcal{Z}_p^{(w_1)}$ $(1 \leq p < \infty)$ and let w_1 and w_2 be two Zygmund moduli of continuity such that w_1/w_2 is non-decreasing function of t. Then*

(i) $\|\varphi_{\cdot}(t)\|_p = \mathcal{O}(w_1(t))$,

(ii) $\|\varphi_{\cdot+h}(t) + \varphi_{\cdot-h}(t) - 2\varphi_{\cdot}(t)\|_p = \mathcal{O}(w_1(t))$,

(iii) $\|\varphi_{\cdot+h}(t) + \varphi_{\cdot-h}(t) - 2\varphi_{\cdot}(t)\|_p = w_2(|h|)\mathcal{O}(w_1(t)/w_2(t))$.

Now we are in able to report the main results.

2 Main Results

At first, we verify the following statement.

Theorem 3 *Let $f \in \mathcal{Z}_p^{(w_1)}$ with $1 \leq p < \infty$ and let $w_1(t)/t$ be a non-increasing function of t. Then*

$$\|\overline{P}_\xi(f;\cdot) - f\|_p = \mathcal{O}\left(w_1(\xi)\right), \tag{5}$$

$$\|Q_\xi(f;\cdot) - f\|_p = \mathcal{O}\left(w_1(\xi)|\ln(1/\xi)|\right), \tag{6}$$

$$\|W_\xi(f;\cdot) - f\|_p = \mathcal{O}\left(w_1(\xi)\xi^{-\frac{1}{2}}\right) \text{ as } \xi \to 0+. \tag{7}$$

Proof. We take into consideration the equality

$$\int_{-\pi}^{\pi} e^{-\frac{|t|}{\xi}} dt = 2(1 - e^{-\frac{\pi}{\xi}})\xi$$

and the truncated Picard singular integral (4) to obtain

$$\overline{P}_\xi(f;x) - f(x) = \frac{1}{(1 - e^{-\frac{\pi}{\xi}})\xi} \int_0^{\pi} \varphi_x(t) e^{-\frac{t}{\xi}} dt.$$

Wherefrom, applying Lemma 1, we get

$$\begin{aligned}
\|\overline{P}_\xi(f;\cdot) - f\|_p &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{(1 - e^{-\frac{\pi}{\xi}})\xi} \int_0^{\pi} \varphi_x(t) e^{-\frac{t}{\xi}} dt \right|^p dx \right\}^{\frac{1}{p}} \\
&\leq \frac{1}{(1 - e^{-\frac{\pi}{\xi}})\xi} \int_0^{\pi} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\varphi_x(t)|^p \, dx \right\}^{\frac{1}{p}} e^{-\frac{t}{\xi}} dt \\
&= \underbrace{\frac{1}{(1 - e^{-\frac{\pi}{\xi}})\xi} \int_0^{\xi} \|\varphi_{\cdot}(t)\|_p \, e^{-\frac{t}{\xi}} dt}_{=:\mathbb{P}_1} \\
&\quad + \underbrace{\frac{1}{(1 - e^{-\frac{\pi}{\xi}})\xi} \int_{\xi}^{\pi} \|\varphi_{\cdot}(t)\|_p \, e^{-\frac{t}{\xi}} dt}_{=:\mathbb{P}_2},
\end{aligned} \tag{8}$$

where $0 < \xi < \pi$.

In light of the fact that $w_1(t)$ is monotonically increasing, Lemma 2 (i) implies

$$\begin{aligned}\mathbb{P}_1 &= \mathcal{O}(1)\frac{1}{(1-e^{-\frac{\pi}{\xi}})\xi}\int_0^{\xi} w_1(t)e^{-\frac{t}{\xi}}dt\\ &= \mathcal{O}(1)\frac{w_1(\xi)}{(1-e^{-\frac{\pi}{\xi}})\xi}\int_0^{\xi} e^{-\frac{t}{\xi}}dt\\ &= \mathcal{O}(1)\frac{w_1(\xi)}{(1-e^{-\frac{\pi}{\xi}})\xi}\cdot\frac{e-1}{e}\xi = \mathcal{O}(w_1(\xi)). \end{aligned} \tag{9}$$

Furthermore, since the function $w_1(t)/t$ is non-increasing, using integration by parts, we obtain

$$\begin{aligned}\mathbb{P}_2 &= \mathcal{O}(1)\frac{1}{\xi}\int_{\xi}^{\pi} w_1(t)e^{-\frac{t}{\xi}}dt\\ &= \mathcal{O}(1)\frac{1}{\xi}\int_{\xi}^{\pi} \frac{w_1(t)}{t}te^{-\frac{t}{\xi}}dt\\ &= \mathcal{O}(1)\frac{w_1(\xi)}{\xi^2}\int_{\xi}^{\pi} te^{-\frac{t}{\xi}}dt\\ &= \mathcal{O}(1)\frac{w_1(\xi)}{\xi^2}\xi\left(2\xi e^{-1}-e^{-\frac{\pi}{\xi}}(\xi+\pi)\right) = \mathcal{O}(w_1(\xi)). \end{aligned} \tag{10}$$

As a result, (8) along with (9) and (10), imply (5).

Based on (2) and some suitable operations, we gain

$$Q_{\xi}(f;x)-f(x) = \frac{2\xi}{\pi}\int_0^{\pi}\frac{\varphi_x(t)}{t^2+\xi^2}dt - f(x)B(\xi), \tag{11}$$

where

$$B(\xi) := -\frac{2\xi}{\pi}\int_0^{\pi}\frac{dt}{t^2+\xi^2}+1.$$

Applying L'hospital's rule we arrive at

$$\lim_{\xi\to 0+}\frac{B(\xi)}{\xi} = \lim_{\xi\to 0+}\left(\frac{1}{\xi}-\frac{2}{\pi\xi}\arctan\left(\frac{\pi}{\xi}\right)\right) = \frac{2}{\pi^2}$$

and thus $(0<\xi\le\pi)$

$$B(\xi) = \mathcal{O}(\xi) = \mathcal{O}\left(\frac{\xi}{w_1(\xi)}w_1(\xi)\right) = \mathcal{O}\left(\frac{\pi}{w_1(\pi)}w_1(\xi)\right) = \mathcal{O}\left(w_1(\xi)\right) \tag{12}$$

since $t/w_1(t)$ is an increasing function of t.

Let us estimate the integral appearing in (11). As a matter of fact, we can write

$$q := \frac{2\xi}{\pi}\int_0^\pi \frac{\varphi_x(t)}{t^2+\xi^2}dt = \frac{2\xi}{\pi}\left(\underbrace{\int_0^\xi \frac{\varphi_x(t)}{t^2+\xi^2}dt}_{:=\mathcal{Q}_1} + \underbrace{\int_\xi^\pi \frac{\varphi_x(t)}{t^2+\xi^2}dt}_{:=\mathcal{Q}_2}\right).$$

Then, using Lemma 2 and $f \in \mathcal{Z}_p^{(w_1)}$, we have

$$\begin{aligned}\|\mathcal{Q}_1\|_p =&\mathcal{O}(1)\int_0^\xi \frac{\|\varphi_\cdot(t)\|_p}{t^2+\xi^2}dt = \mathcal{O}(1)\int_0^\xi \frac{w_1(t)}{t^2+\xi^2}dt\\ &= \mathcal{O}(w_1(\xi))\int_0^\xi \frac{dt}{t^2+\xi^2} = \mathcal{O}(w_1(\xi)/\xi).\end{aligned}$$

Over again, using Lemma 2, $f \in \mathcal{Z}_p^{(w_1)}$, and the assumption, we get

$$\begin{aligned}\|\mathcal{Q}_2\|_p =&\mathcal{O}(1)\int_\xi^\pi \frac{\|\varphi_\cdot(t)\|_p}{t^2+\xi^2}dt = \mathcal{O}(1)\int_\xi^\pi \frac{w_1(t)t}{t(t^2+\xi^2)}dt\\ =&\mathcal{O}\left(w_1(\xi)/\xi\right)\int_\xi^\pi \frac{tdt}{t^2+\xi^2} = \mathcal{O}(w_1(\xi)/\xi)\ln\frac{\pi^2+\xi^2}{2\xi^2}.\end{aligned}$$

Additionally, the limit

$$\lim_{\xi\to 0+}\frac{\ln\frac{\pi^2+\xi^2}{2\xi^2}}{\ln\left(\frac{1}{\xi}\right)} = \lim_{\xi\to 0+}\frac{2\pi^2}{\pi^2+\xi^2} = 2$$

implies

$$\|\mathcal{Q}_2\|_p = \mathcal{O}\left((w_1(\xi)/\xi)|\ln(1/\xi)|\right).$$

Further,

$$\|q\|_p = \mathcal{O}(\xi)\left(\|\mathcal{Q}_1\|_p + \|\mathcal{Q}_2\|_p\right) = \mathcal{O}\left(w_1(\xi)|\ln(1/\xi)|\right). \tag{13}$$

In consequence, relations (11), (12), and (13) give

$$\|Q_\xi(f;\cdot) - f\|_p \le \|q\|_p + \|f\|_p|B(\xi)| = \mathcal{O}\left(w_1(\xi)\ln|1/\xi|\right),$$

which is (6).

First of all, we can write

$$W_\xi(f;x) = \frac{2}{\sqrt{\pi\xi}}\int_0^\pi \varphi_x(t)e^{-\frac{t^2}{\xi}}dt + \frac{2f(x)}{\sqrt{\pi\xi}}\int_0^\pi e^{-\frac{t^2}{\xi}}dt$$

and taking into consideration that $\int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}}dt = \sqrt{\pi\xi} \quad (\xi > 0)$, we get

$$
\begin{aligned}
&W_\xi(f;x) - f(x)\\
&= \underbrace{\frac{2}{\sqrt{\pi\xi}}\int_0^\pi \varphi_x(t)e^{-\frac{t^2}{\xi}}dt}_{:=\mathcal{W}_1} - \underbrace{\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty} - \int_{-\pi}^{\pi}\right) f(x)e^{-\frac{t^2}{\xi}}dt}_{:=\mathcal{W}_2}. \qquad (14)
\end{aligned}
$$

Then, using Lemma 2 and $f \in \mathcal{Z}_p^{(w_1)}$, we obtain

$$
\begin{aligned}
\|\mathcal{W}_1\|_p &= \mathcal{O}\left(\frac{1}{\sqrt{\xi}}\right)\int_0^\pi \|\varphi_\cdot(t)\|_p e^{-\frac{t^2}{\xi}}dt\\
&= \mathcal{O}\left(\frac{1}{\sqrt{\xi}}\right)\left(\underbrace{\int_0^\xi w_1(t)e^{-\frac{t^2}{\xi}}dt}_{=:\mathcal{W}_{11}} + \underbrace{\int_\xi^\pi w_1(t)e^{-\frac{t^2}{\xi}}dt}_{=:\mathcal{W}_{12}}\right). \qquad (15)
\end{aligned}
$$

For $\mathcal{W}_{11}$, we get

$$
\mathcal{W}_{11} \le w_1(\xi)\int_0^\xi e^{-\frac{t^2}{\xi}}dt \le w_1(\xi)\int_0^\infty e^{-\frac{t^2}{\xi}}dt = \frac{\sqrt{\pi}}{2}\xi^{\frac{1}{2}}w_1(\xi) \le \frac{\pi}{2}w_1(\xi), \quad (16)
$$

while for $\mathcal{W}_{12}$, we obtain

$$
\mathcal{W}_{12} \le w_1(\xi)\int_\xi^\pi \frac{t}{\xi}e^{-\frac{t^2}{\xi}}dt \le w_1(\xi)\int_\xi^\infty \frac{t}{\xi}e^{-\frac{t^2}{\xi}}dt = \frac{1}{2e^\xi}w_1(\xi) \le \frac{1}{2}w_1(\xi). \quad (17)
$$

Thereafter, from (15) – (17) we conclude

$$
\|\mathcal{W}_1\|_p = \mathcal{O}\left(\frac{1}{\sqrt{\xi}}\right)(\mathcal{W}_{11} + \mathcal{W}_{12}) = \mathcal{O}\left(\frac{w_1(\xi)}{\sqrt{\xi}}\right). \qquad (18)
$$

For $\mathcal{W}_2$, we can write

$$
\mathcal{W}_2 = \frac{2f(x)}{\sqrt{\pi\xi}}\int_\pi^\infty e^{-\frac{t^2}{\xi}}dt
$$

and, whence, using the assumption that $t/w_1(t)$ is increasing with respect to t, we find that

$$
\begin{aligned}
\|\mathcal{W}_2\|_p &\le \frac{2\|f\|_p}{\pi\sqrt{\pi\xi}}\int_\pi^\infty te^{-\frac{t^2}{\xi}}dt = \frac{\|f\|_p}{\pi\sqrt{\pi\xi}} \cdot \frac{\xi}{e^{\frac{\pi^2}{\xi}}} \cdot \frac{w_1(\xi)}{\xi} \cdot \frac{\xi}{w_1(\xi)}\\
&\le \frac{\|f\|_p}{w_1(\pi)\sqrt{\pi}e^{\frac{\pi^2}{\xi}}} \cdot \frac{w_1(\xi)}{\sqrt{\xi}} = \mathcal{O}\left(\frac{w_1(\xi)}{\sqrt{\xi}}\right) \quad \text{for} \quad 0 < \xi < \pi. \qquad (19)
\end{aligned}
$$

Finally, combining (14), (18), and (19), we arrive at

$$
\|W_\xi(f;\cdot) - f\|_p \le \|\mathcal{W}_1\|_p + \|\mathcal{W}_2\|_p = \mathcal{O}\left(\frac{w_1(\xi)}{\sqrt{\xi}}\right)
$$

which is (7). The proof is completed. □

If in the definition of $\mathcal{Z}_p^{(w_1)}$ space, we take the function t^α $(0 < \alpha \leq 1)$ instead of $w_1(t)$, we obtain the space $\mathcal{Z}_{\alpha,p}$ equipped with the norm denoted as $\|\cdot\|_{\alpha,p} := \|\cdot\|_p$ just to distinguish the norm in this particular space (for details see Remark 8 of [22] at page 20). The condition that $w_1(t)/t$ have to be a non-increasing function of t, is clearly satisfied. Whence, Theorem 3 implies the following.

Corollary 1 *Let $f \in \mathcal{Z}_{p,\alpha}$ with $1 \leq p < \infty$, and $0 < \alpha \leq 1$. Then*

$$\begin{aligned}
&\|\overline{P}_\xi(f;\cdot) - f\|_p = \mathcal{O}\left(\xi^\alpha\right),\\
&\|Q_\xi(f;\cdot) - f\|_p = \mathcal{O}\left(\xi^\alpha |\ln(1/\xi)|\right),\\
&\|W_\xi(f;\cdot) - f\|_p = \mathcal{O}\left(\xi^{\alpha-\frac{1}{2}}\right) \quad as \quad \xi \to 0+\,.
\end{aligned}$$

In the sequel, we are going to prove the homologue statement of Theorem 2.

Theorem 4 *Let $f \in \mathcal{Z}_p^{(w_1)}$ with $1 \leq p < \infty$, $w_1(t)/w_2(t)$ be a non-decreasing function, and $w_1(t)/t$ be a non-increasing function. Then*

$$\|f - \overline{P}_\xi(f;\cdot)\|_p^{(w_2)} = \mathcal{O}\left(\frac{w_1(\xi)}{w_2(\xi)}\right), \tag{20}$$

$$\|f - Q_\xi(f;\cdot)\|_p^{(w_2)} = \mathcal{O}\left(\frac{w_1(\xi)}{w_2(\xi)}|\ln(1/\xi)|\right), \tag{21}$$

$$\|f - W_\xi(f;\cdot)\|_p^{(w_2)} = \mathcal{O}\left(\frac{w_1(\xi)}{w_2(\xi)\sqrt{\xi}}\right) \quad as \quad \xi \to 0+\,. \tag{22}$$

Proof. For the sake of brevity, we denote $D_\xi(f;x) := \overline{P}_\xi(f;x) - f(x)$. Then we can write

$$\begin{aligned}
&D_\xi(f;x{+}h) + D_\xi(f;x-h) - 2D_\xi(f;x)\\
&\quad= \frac{1}{(1-e^{-\frac{\pi}{\xi}})\xi}\int_0^\pi [\varphi_{x+h}(t) + \varphi_{x-h}(t) - 2\varphi_x(t)]e^{-\frac{t}{\xi}}dt\\
&\quad= \frac{1}{(1-e^{-\frac{\pi}{\xi}})\xi}\Bigg(\underbrace{\int_0^\xi [\varphi_{x+h}(t) + \varphi_{x-h}(t) - 2\varphi_x(t)]e^{-\frac{t}{\xi}}dt}_{:=\mathcal{D}_1}\\
&\qquad + \underbrace{\int_\xi^\pi [\varphi_{x+h}(t) + \varphi_{x-h}(t) - 2\varphi_x(t)]e^{-\frac{t}{\xi}}dt}_{:=\mathcal{D}_2}\Bigg),
\end{aligned} \tag{23}$$

where $0 < \xi \leq h \leq \pi$.

Using Lemma 1 and Lemma 2 (iii), we find that

$$\begin{aligned}
\|\mathcal{D}_1\|_p &= \mathcal{O}(1)\int_0^\xi \|\varphi_{x+h}(t)+\varphi_{x-h}(t)-2\varphi_x(t)\|_p e^{-\frac{t}{\xi}}dt\\
&= \mathcal{O}(w_2(|h|))\int_0^\xi \frac{w_1(t)}{w_2(t)}e^{-\frac{t}{\xi}}dt\\
&= \mathcal{O}(w_2(|h|))\frac{w_1(\xi)}{w_2(\xi)}\int_0^\pi e^{-\frac{t}{\xi}}dt\\
&= \mathcal{O}\left((1-e^{-\frac{\pi}{\xi}})\xi w_2(|h|)\frac{w_1(\xi)}{w_2(\xi)}\right). \qquad (24)
\end{aligned}$$

Likewise, using Lemma 1, Lemma 2 (ii), and the assumption that $w_1(t)/t$ is non-increasing function of t, we obtain

$$\begin{aligned}
\|\mathcal{D}_2\|_p &= \mathcal{O}(1)\int_\xi^\pi \|\varphi_{\cdot+h}(t)+\varphi_{\cdot-h}(t)-2\varphi_{\cdot}(t)\|_p e^{-\frac{t}{\xi}}dt\\
&= \mathcal{O}(1)\int_\xi^\pi \frac{\omega_1(t)}{t}te^{-\frac{t}{\xi}}dt\\
&= \mathcal{O}\left(\frac{w_1(\xi)}{\xi}\right)\int_\xi^\pi te^{-\frac{t}{\xi}}dt\\
&= \mathcal{O}\left(\frac{w_1(\xi)}{\xi}\right)\pi\int_0^\pi e^{-\frac{t}{\xi}}dt\\
&= \mathcal{O}\left((1-e^{-\frac{\pi}{\xi}})\xi\frac{w_1(\xi)}{w_2(\xi)}w_2(\xi)\right)\\
&= \mathcal{O}\left((1-e^{-\frac{\pi}{\xi}})\xi w_2(|h|)\frac{w_1(\xi)}{w_2(\xi)}\right) \qquad (25)
\end{aligned}$$

where $0<\xi\le h\le\pi$

Applying Minkowski's inequality in (23), and keeping in mind (24) and (25), we get

$$\begin{aligned}
&\frac{\|D_\xi(f;\cdot+h)+D_\xi(f;\cdot-h)-2D_\xi(f;\cdot)\|_p}{\omega_2(|h|)}\\
&\qquad = \mathcal{O}\left(\frac{1}{(1-e^{-\frac{\pi}{\xi}})\xi}\right)\sum_{j=1}^{2}\|\mathcal{D}_j\|_p = \mathcal{O}\left(\frac{w_1(\xi)}{w_2(\xi)}\right). \qquad (26)
\end{aligned}$$

Based on relation (5) (of Theorem 3), we can write

$$\|\overline{P}_\xi(f;\cdot)-f\|_p = \mathcal{O}\left(w_1(\xi)\right) = \mathcal{O}\left(\frac{w_1(\xi)}{w_2(\xi)}w_2(\xi)\right) = \mathcal{O}\left(\frac{w_1(\xi)}{w_2(\xi)}\right) \qquad (27)$$

since $w_2(\xi)\le w_2(\pi)$.

As a result, using (26) and (27), we obtain

$$\|f - \overline{P}_\xi(f;\cdot)\|_p^{(w_2)} = \mathcal{O}\left(\frac{w_1(\xi)}{w_2(\xi)}\right)$$

which is (20) as requested.

The proofs of (21) and (22) can be done in a similar way, and therefore, will be omitted. □

We can specialize the functions $w_1(t)$ and $w_2(t)$ so that they maintain the conditions of Theorem 2. Indeed, let $w_1(t) = t^\alpha$, $w_2(t) = t^\beta$, $0 \le \beta < \alpha \le 1$, then $w_1(t)/w_2(t) = t^{\alpha-\beta}$ is a non-decreasing and $t^{-1}w_1(t) = t^{\alpha-1}$ is a non-increasing function. Now we can deduce the following.

Corollary 2 *Let $f \in \mathcal{Z}_{p,\alpha}$ with $1 \le p < \infty$, $0 \le \beta < \alpha \le 1$, and $h \in [\xi, \pi]$. Then,*

$$\begin{aligned}
&\|f - \overline{P}_\xi(f;\cdot)\|_{(\beta,p)} = \mathcal{O}\left(\xi^{\alpha-\beta}\right), \\
&\|f - Q_\xi(f;\cdot)\|_{(\beta,p)} = \mathcal{O}\left(\xi^{\alpha-\beta}|\ln(1/\xi)|\right), \\
&\|f - W_\xi(f;\cdot)\|_{(\beta,p)} = \mathcal{O}\left(\xi^{\alpha-\beta-1/2}\right) \quad \text{as} \quad \xi \to 0+ .
\end{aligned}$$

Remark 1 *Even though the integrals $P_\xi(f;x)$ and $\overline{P}_\xi(f;x)$ differ in their limits of integration, we have seen that $\overline{P}_\xi(f;x)$ gives the same order of approximation as $P_\xi(f;x)$ in various metrics (Theorems 1–4).*

3 Conclusion

In this paper, we have introduced the truncated Picard singular integral. The deviations, in the L^p-norm and in the norm $\|\cdot\|_p^{(\cdot)}$, between this integrals and the functions from a generalized Zygmund space are obtained. We have demonstrated that the obtained degrees of approximation are of Jackson's order with exception in the case when the Picard-Cauchy singular integral have been used. The obtained results, in the point of view for future research, open new perspectives for further generalizations.

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Xhevat Z. Krasniqi
University of Prishtina
Erseka Street, no. 23

Prishtina, Kosovo.
xhevat.krasniqi0791@gmail.com

Ram N. Mohapatra
University of Central Florida
Department of Mathematics
Orlando, Florida, USA.
ram.mohapatra@ucf.edu

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