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On Locally Projectively Flat Finsler Space of a Special Exponential Metric with Constant Flag Curvature

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Abstract. From the point of view of Hilbert's fourth problem, Finsler metrics on an open subset of $\mathbb{R}^n$ with positive geodesics that are straight lines are known as locally projectively flat Finsler metrics. In this article, we have studied such projectively flat (α, β)-metrics in the form of the special exponential Finsler metric, where α is a Riemannian metric and β is a differential 1-form. We found that the special exponential metric is locally projectively flat if and only if α is locally projectively flat and β is parallel with respect to α. Furthermore, we obtained the flag curvature and proved that the special exponential metric is locally Minkowskian.

Key Words: Finsler Space, (α, β)-Metric, Special Exponential Metric, Locally Projectively Flat, Flag Curvature, Minkowskian Space
Mathematics Subject Classification 2020: 53B40, 53C60

Introduction

The investigation of the geometric characteristics of locally projectively flat Finsler manifolds is one of the key issues in Finsler geometry. Locally projectively flat metrics have a scalar flag curvature, which means that it depends only on the tangent vectors themselves and not on the tangent planes in which they are included.

The fourth problem of Hilbert characterizes the (not always reversible) distance functions on an open subset of $\mathbb{R}^n$ such that geodesics are straight lines. Projectively flat Finsler metrics are obtained from distance functions with regular straight geodesics. In Riemannian geometry, Beltrami stated that a Riemannian metric is projectively flat if and only if it has constant sectional curvature. The position with Finsler metrics, on the other hand, is far more complicated. The flag curvature in Finsler geometry is a natural generalization of the sectional curvature in the Riemannian case. Locally projectively flat Finsler metrics must be of scalar flag curvature; that is, the flag curvature is a scalar function on the tangent bundle, which may or may not be constant as in the Riemannian case [6]. Finsler metrics that are locally projectively flat are a unique class of Finsler metrics. This metric class has yet to be classified. However, some progress has been made in recent years ([10, 21]) despite some constraints on Finsler metrics.

To find examples of projectively flat Finsler metrics, we consider (α, β)-metrics. An (α, β)-metric is defined by $L = \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ positive scalar function on $(-b_0, b_0)$ satisfying $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ for $|s| \leq b < b_0$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form with $||\beta_x|| < b_0$, $x \in M$ on a manifold M. The simplest example of the (α, β)-metric is the Randers metric $L = \alpha + \beta$. In [13], it is proved that a Randers metric on a manifold is locally projectively flat if and only if α is projectively flat and β is closed. Another important example of the (α, β)-metric is the Berwald metric $L = (\alpha + \beta)^2/\alpha$ on a manifold. In [15], Shen and Yildrium proved that the Berwald metric is projectively flat if and only if the following conditions hold:

(1) $b_{i;j} = \tau\{(1 + 2b^2) - 3b_i b_j\}$,

(2) the spray coefficients $G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$,

where $b = ||\beta_x||_\alpha$, $b_{i;j}$ denote the covariant derivatives of β with respect to α, $\tau = \tau(x)$ is a scalar function, and $\theta_i y^i$ is a 1-form on M. Also, they demonstrated a local structure of L with constant flag curvature. Furthermore, for β not parallel to α, Shen [14] has given an equivalent definition for locally projectively flat (α, β)-metrics. Based on these results, Yu [22] completely characterized the locally projectively flat (α, β)-metrics by β-deformations. Numerous authors (see, for example, [1, 3–5, 7–9, 11, 14, 17–19]) have studied the concept of projectively flat Finsler metrics with different types of (α, β)-metrics. (α, β)-metrics have received more attention due to their ease of use and applicability in physics (see, for exmaple, [2]).

In 2020, Tripathi [16] studied a Finsler space with a special exponential (α, β)-metric and found the fundamental characteristics of this metric and the criteria for Finslerian hypersurfaces. Furthermore, Tripathi et al. [20] considered the same metric and proved the conditions under which the Finsler space with this metric will become a weakly Berwald space.

In the present paper, we consider Finsler space equipped with the above special type of exponential (α, β)-metric on an n-dimensional manifold M and study the conditions under which this metric is a locally projectively flat Finsler metric. Furthermore, we find the flag curvature and prove that the special exponential metric is locally Minkowskian.

1 Preliminaries

A Finsler metric is a scalar field $L(x, y)$ which satisfies the following three conditions:

1. L is C^∞ on TM_0,
2. L is positively 1-homogeneous on the fibers of tangent bundle TM,
3. the Hessian of L^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ is regular on TM_0, i.e., $det\,(g_{ij}) \neq 0$.

The manifold M equipped with a fundamental function $L(x, y)$ is then called a Finsler space $F^n = (M, L)$.

A family of Finsler metrics, known as (α, β)-metrics, is defined by a Riemannian metric and 1-form on a manifold M. These metrics are generally simple and possess significant curvature properties. Furthermore, they are computable. The Finsler space $F^n = (M, L)$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$, where α is a Riemannian metric and β is a differentiable 1-form. The space $\mathbb{R}^n = (M, \alpha)$ is called the associated Riemannian space and the covariant vector field b_i is the associated vector field.

An (α, β)-metric is expressed in the following form:

$$L = \alpha\phi(s), \qquad s = \beta/\alpha$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$. The norm $||\beta_x||_\alpha$ of β with respect to α is defined by

$$||\beta_x||_\alpha = \sup_{y\epsilon T_x M} \beta(x, y), \qquad \alpha(x, y) = a_{ij}(x)b_i(x)b_j(x).$$

In order to define L, β must satisfy the condition $||\beta_x||_\alpha < b_0$ for all $x \in M$.

Let G^i and G^i_α denote the spray coefficients of L and α respectively, defined as follows:

$$G^i = \frac{g^{il}}{4}\left\{[L^2]_{x^k y^l} y^k - [L^2]_{x^k}\right\}, \qquad G^i_\alpha = \frac{a^{il}}{4}\left\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]x^l\right\}.$$

where $g_{ij} = [L^2]_{y^i y^j}/2$ and $(a^{ij}) = (a_{ij})^{-1}$.

A Finsler metric $L = L(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is said to be projectively flat in U if all geodesics are straight lines. This is equivalent to $G^i = P(x, y)y^i$, where $P = L_{x^k} y^k/2L$ is a 1-homogeneous function on TM_0. In this case, L is of scalar curvature with flag curvature

$$K = \frac{P^2 - P_{x^k} y^k}{L^2}. \tag{1}$$

Furthermore, we have the following:

Lemma 1 *[6] The spray coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + H\{-2Q\alpha s_0 + r_{00}\}\left\{b^i - s\frac{y^i}{\alpha}\right\}, \quad (2)$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ H &= \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \\ J &= \frac{\phi'(\phi - s\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \end{aligned} \quad (3)$$

$s = \beta/\alpha$, $b = ||\beta_x||_\alpha$, $s_{ij} = (b_{i;j} - b_{j;i})/2$, $s_{l0} = s_{lj}y^i$, $s_0 = s_{l0}b^l$, $r_{ij} = (b_{i;j} + b_{j;i})/2$, and $r_{00} = r_{ij}y^iy^j$.

From (2), it is easy to see that if α is projectively flat ($G^i_\alpha = \eta y^i$) and β is parallel with respect to $\alpha(r_{ij} = 0, s_{ij} = 0)$, then $G^i = G^i_\alpha = \eta y^i$. Thus, $L = \alpha\phi(\beta/\alpha)$ is a projectively flat Berwald metirc. If $K \neq 0$, then L is a Riemannian metric by Numata's Theorem [12]. If $K = 0$, then L is locally Minkowskian.

Lemma 2 *[6] An (α, β)-metric $L = \alpha\phi(s)$, where $s = \beta/\alpha$, is projectively flat on an open subset $U \subset \mathbb{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_my_l)G^m_\alpha + \alpha^3 Q s_{l0} + \alpha H(-2\alpha Q s_0 + r_{00})(b_l\alpha - sy_l) = 0, \quad (4)$$

where $y_l = a_{lj}y^j$.

2 Locally Projectively Flat Special Exponenial (α, β)-Metric

In this section, we consider a special exponential (α, β)-metric in the following form:

$$L = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}. \quad (5)$$

The above equation can be rewritten as

$$L = \alpha\phi(s), \qquad \phi(s) = e^s + se^{-s}$$

where $s < 1$ so that ϕ must be a positive function. Let $b_0 > 0$ be the largest number such that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad |s| \leq b < b_0,$$

i.e.,

$$e^{-s}[b^2(s + e^{2s} - 2) - s^3 + (3 - e^{2s})s^2 + e^{2s}(1 - s)] > 0.$$

Lemma 3 *The special exponential metric in* (5) *is a Finsler metric.*

Proof. If the special exponential metric in (5) is a Finsler metric, then

$$e^{-s}[b^2(s+e^{2s}-2)-s^3+(3-e^{2s})s^2+e^{2s}(1-s)]>0.$$

Let $s=b$, for any $b<b_0$, $b<1$. Let $b\to b_0$, then $b_0<1$. Thus $||\beta_x||_\alpha<1$. Now if $|s|\leq b<1$, then

$$e^{-s}[b^2(s+e^{2s}-2)-s^3+(3-e^{2s})s^2+e^{2s}(1-s)]>0.$$

Therefore, $L=e^s+se^{-s}$ is a Finsler metric. □

By Lemma 1, the spray coefficients G^i of L are given by (3).

$$\begin{aligned} Q &= \frac{e^{2s}-s+1}{s^2+e^{2s}(1-s)} = \frac{\alpha(\alpha e^{2\beta/\alpha}-\beta+\alpha)}{\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)}, \\ H &= \frac{s+e^{2s}-2}{2[b^2(s+e^{2s}-2)-s^3+(3-e^{2s})s^2+e^{2s}(1-s)]} \\ &= \frac{\alpha^2(\beta+\alpha(e^{2\beta/\alpha}-2))}{2[b^2\alpha^2(\beta+\alpha(e^{2\beta/\alpha}-2))-\beta^3+\alpha\beta^2(3-e^{2\beta/\alpha})+\alpha^2e^{2\beta/\alpha}(\alpha-\beta)]}. \end{aligned} \tag{6}$$

Substituting (6) in (4), we have

$$\begin{aligned} &(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^4\left[\frac{\alpha(e^{2\beta/\alpha}+1)-\beta}{\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)}\right]s_{l0} \\ &+\frac{\alpha^2(\beta+\alpha(e^{2\beta/\alpha}-2)}{2[\alpha^3(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha})+\alpha^2\beta(b^2-e^{2\beta/\alpha})+\alpha\beta^2(3-e^{2\beta/\alpha})-\beta^3]} \\ &\left[-2\alpha^2\left(\frac{\alpha(e^{2\beta/\alpha}+1)-\beta}{\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)}\right)s_0+r_{00}\right](b_l\alpha^2-y_l\beta)=0. \end{aligned} \tag{7}$$

Lemma 4 *If* $(a_{ml}\alpha^2-y_my_l)G^m_\alpha=0$*, then* α *is locally projectively flat.*

Proof. If $(a_{ml}\alpha^2-y_my_l)G^m_\alpha=0$, then

$$a_{ml}\alpha^2G^m_\alpha=y_my_lG^m_\alpha. \tag{8}$$

Contracting (8) with a^{il}, we have

$$\alpha^2G^i_\alpha=y_my^iG^m_\alpha.$$

Let $P(x,y)=(y_m/\alpha^2)\,G^m_\alpha$, then

$$G^i_\alpha=Py^i.$$

Thus, α is projectively flat. □

Now, by Lemma 4, we prove the following

Theorem 1 *A special exponential (α, β)-metric $L = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$ is locally projectively flat if and only if the following conditions hold:*
1. β is parallel with respect to α,
2. α is locally projectively flat, that is, α is of constant curvature.

Proof. Assume that L is locally projectively flat. By rewriting (7) as a polynomial in y^i and α, we get

$$\begin{aligned}&\left(\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)\right)\left[2\left(\alpha^3(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha})+\alpha^2\beta(b^2-e^{2\beta/\alpha})\right.\right.\\&\left.\left.+\alpha\beta^2(3-e^{2\beta/\alpha})-\beta^3\right)\right](a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^4\left(\alpha(e^{2\beta/\alpha}+1)-\beta\right)\\&\left[2\left(\alpha^3(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha})+\alpha^2\beta(b^2-e^{2\beta/\alpha})+\alpha\beta^2(3-e^{2\beta/\alpha})-\beta^3\right)\right]s_{l0}\\&+\alpha^2\left(\beta+\alpha(e^{2\beta/\alpha}-2)\right)\left[-2\alpha^2\left(\alpha(e^{2\beta/\alpha}+1)-\beta\right)s_0\right.\\&\left.+\left(\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)\right)r_{00}\right](b_l\alpha^2-y_l\beta)=0.\end{aligned} \tag{9}$$

Contracting (9) with b^l gives

$$\begin{aligned}&\left(\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)\right)\left[2\left(\alpha^3(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha})+\alpha^2\beta(b^2-e^{2\beta/\alpha})\right.\right.\\&\left.\left.+\alpha\beta^2(3-e^{2\beta/\alpha})-\beta^3\right)\right](b_m\alpha^2-y_m\beta)G^m_\alpha+\alpha^4\left(\alpha(e^{2\beta/\alpha}+1)-\beta\right)\\&\left[2\left(\alpha^3(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha})+\alpha^2\beta(b^2-e^{2\beta/\alpha})+\alpha\beta^2(3-e^{2\beta/\alpha})-\beta^3\right)\right]s_0\\&+\alpha^2\left(\beta+\alpha(e^{2\beta/\alpha}-2)\right)\left[-2\alpha^2\left(\alpha(e^{2\beta/\alpha}+1)-\beta\right)s_0\right.\\&\left.+\left(\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)\right)r_{00}\right](b^2\alpha^2-\beta^2)=0.\end{aligned}$$

Namely,

$$\begin{aligned}&\left(\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)\right)\left[2\left(\alpha^3(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha})+\alpha^2\beta(b^2-e^{2\beta/\alpha})\right.\right.\\&\left.\left.+\alpha\beta^2(3-e^{2\beta/\alpha})-\beta^3\right)\right](b_m\alpha^2-y_m\beta)G^m_\alpha+\alpha^4\left(\alpha(e^{2\beta/\alpha}+1)-\beta\right)\\&\left(2\alpha^3e^{2\beta/\alpha}-2\alpha^2\beta e^{2\beta/\alpha}+2\alpha\beta^2\right)s_0+\alpha^2\left(\beta+\alpha(e^{2\beta/\alpha}-2)\right)\\&\left(\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)\right)r_{00}(b^2\alpha^2-\beta^2)=0.\end{aligned} \tag{10}$$

We have

$$\begin{aligned}&\left(\alpha(e^{\beta/\alpha}+1)-\beta\right)\left(2\alpha^3e^{2\beta/\alpha}-2\alpha^2\beta e^{2\beta/\alpha}+2\alpha\beta^2\right)\\&=\left(\beta^2+\alpha e^{2\beta/\alpha}(\alpha-\beta)\right)\left[2\alpha\left(\alpha(e^{2\beta/\alpha}+1)-\beta\right)\right].\end{aligned} \tag{11}$$

Substituting (11) into (10), we obtain

$$\begin{aligned}&2\left[\alpha^3(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha})+\alpha^2\beta(b^2-e^{2\beta/\alpha})+\alpha\beta^2(3-e^{2\beta/\alpha})-\beta^3\right]\\&(b_m\alpha^2-y_m\beta)G^m_\alpha+2\alpha^5\left(\alpha(e^{2\beta/\alpha}+1)-\beta\right)s_0\\&+\alpha^2\left(\beta+\alpha(e^{2\beta/\alpha}-2)\right)r_{00}(b^2\alpha^2-\beta^2)=0.\end{aligned} \tag{12}$$

Envisaging the coefficients of α necessarily be zero, from (12), we have

$$\begin{aligned}\left[2\alpha^2\left(b^2(e^{2\beta/\alpha}-2)+e^{2\beta/\alpha}\right)+2\beta^2(3-e^{2\beta/\alpha})\right](b_m\alpha^2-y_m\beta)G^m_\alpha \\ =2\alpha^4\beta s_0-\left[\alpha^2(e^{2\beta/\alpha}-2)\right](b^2\alpha^2-\beta^2)r_{00}\end{aligned} \tag{13}$$

and

$$\begin{aligned}\left[2\alpha^2\beta(b^2-e^{2\beta/\alpha})-2\beta^3\right](b_m\alpha^2-y_m\beta)G^m_\alpha \\ =-\left[2\alpha^6(e^{2\beta/\alpha}+1)s_0+\alpha^2\beta(b^2\alpha^2-\beta^2)r_{00}\right].\end{aligned} \tag{14}$$

Subtracting (13) $\times\,\beta$ from (14) $\times\left(e^{2\beta/\alpha}-2\right)$, we obtain

$$\begin{aligned}\left[2\alpha^2\beta\left(e^{4\beta/\alpha}-e^{2\beta/\alpha}\right)+2\beta^3\right](b_m\alpha^2-y_m\beta)G^m_\alpha \\ =\alpha^4\beta^2 s_0+\alpha^6\left(e^{2\beta/\alpha}-e^{4\beta/\alpha}-2\right)s_0.\end{aligned} \tag{15}$$

Let us rewrite (15) in the following form:

$$A+B\alpha^2+C\alpha^4+D\alpha^6=0, \tag{16}$$

where

$$\begin{aligned}A&=2\beta^3(b_m\alpha^2-y_m\beta)G^m_\alpha,\\ B&=2\beta\left(e^{4\beta/\alpha}-e^{2\beta/\alpha}\right)(b_m\alpha^2-y_m\beta)G^m_\alpha,\\ C&=-\beta^2 s_0,\\ D&=-\left(e^{2\beta/\alpha}-e^{4\beta/\alpha}-2\right)s_0.\end{aligned}$$

In (16), A can be divided by α^2, but β^3 cannot. Thus, there is a scalar function $\tau=\tau(x)$ such that $(b_m\alpha^2-y_m\beta)G^m_\alpha=\tau\alpha^2$. Then A and B become

$$\begin{aligned}A&=2\tau\alpha^2\beta^3,\\ B&=2\tau\alpha^2\beta\left(e^{4\beta/\alpha}-e^{2\beta/\alpha}\right).\end{aligned}$$

Replacing them in (16) gives

$$2\beta^3\tau+[2\beta\left(e^{4\beta/\alpha}-e^{2\beta/\alpha}\right)\tau+C]\alpha^2+D\alpha^4=0. \tag{17}$$

In (17), $2\beta^3\tau$ can be divided by α^2, but β^3 cannot. Thus, τ can be divided by α^2. This is impossible unless

$$\tau=0. \tag{18}$$

Hence, from $(b_m\alpha^2-y_m\beta)G^m_\alpha=\tau\alpha^2$ and (18), we have $(b_m\alpha^2-y_m\beta)G^m_\alpha=0$, which implies

$$(a_{ml}\alpha^2-y_m y_l)G^m_\alpha=0. \tag{19}$$

Thus, according to Lemma 4, α is locally projectively flat, i.e., $G^i_\alpha = 0$. Now, using equations (13) and (14), we obtain

$$2\alpha^4\beta s_0 - \left[\alpha^2(e^{2\beta/\alpha} - 2)\right](b^2\alpha^2 - \beta^2)r_{00} = 0$$

and

$$2\alpha^6(e^{2\beta/\alpha} + 1)s_0 + \alpha^2\beta(b^2\alpha^2 - \beta^2)r_{00} = 0.$$

Thus,

$$A.X = \begin{bmatrix} 2\alpha^4\beta & -\alpha^2(e^{2\beta\alpha} - 2)(b^2\alpha^2 - \beta^2) \\ 2\alpha^6(e^{2\beta/\alpha} + 1) & \alpha^2\beta(b^2\alpha^2 - \beta^2) \end{bmatrix} \begin{bmatrix} s_0 \\ r_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $A \neq 0$, we have

$$s_0 = 0, \qquad r_{00} = 0. \tag{20}$$

Substituting (19) and (20) in (7), we obtain

$$\alpha^4\left[\alpha\left(e^{2\beta/\alpha} + 1\right) - \beta\right]s_{l0} = 0.$$

Since $\alpha^4 \neq 0$ and $\left[\alpha\left(e^{2\beta/\alpha} + 1\right) - \beta\right] \neq 0$, we have $s_{l0} = 0$.

In view of equation (20) and $s_{l0} = 0$, we have $b_{i;j} = 0$, i.e., β is parallel with respect to α.

On the other hand, if β is parallel with respect to α and α is projectively flat, then by Lemma 1 and 2, we see that L is locally projectively flat. □

Lemma 5 *Let $L = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$ be locally projectively flat with constant flag curvature $\boldsymbol{K} = \lambda$. Then $\lambda = 0$.*

Proof. By Theorem 1, $G^i = \eta y^i$; then by (1), the equation $\mathbf{K} = \lambda$ gives

$$\alpha^2\left(e^{\beta/\alpha} + \beta/\alpha e^{-\beta/\alpha}\right)^2\lambda = \eta^2 - \eta_{x^k}y^k.$$

Using Taylor expansion of $e^{\beta/\alpha}$ and $e^{-\beta/\alpha}$, we get

$$\alpha^2\left(1 + 2\beta/\alpha - \beta^2/\alpha^2\right)^2\lambda = \eta^2 - \eta_{x^k}y^k. \tag{21}$$

Replacing y with $-y$, we get

$$\alpha^2\left(1 - 2\beta/\alpha - \beta^2/\alpha^2\right)^2\lambda = \eta^2 - \eta_{x^k}y^k. \tag{22}$$

Adding (21) and (22), we have

$$\alpha^2\lambda\left[\left(1 + 2\beta/\alpha - \beta^2/\alpha^2\right)^2 + \left(1 - 2\beta/\alpha - \beta^2/\alpha^2\right)^2\right] = 2(\eta^2 - \eta_{x^k}y^k). \tag{23}$$

From equation (23), we obtain

$$(\alpha^2 + \beta^2)^2\lambda = \alpha^2(\eta^2 - \eta_{x^k}y^k). \tag{24}$$

The left side of (24) is purely quadratic while right side is not, thus, $\lambda = 0$. □

Proposition 1 *Let $L = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$ be projectively flat with zero flag curvature. Then α is a flat metric and β is parallel with respect to α. In this case, L is locally Minkowskian.*

Proof. According to Theorem 1 and Lemma 5, β is parallel with respect to α and L has zero flag curvature, thus α has zero sectional curvature. As a result, α is a flat metric, that is, it is locally isometric to the Euclidean metric, and hence, $G^i_\alpha = 0$. Thus, $G^i = 0$, that is, L is locally Minkowskian. □

3 Conclusion

In the regular case, Hilbert's fourth problem is to study and describe projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$. Therefore, it is important to study locally projectively flat Finsler metrics. Also, it is well known that any Finsler metric that is locally projectively flat has a scalar flag curvature. This means that the flag curvature is a scalar function on the tangent bundle, and it may or may not be constant like in the Riemannian case [6]. Thus, locally projectively flat metrics form a rich class of Finsler metrics.

The main objective of this paper is to investigate locally projectively flat Finsler metric in the form of the special exponential (α, β)-metric. Furthermore, we identified flag curvature and proved that the metric mentioned above is locally Minkowskian.

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