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New inequalities of Grüss–Lupaş type and Applications for Selfadjoint Operators

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Abstract. In this paper, some Čebyšev–Lupaş type inequalities are proved. New inequalities of Grüss type for Riemann–Stieltjes integral are also obtained. Applications for functions of selfadjoint operators on complex Hilbert spaces are provided as well.

Key Words: Grüss inequality, Čebyšev inequality
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1 Introduction

In recent years the approximation problem of the Riemann–Stieltjes integral $\int_a^b f(t)du(t)$ via the famous Čebyšev functional

$$\mathcal{T}(f,g) = \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt. \quad (1)$$

increasingly became essential. In 1882, Čebyšev [12] derived an interesting result involving two absolutely continuous functions whose first derivatives are continuous and bounded, and is given by

$$|\mathcal{T}(f,g)| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \quad (2)$$

and the constant $\frac{1}{12}$ is the best possible.

In 1935, Grüss [32] proved another result for two integrable mappings f, g such that $\phi \le f(x) \le \Phi$ and $\gamma \le f(x) \le \Gamma$, the inequality

$$|\mathcal{T}(f,g)| \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma) \quad (3)$$

holds, and the constant $\frac{1}{4}$ is the best possible.

In [11] Beesack et al. have proved the following Čebyšev inequality for absolutely continuous functions whose first derivatives belong to L_p spaces (see also [34]):

$$|\mathcal{T}(f,g)| \leq \frac{b-a}{4}\left(\frac{2^p-1}{p(p+1)}\right)^{\frac{1}{p}}\left(\frac{2^q-1}{q(q+1)}\right)^{\frac{1}{q}}\|f'\|_p\|g'\|_q, \tag{4}$$

where $\|h\|_p := \left(\int_a^b |h(t)|^p\,dt\right)^{1/p}$, $\forall p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

For the constant

$$\omega(p,q) := \frac{1}{4}\left(\frac{2^p-1}{p(p+1)}\right)^{1/p}\left(\frac{2^q-1}{q(q+1)}\right)^{1/q}$$

we have

$$\frac{1}{8} \leq \omega(p,q) \leq \frac{1}{4}$$

for all $q=\frac{p}{p-1}$, $p>1$. Furthermore, we have the following particular cases in (4).

1. If $p=q=2$, we have

$$|\mathcal{T}(f,g)| \leq \frac{b-a}{8}\|f'\|_2\|g'\|_2. \tag{5}$$

2. If $q \longrightarrow \infty$, we have

$$|\mathcal{T}(f,g)| \leq \frac{b-a}{4}\|f'\|_1\|g'\|_\infty. \tag{6}$$

In 1970, A.M. Ostrowski [35] has proved the following combination of the Čebyšev and Grüss results

$$|\mathcal{T}(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty. \tag{7}$$

where, g is absolutely continuous with $g' \in L_\infty[a,b]$ and f is Lebesgue integrable on [a, b] and satisfying $m \leq f(t) \leq M$, for all $t \in [a,b]$. The constant $\frac{1}{8}$ is the best possible.

In 1973, Laupş has improved Beesack et al. inequality (5), as follows:

$$|\mathcal{T}(f,g)| \leq \frac{(b-a)}{\pi^2}\|f'\|_2\|g'\|_2, \tag{8}$$

provided that f,g are two absolutely continuous functions on $[a,b]$ with $f',g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

In 2012, Dragomir [16] used the identity ([34], p. 246),

$$\mathcal{T}(f,g) = \frac{1}{b-a}\int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right] \cdot \\ \cdot \left[g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right] dt, \quad (9)$$

and has proved the following inequality:

Theorem 1 *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$, then*

$$|\mathcal{T}(f,g)| \leq \frac{1}{2}\bigvee_a^b (f) \cdot \frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \quad (10)$$

where, $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in (10).

Another result when both functions are of bounded variation, was considered in the same paper [16], as follows:

Theorem 2 *If $f,g : [a,b] \to \mathbb{C}$ are of bounded variation on $[a,b]$, then*

$$|\mathcal{T}(f,g)| \leq \frac{1}{4}\bigvee_a^b (f) \cdot \bigvee_a^b (g) \quad (11)$$

The constant $\frac{1}{4}$ is best possible in (11).

After that many authors have studied the functional (1) and therefore, several bounds under various assumptions had been obtained, for more new results and generalizations the reader may refer to [1],[2], [7]–[10], [13]–[17].

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(x)\,du(x)$ by the Riemann integral $\int_a^b f(t)\,dt$, Dragomir and Fedotov [21], have introduced the following functional:

$$\mathcal{D}(f;u) := \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt, \quad (12)$$

provided that the Riemann-Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

In the same paper [21], the authors have proved the following result:

$$|\mathcal{D}(f;u)| \leq \frac{1}{2}K(b-a)\bigvee_a^b (u), \quad (13)$$

provided that u is of bounded variation and f is Lipschitzian with the constant $K > 0$. Then we have the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In [22], Dragomir and Fedotov have obtained the following inequality:

$$|\mathcal{D}(f;u)| \le \frac{1}{2}L(M-m)(b-a), \tag{14}$$

provided that u is L-Lipschitzian on $[a,b]$ and f is Riemann integrable on $[a,b]$ such that $m \le f(x) \le M$ for any $x \in [a,b]$ where, $m, M > 0$. The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity. For more result of this type see [3]–[6], [14], [15] and [19]–[23].

This paper is organized as follows: in section 2, we obtain new bounds for the Čebyšev functional $\mathcal{T}(f,g)$ where f is assumed to be of p–Hölder type or of bounded variation on $[a,b]$ while the function $g \in L_2[a,b]$. In section 3, two inequalities of Grüss type for Riemann–Stieltjes integral are proved. Finally, in section 4, applications for functions of selfadjoint operators on complex Hilbert spaces are provided.

2 The results

We may start with the following result:

Theorem 3 *Let $f : [a,b] \to \mathbb{C}$ be a p-H_f-Hölder continuous on $[a,b]$, where $p \in (0,1]$ and $H_f > 0$ are given. Let $g : [a,b] \to \mathbb{C}$ be such that $g' \in L^2[a,b]$, then*

$$|\mathcal{T}(f,g)| \le \frac{H_f}{2\pi}(b-a)^{p+\frac{1}{2}}\|g'\|_2. \tag{15}$$

Proof. Taking the modulus in (9), utilizing the triangle inequality and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\mathcal{T}(f,g)| &== \left|\frac{1}{b-a}\int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right]\cdot\right.\\ &\left.\cdot\left[g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right]dt\right| \le \frac{1}{b-a}\int_a^b \left|f(t) - \frac{f(a)+f(b)}{2}\right|\cdot\\ &\cdot\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \le \sup_{t\in[a,b]}\left|f(t) - \frac{f(a)+f(b)}{2}\right|\cdot\\ &\cdot\left(\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\right|^2 dt\right)^{1/2}. \end{aligned} \tag{16}$$

Now, by (8), we have

$$\mathcal{T}(g,g) := \frac{1}{b-a}\int_a^b \left(g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right)^2 dt \le \frac{b-a}{\pi^2}\left\|g'\right\|_2^2. \quad (17)$$

As f is p-H_f–Holder continuous on $[a,b]$, then

$$\begin{aligned}\left|f(t) - \frac{f(a)+f(b)}{2}\right| &\le \frac{1}{2}\left[\left|f(t)-f(a)\right| + \left|f(t)-f(b)\right|\right] \\ &\le \frac{H_f}{2}\left[(t-a)^p + (b-t)^p\right],\end{aligned}$$

which implies that

$$\sup_{t\in[a,b]}\left|f(t) - \frac{f(a)+f(b)}{2}\right| \le \frac{H_f}{2}(b-a)^p. \quad (18)$$

Substituting (17) and (18) in (16), we get the required result (15). □

Corollary 1 *Let g be as in Theorem 3. If $f:[a,b]\to\mathbb{R}$ is L_f–Lipschitzian on $[a,b]$, then*

$$\left|\mathcal{T}(f,g)\right| \le \frac{L_f}{2\pi}(b-a)^{3/2}\left\|g'\right\|_2. \quad (19)$$

Theorem 4 *Let g be as in Theorem 3. Let $f:[a,b]\to\mathbb{C}$ be a mapping of bounded variation on $[a,b]$, then we have*

$$\left|\mathcal{T}(f,g)\right| \le \frac{1}{2\pi}(b-a)^{1/2}\left\|g'\right\|_2 \bigvee_a^b(f) \quad (20)$$

Proof. As in Theorem 3, we have observed that

$$\begin{aligned}&\left|\mathcal{T}(f,g)\right| \\ &\le \frac{1}{b-a}\sup_{x\in[a,b]}\left|f(t) - \frac{f(a)+f(b)}{2}\right|\int_a^b\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \quad (21)\end{aligned}$$

As f is of bounded variation on $[a,b]$, we have

$$\begin{aligned}\sup_{t\in[a,b]}\left|f(t) - \frac{f(a)+f(b)}{2}\right| &\le \frac{1}{2}\sup_{t\in[a,b]}\left[\left|f(t)-f(a)\right| + \left|f(t)-f(b)\right|\right] \\ &\le \frac{1}{2}\bigvee_a^b(f).\end{aligned}$$

Since g is absolutely continuous and $g'\in L_2[a,b]$, then (17) holds. Combining the above inequality with (17) and then substituting in (21) we get the required result. □

Theorem 5 *Let $f:[a,b]\to\mathbb{C}$ be such that f is absolutely continuous and $f'\in L_2[a,b]$ and $g:[a,b]\to\mathbb{R}$ satisfies the condition that there exists $m,M>0$ such that $m\le g(t)\le M$, for all $t\in[a,b]$, then*

$$\left|\mathcal{T}(f,g)\right|\le\frac{(b-a)}{2\pi}(M-m)\left\|f'\right\|_2. \tag{22}$$

Proof. Taking the modulus in (9) and utilizing the triangle inequality, we get

$$\begin{aligned}\left|\mathcal{T}(f,g)\right|&=\left|\frac{1}{b-a}\int_a^b\left[f(t)-f\left(\frac{a+b}{2}\right)\right]\left[g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right]dt\right|\\&\le\frac{1}{b-a}\int_a^b\left|f(t)-f\left(\frac{a+b}{2}\right)\right|\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt\\&\le\frac{1}{b-a}\left(\int_a^b\left|f(t)-f\left(\frac{a+b}{2}\right)\right|^2dt\right)^{\frac{1}{2}}\\&\qquad\left(\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\right|^2dt\right)^{\frac{1}{2}},\end{aligned}$$

which follows by the Cauchy-Schwarz inequality. Since f is absolutely continuous on $[a,b]$, then

$$\int_a^b\left|f(t)-f\left(\frac{a+b}{2}\right)\right|^2dt\le\frac{(b-a)^2}{\pi^2}\int_a^b\left|f'(t)\right|^2dt. \tag{23}$$

Now, we define

$$I(g):=\frac{1}{b-a}\int_a^b\left(g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right)^2dt.$$

Then, we have

$$\begin{aligned}I(g)&:=\frac{1}{b-a}\int_a^b\left[g^2(t)-2g(t)\frac{1}{b-a}\int_a^b g(s)\,ds+\right.\\&\left.+\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right)^2\right]dt=\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right)^2\end{aligned}$$

and

$$\begin{aligned}I(g)&:=\left(M-\frac{1}{b-a}\int_a^b g(s)\,ds\right)\left(\frac{1}{b-a}\int_a^b g(s)\,ds-m\right)\\&\qquad-\frac{1}{b-a}\int_a^b(M-g(t))(g(t)-m)\,dt.\end{aligned}$$

As $m \le g(t) \le M$, for all $t \in [a,b]$, then

$$\int_a^b (M - g(t))(g(t) - m)\, dt \ge 0,$$

which implies

$$\begin{aligned} I(g) &\le \left(M - \frac{1}{b-a}\int_a^b g(s)\, ds\right)\left(\frac{1}{b-a}\int_a^b g(s)\, ds - m\right) \\ &\le \frac{1}{4}\left[\left(M - \frac{1}{b-a}\int_a^b g(s)\, ds\right) + \left(\frac{1}{b-a}\int_a^b g(s)\, ds - m\right)\right]^2 \\ &= \frac{1}{4}(M-m)^2. \end{aligned} \tag{24}$$

Using the CBS inequality we have

$$I(g) \ge \left[\frac{1}{b-a}\left|\int_a^b g(t) - \frac{1}{b-a}\int_a^b g(s)\, ds\right| dt\right]^2$$

we get from (24), that

$$\left(\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\right|^2 dt\right)^{1/2} \le \frac{1}{2}(M-m)(b-a). \tag{25}$$

Thus, by (23) and (25) we get the desired result. □

Theorem 6 *Let $f, g : [a,b] \to \mathbb{C}$ be two continuous functions such that $f' \in L_\infty[a,b]$ and $g' \in L_2[a,b]$, then*

$$|\mathcal{T}(f,g)| \le \frac{(b-a)^{3/2}}{\pi\sqrt{12}} \|f'\|_\infty \|g'\|_2 . \tag{26}$$

Proof. We use the inequality

$$|\mathcal{T}(f,g)| \le \mathcal{T}^{1/2}(f,f) \cdot \mathcal{T}^{1/2}(g,g). \tag{27}$$

Since $f' \in L_\infty[a,b]$ and $g' \in L_2[a,b]$, by (2) and (8), we respectively have

$$\mathcal{T}(f,f) \le \frac{(b-a)^2}{12} \|f'\|_\infty^2 ,$$

and

$$\mathcal{T}(g,g) \le \frac{b-a}{\pi^2} \|g'\|_2^2 .$$

Combining the above two inequalities with (27) we get the desired inequality (26). □

3 Bounding the Dragomir–Fedotov functional

We may state our first result regarding Dragomir–Fedotov functional as follows:

Theorem 7 *Let $u, f : [a,b] \to \mathbb{R}$ be such that u is L–Lipschitzian on $[a,b]$, and f is Lebesgue integrable on $[a,b]$, then*

$$|\mathcal{D}(f;u)| \le L(b-a) \cdot \mathcal{T}^{1/2}(f,f) \tag{28}$$

Proof. It is well-known that for a Riemann integrable function $p : [a,b] \to \mathbb{R}$ and L–Lipschitzian function $\nu : [a,b] \to \mathbb{R}$, one has the inequality

$$\left| \int_a^b p(t)\, d\nu(t) \right| \le L \int_a^b |p(t)|\, dt. \tag{29}$$

Therefore, as u is L-Lipschitzian on $[a,b]$, we have

$$\begin{aligned} |\mathcal{D}(f;u)| &= \left| \int_a^b \left[f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right] du(x) \right| \\ &\le L \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| dx, \end{aligned} \tag{30}$$

as (17), we have shown that

$$\left[\frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt \right]^2 \le \mathcal{T}(f,f), \tag{31}$$

simple calculations with (30) yield the required result. □

Theorem 8 *Let $u, f : [a,b] \to \mathbb{R}$ be such that u is L–Lipschitzian on $[a,b]$, and f is absolutely continuous on $[a,b]$ with $f' \in L_2[a,b]$, then*

$$|\mathcal{D}(f;u)| \le \frac{L}{\pi} (b-a)^{3/2} \|f'\|_2 . \tag{32}$$

Proof. As we shown in Theorem 7, we have

$$\begin{aligned} |\mathcal{D}(f;u)| &= \left| \int_a^b \left[f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right] du(x) \right| \\ &\le L \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| dx \end{aligned} \tag{33}$$

and

$$\int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| dx \le (b-a)\, \mathcal{T}^{1/2}(f,f). \tag{34}$$

On the other hand, by Lupaş inequality (8), we have

$$\mathcal{T}(f,f) \le \frac{(b-a)}{\pi^2}\|f'\|_2^2,$$

which implies that

$$|\mathcal{D}(f;u)| \le \frac{L}{\pi}(b-a)^{3/2}\|f'\|_2,$$

and the theorem is proved. □

Corollary 2 *Let $u, f : [a,b] \to \mathbb{R}$ be such that u is L–Lipschitzian on $[a,b]$, and f is absolutely continuous such that $f' \in L_\infty[a,b]$, then*

$$|\mathcal{D}(f;u)| \le \frac{L}{\sqrt{12}}(b-a)^2\|f'\|_\infty. \tag{35}$$

Proof. By (33), (34) and Čebyšev inequality (2), we have

$$\mathcal{T}(f,f) \le \frac{1}{12}(b-a)^2\|f'\|_\infty^2,$$

which implies that

$$|\mathcal{D}(f;u)| \le \frac{L}{\sqrt{12}}(b-a)^2\|f'\|_\infty,$$

and the inequality is proved. □

4 Applications for Selfadjoint Operators

Let U be a bounded selfadjoint operator on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any continuous function $f : [m,M] \to \mathbb{R}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U)x, y\rangle = \int_{m+0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle \tag{36}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of bounded variation on the interval $[m,M]$ and $g_{x,y}(m+0) = 0$ and $g_{x,y}(M) = \langle x, y\rangle$, for any $x, y \in H$.

It is also well known that $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$, is monotonic nondecreasing and right continuous on $[m,M]$.

Theorem 9 *Let A be a bounded selfadjoint operator on the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

(i) If $u : [m, M] \to \mathbb{C}$ is a L–Lipschitzian function on $[m, M]$, where $L > 0$ and $f : [m, M] \to \mathbb{C}$ is Lebesgue integrable on $[m, M]$, then

$$\left|\langle f(A)x, y\rangle - \langle x, y\rangle \cdot \frac{1}{M-m}\int_m^M f(s)\,ds\right| \le L(M-m)\left|C(f, f; A; x)\right|^{1/2}, \quad (37)$$

where $C(f, g; A; x)$ denotes the following Čebyšev functional

$$C(f, g; A; x) := \langle f(A)g(A)x, x\rangle - \langle f(A)x, x\rangle \cdot \langle g(A)x, x\rangle$$

and $x \in H$ with $\|x\| = 1$.

(ii) If $f : [m, M] \to \mathbb{C}$ is absolutely continuous and $f' \in L_2[m, M]$ while u is as in (i), then

$$\left|\langle f(A)x, y\rangle - \langle x, y\rangle \cdot \frac{1}{M-m}\int_m^M f(s)\,ds\right| \le \frac{L}{\pi}(M-m)^{3/2}\|f'\|_2 \quad (38)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. (i) Fix $x \in H$ with $\|x\| = 1$. Let $s > 0$ and extend by continuity the function f to the interval $[m+s, M]$ by preserving its properties from $[m, M]$. Consider also the monotonic function $u(\lambda) := \langle E_\lambda x, x\rangle$ which is monotonic nondecreasing and right continuous on $[m+s, M]$.

Utilizing the spectral representation (36) we have the following equality of interest

$$\langle f(A)x, y\rangle - \langle x, y\rangle \cdot \frac{1}{M-m}\int_m^M f(s)\,ds = \int_{m+s}^M \left[f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds\right] d\langle E_t x, y\rangle \quad (39)$$

for any $x, y \in H$.

Applying Theorem 7, we get

$$\begin{aligned}
&\left|\langle f(A)x, y\rangle - \langle x, y\rangle \cdot \frac{1}{M-m}\int_m^M f(s)\,ds\right| \\
&= \left|\int_{m+s}^M \left[f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds\right] d\langle E_t x, y\rangle\right| \\
&\le L\int_{m+s}^M \left|f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds\right| dt \\
&\le L(M-m)\left|C(f, f; A; x)\right|^{1/2}
\end{aligned}$$

Finally, letting $s \to 0+$ and utilizing the representation (36) and the fact that $u(M) = 1$, $u(m+0) = 0$, we get the required result.

The proof of (ii) may be done by applying Theorem 8 and similar to that one in Theorem 7, we omit the details. □

Remark 1 *The interested reader in operator inequalities may be able to find other recent results providing various bounds for the Čebyšev functional* $C(f, g; A; x)$ *in the papers [24]-[30] and the monograph [31].*

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